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Bifurcation analysis of a population model and the resulting SIS epidemic model with delay☆

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Abstract

This paper deals with the model for matured population growth proposed in Cooke et al. [Interaction of matiration delay and nonlinear birth in population and epidemic models, J. Math. Biol. 39 (1999) 332–352] and the resulting SIS epidemic model. The dynamics of these two models are still largely undetermined, and in this paper, we perform some bifurcation analysis to the models. By applying the *global bifurcation theory* for functional differential equations, we are able to show that the population model allows multiple periodic solutions. For the SIS model, we obtain some local bifurcation results and derive formulas for determining the bifurcation direction and the stability of the bifurcated periodic solution.

Keywords: Population; Epidemic; SIS model; Delay; Stability; Hopf bifurcation; Periodic solution

1. Introduction

For single species population growth, Cooke et al. [2] proposed the following model:

$$\dot{N}(t) = B(N(t-\tau))N(t-\tau)e^{-d_1\tau} - dN(t). \tag{1.1}$$

Here, $N(t)$ denote the mature population of the species, and $d_1 \geqslant 0$ and $d \geqslant 0$ are the death rate of the immature and mature population, respectively, and the delay $\tau \geqslant 0$ is the maturation time. The birth rate function $B(N)$ is assumed to satisfy the basic assumptions for $N \in (0, \infty)$:

(A1) $B(N) > 0$;
(A2) $B(N)$ is continuously differentiable with $B'(N) < 0$;
(A3) $B(0^+) > de^{d_1\tau} > B(\infty)$.

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One important feature of (1.1) is that the delay τ also appears in the coefficient (if $d_0 > 0$) in addition to appearing in the unknown function $N(t)$. Such a feature has brought in new phenomena in the dynamics behavior. For example, using τ as a parameter, and choosing the frequently used Rick function

$$B(N) = b\mathrm{e}^{-aN}. \tag{1.2}$$

Cooke et al. [2] observed switches of stability–instability–stability when τ increases. The recent paper Beretta and Kuang [1] give more detailed results on such delay differential equations with delay appearing in the coefficients.

The analysis of (1.1) with (1.2) is far from complete. One purpose of this paper is to perform a more thorough bifurcation analysis on (1.1) in Section 2. Unlike in Cooke et al. [2], where the delay τ is used as the bifurcation parameter, here we will use b as bifurcation parameter. Our global Hopf bifurcation analysis shows that the model (1.1) with (1.2) actually allows multiple periodic solutions as b increases. The bifurcation diagram on b–τ plane would easily reproduce the double switch of the stability for (1.1)–(1.2) obtained in Cooke et al. [2] (see the Remark in Section 2).

Using (1.1) as the basis population model, Cooke et al. [2] also constructed and studied the following SIS epidemic model:

$$\begin{cases} \dot{I}(t) = \mu(N(t) - I(t))\dfrac{I(t)}{N(t)} - (d + \varepsilon + \gamma)I(t), \\ \dot{N}(t) = b\mathrm{e}^{-aN(t-\tau)}N(t-\tau)\mathrm{e}^{-d_1\tau} - dN(t) - \varepsilon I(t). \end{cases} \tag{1.3}$$

Here the total population is divided into susceptible and infective classes, with the size of each class given by $S(t)$ and $I(t)$, respectively, so that $N(t) = S(t) + I(t)$. The parameter $\mu > 0$ is the contact rate constant, $\gamma \geqslant 0$ is the recovery rate constant, and $\varepsilon \geqslant 0$ is the disease induced death rate constant. When $B(N)N$ is increasing, the dynamics of (1.3) have been well determined in Cooke et al. [2] by applying monotone dynamics system theory and the theory for asymptotically autonomous systems. However, when $B(N)N$ does not possess the monotonicity (e.g., when $B(N)$ is the Rick function), the dynamics of system (1.3) become a hard problem. Only for the special case $\varepsilon = 0$, when $N(t)$ equation is decoupled from $I(t)$ equations, was the stability the endemic equilibrium (EE) established in Cooke et al. [2]; and little bit later, using a perturbation technique, the result was extended in Zhao and Zou [8] to the case when ε is sufficiently small. Overall, the dynamics of this SIS model (1.3) still largely remain undetermined. Thus, our second goal in this paper is to further analyze the SIS model (1.3) to gain more knowledge about the dynamics of the model, and thereby, obtain more insight into the model. The focus is the existence and stability of Hopf bifurcation, and this is done in Section 3.

As mentioned above, the dynamics of (1.1) and (1.3) with increasing $B(N)N$ have been well determined, but the case when $B(N)N$ does not possess the monotonicity remains largely undetermined. Therefore, we are only interested in the later case. For concreteness, in the rest of the paper, we will consider the Rick function for $B(N)$ which is given by (1.2).

2. Local and global Hopf bifurcation for (1.1) with $B(N)$ being the Rick function

In this section, we consider the population model with delay

$$\dot{N}(t) = b\mathrm{e}^{-aN(t-\tau)}N(t-\tau)\mathrm{e}^{-d_1\tau} - dN(t). \tag{2.1}$$

Throughout this session, we assume

(H1) $b > d\mathrm{e}^{d_1\tau}$,

under which, (2.1) has two biological meaningful equilibria:

$$N = 0 \quad \text{and} \quad N^* = \frac{1}{a}\ln\frac{b}{d\mathrm{e}^{d_1\tau}}.$$

Set $x(t) = N(t) - N^*$. Then Eq. (2.1) becomes

$$\dot{x}(t) = -dx(t) - dN^* + d[N^* + x(t-\tau)]\mathrm{e}^{-ax(t-\tau)}. \tag{2.2}$$

The linearization of Eq. (2.2) at $x=0$ is

$$\dot{x}(t) = -dx(t) + d\left[1 - \ln\frac{b}{de^{d_1\tau}}\right]x(t-\tau),$$

whose characteristic equation is

$$\lambda = -d + d\left[1 - \ln\frac{b}{de^{d_1\tau}}\right]e^{-\lambda\tau}. \tag{2.3}$$

Firstly, employing the result in Ruan and Wei [5], we investigate the distribution of roots of the characteristic equation (2.3), and give the bifurcation diagram in (τ, b)-plane. Then, combining the global Hopf bifurcation theorem due to Wu [7] and high-dimensional Bendixson's criterion established in Li and Muldowney [4], we obtain a set of conditions for global existence of periodic solutions to model (2.1).

Lemma 2.1. *If*

$$d < b \leqslant de^{d_1\tau+2}, \tag{2.4}$$

then all roots of Eq. (2.3) *have negative real parts.*

Proof. First of all, we know that the root of Eq. (2.3) with $\tau = 0$ satisfies

$$\lambda = -d\ln\frac{b}{d} < 0.$$

For $\tau > 0$, clearly $\lambda = 0$ is not a root to Eq. (2.3). Let $i\omega(\omega > 0)$ be a root of Eq. (2.3). Then we have

$$i\omega = -d + d\left[1 - \ln\frac{b}{de^{d_1\tau}}\right](\cos(\omega\tau) - i\sin(\omega\tau)).$$

Separating the real and imaginary parts gives

$$\begin{cases} d = d\left[1 - \ln\frac{b}{de^{d_1\tau}}\right]\cos(\omega\tau), \\ \omega = -d\left[1 - \ln\frac{b}{de^{d_1\tau}}\right]\sin(\omega\tau). \end{cases} \tag{2.5}$$

It follows that

$$\omega^2 = d^2\ln\frac{b}{de^{d_1\tau}}\left(\ln\frac{b}{de^{d_1\tau}} - 2\right). \tag{2.6}$$

Eq. (2.6) makes sense if and only if

$$b > de^{d_1\tau+2}$$

holds. Thus, under inequality (2.4), Eq. (2.3) has no imaginary root. In other words, Eq. (2.3) has no root appearing on the imaginary axis for $b \in (d, de^{d_1\tau+2}]$. Recalling that the root of Eq. (2.3) with $\tau = 0$ has negative real part, and applying Corollary 2.4 in Ruan and Wei [5], the conclusion follows. □

Now we regard b as the bifurcation parameter. Then we can prove the following transversality result.

Lemma 2.2. *Let*

$$\lambda(b) = \alpha(b) + i\omega(b)$$

be a root of Eq. (2.3) *satisfying* $\alpha(b_0) = 0$ *and* $\omega(b_0) = \omega_0 > 0$ *for certain* $b_0 > de^{d_1\tau}$. *Then* $\alpha'(b_0) > 0$.

Proof. In fact, substituting $\lambda(b)$ into Eq. (2.3) and taking the derivative with respect to b, we have

$$\frac{\mathrm{d}\lambda(b)}{\mathrm{d}b} = -\frac{(\lambda + d)}{b(1+\tau(\lambda+d))(1-\ln(b/d\mathrm{e}^{d_1\tau}))},$$

and hence,

$$\frac{\mathrm{d}\lambda(b_0)}{\mathrm{d}b} = \frac{-1}{\Delta b_0(1-\ln(b_0/d\mathrm{e}^{d_1\tau}))}[d(1+\tau d)+\tau\omega_0^2+\mathrm{i}\omega_0],$$

where $\Delta = (1+\tau d)^2 + \tau^2\omega_0^2$. Thus,

$$\alpha'(b_0) = -\frac{d(1+\tau d)+\tau\omega_0^2}{\Delta b_0(1-\ln(b_0/d\mathrm{e}^{d_1\tau}))}. \tag{2.7}$$

By $b_0 > d\mathrm{e}^{d_1\tau}$, we have $\alpha'(b_0) > 0$. □

Next lemma identifies the critical values of b at which, Hopf bifurcation may occur.

Lemma 2.3. *For each fixed* $\tau > 0$, *there exists a sequence of values for* b:

$$d\mathrm{e}^{d_1\tau+2} < b_0 < b_1 < \cdots < b_j < \cdots$$

such that

(i) *Eq.* (2.3) *has a pair of simple purely imaginary roots* $\pm\mathrm{i}\omega_j$ *when* $b = b_j$, $j = 0, 1, 2, \ldots$;
(ii) *all the roots of Eq.* (2.3) *have negative real parts when* $b \in (d\mathrm{e}^{d_1\tau}, b_0)$;
(iii) *Eq.* (2.3) *has at least one pair of roots with positive real parts when* $b > b_0$.

Here

$$\omega_j \in \left(\frac{(2j+1/2)\pi}{\tau}, \frac{(2j+1)\pi}{\tau}\right), \quad j = 0, 1, 2, \ldots, \tag{2.8}$$

are the roots of the equation

$$-\frac{\omega}{d} = \tan\omega\tau, \tag{2.9}$$

and

$$b_j = d\mathrm{e}^{d_1\tau+(1-1/(\cos\omega_j\tau))}, \quad j = 0, 1, 2, \ldots. \tag{2.10}$$

Proof. Eq. (2.9) is directly from (2.5). If $\bar{\omega}$ solves (2.9), then $\cos\bar{\omega}\tau < 0$ and $\sin\bar{\omega}\tau > 0$. Let ω_j satisfy (2.8) and be the root of Eq. (2.9), and define b_j as in (2.10). Then (b_j, ω_j) $(j = 0, 1, 2, \ldots)$ is a solution of Eq. (2.5). Thus $\pm\mathrm{i}\omega_j$ is a pair of purely imaginary roots of Eq. (2.3) when $b = b_j$. Clearly, Eq. (2.3) has no other purely imaginary root and

$$\omega_{j+1}\tau - (2(j+1)+\tfrac{1}{2})\pi < \omega_j\tau - (2j+\tfrac{1}{2})\pi$$

(see Fig. 1) which implies that $|\cos\omega_{j+1}\tau| < |\cos\omega_j\tau|$, and hence,

$$\mathrm{e}^{1-1/(\cos\omega_{j+1}\tau)} > \mathrm{e}^{1-1/(\cos\omega_j\tau)}.$$

Therefore,

$$b_{j+1} = d\mathrm{e}^{d_1\tau+(1-1/(\cos\omega_{j+1}\tau))} > d\mathrm{e}^{d_1\tau+(1-1/(\cos\omega_j\tau))} = b_j$$

for $j = 0, 1, 2, \ldots$. This implies that b_0 is the first value of b such that Eq. (2.3) has roots appearing on the imaginary axis. Noticing that $b_0 > d\mathrm{e}^{d_1\tau+2}$ and applying Lemma 2.1 and Corollary 2.4 in Ruan and Wei [5], we conclude that all

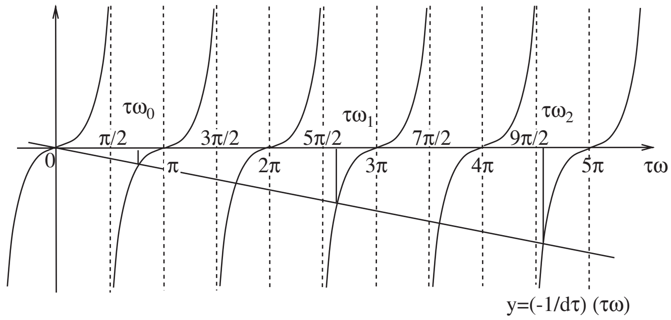

Fig. 1. Illustration of roots of (2.8)–(2.9).

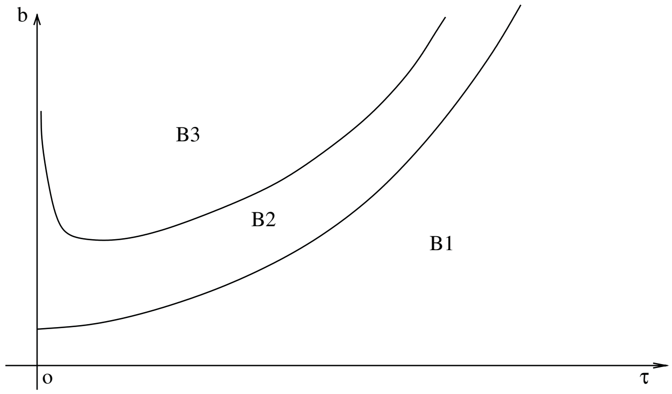

Fig. 2. The curves $b = de^{d_1\tau}$ (lower) and $b = b_0(\tau)$ (upper) divide the first quadrant of (τ, b)-plane into three regions: $B1$, $B2$ and $B3$. When $(\tau, b) \in B1$, there is no positive equilibrium and the trivial equilibrium $N = 0$ is asymptotically stable; when $(\tau, b) \in B2$, $N = 0$ becomes unstable and there is a positive equilibrium N^* which is asymptotically stable; and when (τ, b) crosses the upper curve $b = b_0(\tau)$ from B_2 to $B3$, N^* loses its stability and Hopf bifurcation occurs.

roots of Eq. (2.3) have negative real parts when $b \in [de^{d_1\tau}, b_0)$. The last conclusion follows from Lemma 2.2, and this completes the proof. □

Applying Lemmas 2.3 and 2.2, we have the following stability and bifurcation results to Eq. (2.1).

Theorem 2.4. *Suppose that* (H1) *holds. Then for each fixed* $\tau > 0$, *there exists a sequence values of* b:

$$de^{d_1\tau+2} < b_0 < b_1 < \cdots < b_j < \cdots$$

such that the positive equilibrium N_* *is asymptotically stable when* $b \in (de^{d_1\tau}, b_0)$, *and unstable when* $b > b_0$. *Furthermore, Eq.* (2.1) *undergoes a Hopf bifurcation at* N^* *when* $b = b_j$, $j = 0, 1, 2, \ldots,$ *where* b_j *is defined by* (2.9)–(2.10).

According to Theorem 2.4, we can draw the bifurcation diagram in the τ–b plane, as is shown in Fig. 2. The shape of the curve $b = de^{d_1\tau}$ is easily determined by calculus. Now we explore a bit about the shape of the curve $b = b_0(\tau)$. From $\omega = -d \tan \omega\tau$, it is easily seen (see Fig. 1) that $\omega_0\tau \to (\pi/2)^+$ as $\tau \to 0^+$, and $\omega_0\tau \to \pi^+$ as $\tau \to \infty$. From these two limits and (2.10), we see that

$$\lim_{\tau\to 0^+} b_0(\tau) = \infty \quad \text{and} \quad \lim_{\tau\to\infty} b_0(\tau) = \infty$$

giving the shape in Fig. 2.

Remark. In Cooke et al. [2], b is fixed and τ is used as the bifurcation parameter. When τ increases, double switches of stability of N^* is obtained in the pattern of "stable *to* unstable $\to$ stable". Such a switching pattern is readily obtained from the bifurcation diagram Fig. 2, by drawing horizontal lines intercepting the upper curve.

In what follows, we shall study the *global* existence of the bifurcated periodic solutions, by applying the global bifurcation theory for functional differential equations, established in Wu [7]. To this end, we need the following lemmas.

Lemma 2.5. *All periodic solutions to Eq.* (2.2) *for* $b \geqslant d\mathrm{e}^{d_1\tau}$ *are uniformly bounded.*

Proof. Let $x(t)$ be a non-constant periodic solution to Eq. (2), and $x(t_1)=M$, $x(t_2)=m$ be its maximum and minimum, respectively. Then $x'(t_1)=x'(t_2)=0$, and by Eq. (2.2)

$$M = x(t_1-\tau)\mathrm{e}^{-ax(t_1-\tau)} - N^*(1-\mathrm{e}^{-ax(t_1-\tau)}), \tag{2.11}$$

and

$$m = x(t_2-\tau)\mathrm{e}^{-ax(t_2-\tau)} - N^*(1-\mathrm{e}^{-ax(t_2-\tau)}). \tag{2.12}$$

We claim that $x(t_1-\tau)<0$ and $x(t_2-\tau)>0$. In fact, if $x(t_1-\tau)=0$, then $M=0$, by (2.11), and thus $m<M=0$ (as $x(t)$ is non-constant) and $x(t_2-\tau)\leqslant M=0$. This in turn implies $x(t_2-\tau)<0$, because otherwise, $x(t_2-\tau)=0$ would lead to $m=0$ by (2.12). If $x(t_1-\tau)>0$, then by (2.11), we arrive at

$$M \leqslant M - N^*(1-\mathrm{e}^{-ax(t_1-\tau)}) < M,$$

also a contradiction. Hence $x(t_1-\tau)<0$. A similar argument can show that $x(t_2-\tau)>0$. The above claim implies that $m<0$ and $M>0$. Applying this claim to (2.12) and (2.11), respectively, leads to

$$m > N^*(\mathrm{e}^{-aM}-1) > -N^* = \frac{-1}{a}\ln\frac{b}{d\mathrm{e}^{d_1\tau}} \tag{2.13}$$

and

$$\begin{aligned} M &= -N^* + [x(t_1-\tau)+N^*]\mathrm{e}^{-ax(t_1-\tau)} \\ &= -N^* + \mathrm{e}^{aN^*}[x(t_1-\tau)+N^*]\mathrm{e}^{-a[x(t_1-\tau)+N^*]} \\ &\leqslant -N^* + \frac{\mathrm{e}^{aN^*}}{a\mathrm{e}} = \frac{1}{a}\left[\frac{b}{d\mathrm{e}^{d_1\tau+1}} - \ln\frac{b}{d\mathrm{e}^{d_1\tau}}\right]. \end{aligned} \tag{2.14}$$

Here in (2.14), we have used the fact that $x\mathrm{e}^{-ax} < 1/a\mathrm{e}$. Obviously, (2.13) and (2.14) give uniform boundedness of all periodic solutions of (2.2), and thereby, complete the proof. □

Lemma 2.6. Eq. (2.2) *does not have a non-constant periodic solution when* $b = d\mathrm{e}^{d_1\tau}$.

Proof. Let $x(t)$ be a non-constant periodic solution, and $x(t_1)=M$, $x(t_2)=m$ be its maximum and minimum, respectively. Then $x'(t_1)=x'(t_2)=0$, and as is shown in the proof of Lemma 2.5, $x(t_1-\tau)<0$ and $x(t_2-\tau)>0$. Note that the critical case $b=d\mathrm{e}^{d_1\tau}$ corresponds to $N^*=0$. Hence, by (2.2) and $x(t_1-\tau)<0$, we obtain

$$M = x(t_1-\tau)\mathrm{e}^{-ax(t_1-\tau)} < x(t_1-\tau) \leqslant M, \tag{2.15}$$

a contradiction. This completes the proof. □

Eq. (2.1) only has the trivial equilibrium $N_*=0$ when $b\leqslant d\mathrm{e}^{d_1\tau}$, and has two equilibria $N_*=0$ and $N^*=(1/a)\ln(b/d\mathrm{e}^{d_1\tau})$ when $b>d\mathrm{e}^{d_1\tau}$. The linearization of Eq. (2.1) around $N_*=0$ is

$$\dot N(t) = -dN(t) + b\mathrm{e}^{-d_1\tau}N(t-\tau),$$

and its characteristic equation is

$$\lambda = -d + b\mathrm{e}^{-d_1\tau}\mathrm{e}^{-\lambda\tau}. \tag{2.16}$$

Let $\lambda(b) = \alpha(b) + \mathrm{i}\omega(b)$ be the root of Eq. (2.16) with $\alpha(b_0) = 0$ and $\omega(b_0) = \omega_0 > 0$ for certain $b_0 > 0$. Similar to Lemma 2.2, we have the following lemma.

Lemma 2.7. $\alpha'(b_0) > 0$.

The proof of following lemma is trivial, but for completeness, we also give its proof.

Lemma 2.8. *Eq.* (2.2) *has no* τ-period *solution*.

Proof. We only need to show that (2.1) has no τ-*period* solution. Let $N(t)$ be a τ-*period* solution of Eq. (2.1), then $N(t)$ solves the following ordinary differential equation:

$$\dot{x}(t) = -dx(t) + b\mathrm{e}^{-d_1\tau}x(t)\mathrm{e}^{-ax(t)}. \tag{2.17}$$

But (2.17) does not allow non-constant periodic solution. In fact, if $x(t)$ is a periodic solution of Eq. (2.17) and let $x(t_1) = M$, $x(t_2) = m$ be its maximum and minimum, respectively. Then $x'(t_1) = x'(t_2) = 0$, and it follows from (2.17) that

$$1 = \frac{b}{d\mathrm{e}^{d_1\tau}}\mathrm{e}^{-aM}, \quad \text{and} \quad 1 = \frac{b}{d\mathrm{e}^{d_1\tau}}\mathrm{e}^{-am}.$$

Thus $M = m$, implying that $x(t)$ is indeed a constant function. Hence, Eq. (2.1) has no τ-*period* solution, and this completes the proof. □

Using a general Bendixson's criterion for higher dimensional systems in Li and Muldowney [4], Wei and Li [6] proved that if $\gamma\mathrm{e}^2 < p < \sqrt{2}\gamma\mathrm{e}^2$, then the equation

$$N'(t) = -\gamma N(t) + pN(t-\tau)\mathrm{e}^{-aN(t-\tau)}, \quad t > 0$$

has no 4τ-period solution. Applying the conclusion to Eq. (2.1), we have the following lemma.

Lemma 2.9. *Assume that*

$$d\mathrm{e}^{d_1\tau+2} < b < \sqrt{2}d\mathrm{e}^{d_1\tau+2}.$$

Then Eq. (2.1) *has no* 4τ-period *solution* (*equivalently*, *Eq.* (2.2) *has no* 4τ-period *solution also*).

Now we are in the position to state some results on the global existence of a periodic solution to Eq. (2.2) (or equivalently (2.1)). We shall use the following notations:

$$X = C([-\tau, 0], R),$$

$$\Sigma = cl\{(x, b, T) : x \text{ is a } T\text{-periodic solution of Eq. (2.2)}\} \subset X \times R_+ \times R_+.$$

$$\Omega = \left\{(\hat{x}, b, T) : \hat{x} + \frac{1}{a}\ln\frac{b}{d\mathrm{e}^{d_1\tau}} - \left[\frac{1}{a}\ln\frac{b}{d\mathrm{e}^{d_1\tau}} + \hat{x}\right]\mathrm{e}^{-a\hat{x}} = 0\right\}.$$

Let $\mathscr{C}(0, b_j.2\pi/\omega_j)$ denote the connected component of $(0, b_j.2\pi/\omega_j)$ in Σ, where b_j and ω_j are defined in (2.9) and (2.10), respectively.

Theorem 2.10. *Suppose that* (H_1) *holds. Then*

(i) *When* $b > b_j$, $j = 1, 2, \ldots$, *Eq.* (2.2) *has at least j non-constant periodic solutions*.

(ii) *If* $b_1 < \sqrt{2}d\mathrm{e}^{d_1\tau+2}$, *then Eq.* (2.2) *has at least one non-constant periodic solution when* $b > b_0$.
(iii) *If* $b_0 < \sqrt{2}d\mathrm{e}^{d_1\tau+2} < b_1$, *then Eq.* (2.2) *has at least one non-constant periodic solution when* $b \in (b_0, \sqrt{2}d\mathrm{e}^{d_1\tau+2})$.

Proof. We will apply the global bifurcation result for functional differential equations established in Wu [7]. First, note that

$$F(x_t, b) := -dx(t) + d\left[\frac{1}{a}\ln\frac{b}{d\mathrm{e}^{d_1\tau}} + x(t-\tau)\right]\mathrm{e}^{-ax(t-\tau)} - \frac{d}{a}\ln\frac{b}{d\mathrm{e}^{d_1\tau}}$$

satisfies the hypotheses (A_1), (A_2) and (A_3) in Wu [7], with

$$(\hat{x}_0, \alpha_0, p_0) = \left(0, b_j, \frac{2\pi}{\omega_j}\right),$$

and

$$\Delta\left(0, b_j, \frac{2\pi}{\omega_j}\right)(z) = z - d\left[1 - \ln\frac{b}{d\mathrm{e}^{d_1\tau}}\right]\mathrm{e}^{-z\tau} + d.$$

It can also be verified that $(0, b_j, 2\pi/\omega_j)$ is an isolated center (see Wu [7]) for $j = 0, 1, 2, \ldots$. By Lemma 2.2 there exist $\varepsilon > 0$, $\delta > 0$ and a smooth curve $z : (b_j - \delta, b_j + \delta) \to C$ such that $\Delta(z(b)) = 0$, $|z(b) - \mathrm{i}\omega_0| < \varepsilon$ for all $b \in [b_j - \delta, b_j + \delta]$, and

$$z(b_j) = \mathrm{i}\omega_j, \qquad \left.\frac{\mathrm{dRe}\, z(b)}{\mathrm{d}b}\right|_{b=b_j} > 0.$$

Denote $T_j = 2\pi/\omega_j$, and

$$\Omega_\varepsilon = \{(0, T) : 0 < u < \varepsilon, |T - T_j| < \varepsilon\}.$$

Clearly, if $|b - b_j| \leqslant \delta$ and $(u, p) \in \partial\Omega_\varepsilon$ satisfying $\Delta_{(0,b,T)}(u + 2\mathrm{i}\pi/T) = 0$, then $b = b_j$, $u = 0$, and $T = T_j$. This verifies assumption (A_4) in Wu [7] for $m = 1$. Moreover, if we put

$$H^{\pm}\left(0, b_j, \frac{2\pi}{\omega_j}\right)(u, T) = \Delta_{(0, b_j \pm \delta, T)}\left(u + \mathrm{i}\frac{2\pi}{T}\right),$$

then we have the crossing number (see Wu [7])

$$\gamma_1\left(0, b_j, \frac{2\pi}{\omega_j}\right) = deg_B\left(H^-\left(0, b_j, \frac{2\pi}{\omega_j}\right), \Omega_\varepsilon\right) - deg_B\left(H^+\left(0, b_j, \frac{2\pi}{\omega_j}\right), \Omega_\varepsilon\right) = -1.$$

By Theorem 3.3 in Wu [7], we conclude that the connected component $\mathscr{C}(0, b_j, 2\pi/\omega_j)$ through $(0, b_j, 2\pi/\omega_j)$ in Σ is non-empty. Moreover, from Lemma 2.8 know that

$$\sum_{(\hat{x}, b, T) \in \mathscr{C}(0, b_j, 2\pi/\omega_j)} \gamma_m(\hat{x}, b, T) < 0,$$

and hence $\mathscr{C}(0, b_j, 2\pi/\omega_j)$ is unbounded.

Now Lemma 2.5 implies that the projection of $\mathscr{C}(0, b_j, 2\pi/\omega_j)$ onto the *x-space* is bounded. From (2.8)–(2.10) (see Fig. 1), we know that

$$\tau\omega_j > 2\pi, \quad \text{for } j = 1, 2, \ldots.$$

Hence,

$$\frac{\tau}{n_j} < \frac{2\pi}{\omega_j} < \tau$$

for some integer n_j. Applying Lemma 2.8, we know that $\tau/n_j < T < \tau$ if $(x, b, T) \in \mathscr{C}(0, b_j, 2\pi/\omega_j)$. This shows that in order for $\mathscr{C}(0, b_j, 2\pi/\omega_j)$ to be unbounded, its projection onto the *b-space* must be unbounded. However,

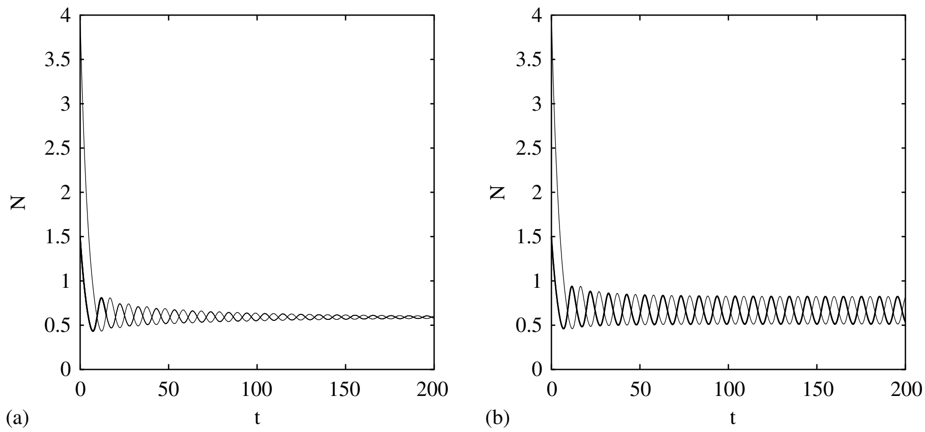

Fig. 3. Numerical simulation of (2.1) with parameters $d=0.2$, $a=7$, $\tau=3$, $d_1\tau=3$, which confirms that the periodic solution bifurcates from the positive equilibrium. On the left, $b=250<b_0$; and on the right, $b=350>b_0$.

Lemma 2.6 implies that the projection of $\mathscr{C}(0, b_j, 2\pi/\omega_j)$ onto the *b-space* is bounded below by $b=d\mathrm{e}^{d_1\tau}$. Consequently, the projection of $\mathscr{C}(0, b_j, 2\pi/\omega_j)$ onto the *b-space* must includes $[b_j, \infty)$, $j\geqslant 1$. This shows that, for every $b>b_j$, Eq. (2.2) has a non-constant periodic solution with period in $(\tau/n_j, \tau)$. Finally, counting the total number of connected components $\mathscr{C}(0, b_j, 2\pi/\omega_j)$ leads to the conclusion in part (i) of the theorem.

Noticing that $\tau\omega_0>\pi/2$, and hence $2\pi/\omega_0<4\tau$, there exists an integer n_0 such that

$$\frac{\tau}{n_0}<\frac{2\pi}{\omega_0}<4\tau.$$

Applying Lemma 2.9, we see that $\tau/n_0<T<4\tau$ when $b\in(d\mathrm{e}^{d_1\tau+2}, \sqrt{2}d\mathrm{e}^{d_1\tau+2})$ and $(x, b, T)\in\mathscr{C}(0, b_0, 2\pi/\omega_0)$. By an argument similar to the above, we know that the projection of $\mathscr{C}(0, b_0, 2\pi/\omega_0)$ on b-space includes the interval $[b_0, \sqrt{2}d\mathrm{e}^{d_1\tau+2})$. Combining this with the conclusion in (i), we know that Eq. (2.2) has at least one non-constant periodic solution for $b\in(b_0, \sqrt{2}d\mathrm{e}^{d_1\tau+2})\cup(b_1, \infty)$, concluding (ii) and (iii). This completes the proof. □

In order to illustrate the results obtained above, we choose $d=0.2$, $a=7$, $\tau=3$, $d_1\tau=3$. With these parameter values, b_0 can be numerically computed as $b_0=292.6126$. Hence, the positive equilibrium N^* is asymptotically stable when $b<292.6126$, and unstable when $b>292.6126$ in which case, there is a periodic solution around N^* (see Fig. 3).

3. Bifurcation analysis of the SIS epidemic model (1.3) with Rick birth rate function

In this section we consider the SIS model (1.3) with the birth function given by (1.2), that is, the following delayed system:

$$\begin{cases} \dot{I}(t)=\mu(N(t)-I(t))\dfrac{I(t)}{N(t)}-(d+\varepsilon+\gamma)I(t), \\ \dot{N}(t)=b\mathrm{e}^{-aN(t-\tau)}N(t-\tau)\mathrm{e}^{-d_1\tau}-dN(t)-\varepsilon I(t). \end{cases} \tag{3.1}$$

Parallel to (H1) in Section 2 and as in Cooke et al. [2], throughout this section we assume a strengthened version of (H1):

(H1′) $b>(d+\varepsilon)\mathrm{e}^{d_1\tau}$.

As in Section 2, we shall regard b as the bifurcation parameter in analyzing the bifurcation of system (3.1). First, in Section 3.1, we will give a set of bifurcation values for system (3.1) via analyzing the distribution of the roots of

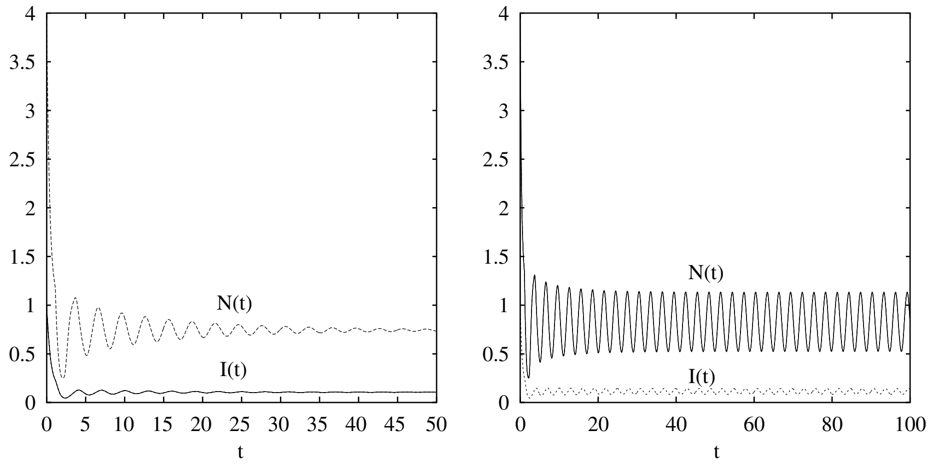

Fig. 4. Numerical simulation of (3.1) with parameters $\mu = 14$, $\varepsilon = 1$, $d = 2$, $\gamma = 9$, $a = 3$, $\tau = 1.1$ and $d_1 = 2$. On the left, $b = 150 < b_0$; and on the right, $b = 250 > b_0$.

the characteristic equation associated with (3.1). Then, by employing the normal form method and center manifold theorem, we derive formulas for determining the properties of Hopf bifurcation. Finally, we perform some numerical simulations in Section 3.2 (Fig. 4).

3.1. Stability analysis and existence of Hopf bifurcation

In Cooke et al. [2], the basic reproduction number for (3.1) has been identified as

$$R_0 = \frac{\mu}{d + \varepsilon + \gamma}.$$

It has been shown that when $R_0 \leqslant 1$, (3.1) only has the disease free equilibrium (DFE) (which is globally asymptotically stable, while when $R_0 > 1$, the DFE becomes unstable and an EE (I^*, N^*) is bifurcated where

$$I^* = (1 - 1/R_0)N^* \quad \text{and} \quad N^* = \frac{1}{a} \ln \frac{b}{[d + \varepsilon(1 - 1/R_0)]\mathrm{e}^{d_1\tau}}. \tag{3.2}$$

As mentioned in the Introduction, the stability of (I^*, N^*) still largely remains unsolved.

For convenience, let $Q_0 = 1/R_0$. Then the condition $R_0 > 1$ is equivalent to $Q_0 < 1$, which will be assumed in the rest of this section. The linearization of Eq. (3.1) at (I^*, N^*) is

$$\begin{cases} \dot{x_1}(t) = -\mu(1 - Q_0)x_1(t) + \mu(1 - Q_0)^2 x_2(t), \\ \dot{x_2}(t) = -\varepsilon x_1(t) - d x_2(t) + b\mathrm{e}^{-d_1\tau}\mathrm{e}^{-aN^*}(1 - aN^*)x_2(t - \tau), \end{cases} \tag{3.3}$$

and the characteristic equation associated with Eq. (3.3) is

$$\begin{aligned} &\lambda^2 + [d + \mu(1 - Q_0)]\lambda - b(1 - aN^*)\mathrm{e}^{-(aN^* + d_1\tau)}\lambda \mathrm{e}^{-\lambda\tau} \\ &\quad - \mu(1 - Q_0)b(1 - aN^*)\mathrm{e}^{-(aN^* + d_1\tau)}\mathrm{e}^{-\lambda\tau} + \varepsilon\mu(1 - Q_0)^2 + d\mu(1 - Q_0) = 0. \end{aligned} \tag{3.4}$$

Using the definition of N^*, (3.4) becomes

$$\begin{aligned} &\lambda^2 + [d + \mu(1 - Q_0)]\lambda - (d + \varepsilon(1 - Q_0))\left(1 - \ln \frac{b}{[d + \varepsilon(1 - Q_0)]\mathrm{e}^{d_1\tau}}\right)\lambda \mathrm{e}^{-\lambda\tau} \\ &\quad - \mu(1 - Q_0)(d + \varepsilon(1 - Q_0))\left(1 - \ln \frac{b}{[d + \varepsilon(1 - Q_0)]\mathrm{e}^{d_1\tau}}\right)\mathrm{e}^{-\lambda\tau} \\ &\quad + \mu(1 - Q_0)(d + \varepsilon(1 - Q_0)) = 0. \end{aligned} \tag{3.5}$$

Let

$$a_1 = d + \mu(1 - Q_0), \quad a_2 = -\left(1 - \ln \frac{b}{[d + \varepsilon(1 - Q_0)]\mathrm{e}^{d_1\tau}}\right),$$
$$a_3 = d + \varepsilon(1 - Q_0) \quad \text{and} \quad a_4 = \mu(1 - Q_0).$$

Then (3.5) is further rewritten as

$$\lambda^2 + a_1\lambda + a_2a_3\lambda\mathrm{e}^{-\lambda\tau} + a_2a_3a_4\mathrm{e}^{-\lambda\tau} + a_3a_4 = 0. \tag{3.6}$$

Let $\mathrm{i}\omega(\omega > 0)$ be a root of (3.6). Then we have

$$-\omega^2 + \mathrm{i}a_1\omega + \mathrm{i}a_2a_3\omega(\cos\omega\tau - \mathrm{i}\,\sin\omega\tau) + a_2a_3a_4(\cos\omega\tau - \mathrm{i}\,\sin\omega\tau) + a_3a_4 = 0.$$

Separating the real and imaginary parts gives

$$\begin{cases} \omega^2 - a_3a_4 = a_2a_3\omega\sin\omega\tau + a_2a_3a_4\cos\omega\tau, \\ a_1\omega = -a_2a_3\omega\cos\omega\tau + a_2a_3a_4\sin\omega\tau. \end{cases} \tag{3.7}$$

In order to proceed further, we need the following assumption:

(A1) $\varepsilon > 0$ satisfies

$$(1 - Q_0)^2[\mu^2 - 2\varepsilon\mu - a_2^2\varepsilon^2] - 2\varepsilon d a_2^2(1 - Q_0) \geqslant 0$$

for $b \in ([d + \varepsilon(1 - Q_0)])\mathrm{e}^{d_1\tau}, [d + \varepsilon(1 - Q_0)]\mathrm{e}^{d_1\tau+1}]$.

Remark. If

$$\varepsilon < \sqrt{\left(\mu + \frac{d}{1 - Q_0}\right)^2 + \mu^2} - \left(\mu + \frac{d}{1 - Q_0}\right), \tag{3.8}$$

then assumption (A1) is satisfied. In fact, the inequality (3.12) implies that

$$\varepsilon^2 + 2\left(\mu + \frac{d}{1 - Q_0}\right)\varepsilon - \mu^2 \leqslant 0.$$

From the definition of a_2 and the fact that $b \in ([d + \varepsilon(1 - Q_0)]\mathrm{e}^{d_1\tau}, [d + \varepsilon(1 - Q_0)]\mathrm{e}^{d_1\tau+1}])$, it follows that $|a_2| < 1$, and hence

$$a_2^2\varepsilon^2 + 2\left(\mu + \frac{da_2^2}{1 - Q_0}\right)\varepsilon - \mu^2 \leqslant 0,$$

and the conclusion follows.

Lemma 3.1. *Assume that* (A1) *holds. Then Eq.* (3.6) *has no purely imaginary root.*

Proof. Let $\mathrm{i}\omega$ be a root of Eq. (3.6). Then, squaring and adding the two equations in (3.7) gives

$$\omega^4 + (a_1^2 - 2a_3a_4 - a_2^2a_3^2)\omega^2 + a_3^2a_4^2 - a_2^2a_3^2a_4^2 = 0,$$

which yields

$$\omega^2 = \frac{1}{2}\left[-(a_1^2 - 2a_3a_4 - a_2^2a_3^2) \pm \sqrt{(a_1^2 - 2a_3a_4 - a_2^2a_3^2)^2 - 4a_3^2a_4^2(1 - a_2^2)}\right]. \tag{3.9}$$

By the definitions of a_1, a_3 and a_4, we have

$$a_1^2 - 2a_3a_4 - a_2^2a_3^2 = (1 - a_2^2)d^2 + (\mu^2 - 2\mu\varepsilon - a_2^2\varepsilon^2)(1 - Q_0)^2 - 2\varepsilon d a_2^2(1 - Q_0). \tag{3.10}$$

From the definition of a_2 and the assumption $b \in ([d+\varepsilon(1-Q_0))]\mathrm{e}^{d_1\tau}, [d+\varepsilon(1-Q_0)]\mathrm{e}^{d_1\tau+1}]$, we see that $-1 \leqslant a_2 \leqslant 0$, and hence, from (3.10),

$$a_1^2 - 2a_3a_4 - a_2^2a_3^2 \geqslant 0.$$

This implies that (3.9) is meaningless, and thus, completes the proof. □

Eq. (3.7) is further equivalent to

$$\begin{cases} \omega(\omega^2 - a_3a_4 + a_1a_4) = a_2a_3(\omega^2 + a_4^2)\sin\omega\tau, \\ (a_4 - a_1)\omega^2 - a_3a_4^2 = a_2a_3(a_4^2 + \omega^2)\cos\omega\tau, \end{cases} \tag{3.11}$$

from which, we obtain

$$\frac{\omega(\omega^2 + a_4(a_1 - a_3))}{(a_4 - a_1)\omega^2 - a_3a_4^2} = \tan\omega\tau. \tag{3.12}$$

Substituting a_1, a_3 and a_4 into (3.12) yields

$$\frac{\omega(\omega^2 + \mu(\mu - \varepsilon)(1 - Q_0)^2)}{-d\omega^2 - \mu^2(1 - Q_0)^2(d + \varepsilon(1 - Q_0))} = \tan\omega\tau. \tag{3.13}$$

The assumption $Q_0 < 1$ implies $\mu > \varepsilon$. Hence

$$y := \frac{\omega(\omega^2 + \mu(\mu - \varepsilon)(1 - Q_0)^2)}{-d\omega^2 - \mu^2(1 - Q_0)^2(d + \varepsilon(1 - Q_0))} \leqslant 0, \tag{3.14}$$

and $y = 0$ if and only if $\omega = 0$.

Let

$$\omega_j \in \left(\frac{(2j + 1/2)\pi}{\tau}, \frac{(2j + 1)\pi}{\tau}\right), \quad j = 0, 1, 2, \ldots, \tag{3.15}$$

be the root of Eq. (3.12), and define

$$b_j = (d + \varepsilon(1 - Q_0))\mathrm{e}^{d_1\tau+1+\delta} \quad \text{where } \delta = \frac{(a_4 - a_1)\omega_j^2 - a_3a_4^2}{a_3(a_4^2 + \omega_j^2)\cos\omega_j\tau}. \tag{3.16}$$

Lemma 3.2. *Assume that* (A1) *holds. Then there exists a sequence of values of b such that*

$$(d + \varepsilon(1 - Q_0))\mathrm{e}^{d_1\tau+1} < b_0 < b_1 < \cdots < b_j < \cdots,$$

and Eq. (3.6) *with* $b = b_j$ *has a pair of pure imaginary roots* $\pm \mathrm{i}\omega_j$. *Moreover, the roots of Eq.* (3.6) *have negative real parts when* $b \in ([d + \varepsilon(1 - Q_0)]\mathrm{e}^{d_1\tau}, b_0)$.

Proof. Firstly, from (3.13) and (3.14), we know that the roots of (3.13) satisfy (3.15), and hence, by the definition of b_j in (3.16), (ω_j, b_j) is a solution of (3.6). This implies that $\pm\mathrm{i}\omega_j$ is a pair of purely imaginary roots to Eq. (3.6) when $b = b_j$. Clearly, Eq. (3.6) has no purely imaginary root when $b \neq b_j$, and Lemma 3.1 implies that $b_0 > [d + \varepsilon(1 - Q_0)]\mathrm{e}^{d_1\tau+1}$.

Secondly, we know that

$$(a_4 - a_1)\omega^2 - a_3a_4^2 = -d\omega^2 - \mu^2(1 - Q_0)^2(d + \varepsilon(1 - Q_0)) < 0.$$

Hence, $\cos\omega_j\tau < 0$, and $|\cos\omega_j\tau| > |\cos\omega_{j+1}\tau|$. This means that $b_{j+1} > b_j$, and b_0 is the first value of b such that Eq. (3.6) has roots appearing on the imaginary axis. Meanwhile, the roots of Eq. (3.16) with $b = [d + \varepsilon(1 - Q_0)]\mathrm{e}^{d_1\tau+1}$ are

$$\lambda_{1,2} = \frac{1}{2}\left[-a_1 \pm \sqrt{a_1^2 - 4a_3a_4}\right],$$

which implies that $\operatorname{Re}\lambda_{1,2} < 0$ from $a_1 > 0$ and $a_3a_4 > 0$. Therefore, all the roots of Eq. (3.16) have negative real parts when $b \in ([d + \varepsilon(1 - Q_0)]\mathrm{e}^{d_1\tau}, b_0)$. The proof is completed. □

Let

$$\lambda(b) = \alpha(b) + \mathrm{i}\omega(b)$$

be the root of Eq. (3.16) satisfying

$$\alpha(b_j) = 0, \quad \omega(b_j) = \omega_j.$$

Lemma 3.3. *If* (A1) *is satisfied, then* $\alpha'(b_j) > 0$.

Proof. Notice that a_1, a_3, and a_4 are all independent of b and a_2 is increasing in b. Substituting $\lambda(b)$ into Eq. (3.6) and taking derivative with respect to a_2, we have

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}a_2}\right)^{-1} = -\frac{2\lambda + a_1}{a_3(a_4+\lambda)\mathrm{e}^{-\lambda\tau}} - \frac{a_2 a_3}{a_3(a_4+\lambda)} + a_2\tau.$$

By $a_2 a_3(a_4+\lambda)\mathrm{e}^{-\lambda\tau} = -(\lambda^2 + a_1\lambda + a_3 a_4)$, we obtain

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}a_2}\right)^{-1} = \frac{a_2(2\lambda + a_1)}{\lambda^2 + a_1\lambda + a_3 a_4} - \frac{a_2 a_3}{a_3(a_4+\lambda)} + a_2\tau.$$

Thus,

$$\begin{aligned}\left.\left(\frac{\mathrm{d}\lambda}{\mathrm{d}a_2}\right)^{-1}\right|_{b=b_j} &= \frac{a_1 a_2 + \mathrm{i}2a_2\omega_j}{a_3 a_4 - \omega_j^2 + \mathrm{i}a_1\omega_j} - \frac{a_2 a_3}{a_3 a_4 + \mathrm{i}a_3\omega_j} + a_2\tau \\ &= \frac{1}{\Delta_1}[a_1 a_2(a_3 a_4 - \omega_j^2) + 2a_1 a_2\omega_j^2] - \frac{1}{\Delta_2}a_2 a_3^2 a_4 + a_2\tau \\ &\quad + \mathrm{i}\left\{\frac{1}{\Delta_1}[2a_2\omega_j(a_3 a_4 - \omega_j^2) - a_1^2 a_2\omega_j] + \frac{1}{\Delta_2}a_2 a_3^2\omega_j\right\},\end{aligned}$$

where $\Delta_1 = (a_3 a_4 - \omega_j^2)^2 + a_1^2\omega_j^2$, and $\Delta_2 = a_3^2 a_4^2 + a_3^2\omega_j^2$. Direct computation gives

$$\mathrm{Re}\left.\left(\frac{\mathrm{d}\lambda}{\mathrm{d}a_2}\right)^{-1}\right|_{b=b_j} = \frac{a_2 a_3^2}{\Delta_1\Delta_2}[a_3 a_4^3(\mu-\varepsilon)(1-Q_0) + a_4(a_1\varepsilon(1-Q_0) + 2a_3 a_4)\omega_j^2 + a_3^2 d\omega_j^4] + a_2\tau > 0.$$

Thus

$$\mathrm{sign}\left(\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}a_2}\right)\right) = \mathrm{sign}\left(\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}a_2}\right)^{-1}\right) > 0.$$

This together with the fact that $\mathrm{d}a_2/\mathrm{d}b > 0$ gives $\alpha'(b_j) > 0$, and thereby completes the proof. □

Combining Lemmas 3.2 and 3.3, we obtain the following main results on the stability and Hopf bifurcations of (3.1).

Theorem 3.4. *Assume that* (A1) *holds. Then*,

(i) *the positive equilibrium* (I^*, N^*) *is asymptotically stable when* $b \in ([d+\varepsilon(1-Q_0)]\mathrm{e}^{d_1\tau}, b_0)$, *and unstable when* $b > b_0$;
(ii) *Eq.* (3.1) *undergoes a Hopf bifurcation at* (I^*, N^*) *when b passes* b_j, $j = 0, 1, 2, \ldots$, *were* b_j, $j = 0, 1, 2, \ldots$, *are defined by* (3.16).

Based on this theorem, we can draw a bifurcation diagram for (3.1) in the τ–b plane. Since the diagram is similar to Fig. 2, we omit it here.

In Fig. 4, we show some numeric simulations about the Hopf bifurcation obtained in Theorem 3.4. The parameters are chosen as $\mu=14$, $\varepsilon=1$, $d=2$, $\gamma=9$, $a=3$, $\tau=1.1$ and $d_1=2$. Then numeric computation gives $b_0=198.5389$. Therefore, (3.1) has a periodic solution when $b>198.5389$, as is shown in Fig. 4.

3.2. Properties of Hopf bifurcations

In Section 3.1, we have obtained a set of conditions for system (3.1) to undergo Hopf bifurcations at the positive equilibrium (I^*, N^*) when b passes b_j, $j=0,1,2,\ldots$, where b_j, $j=0,1,2,\ldots$ are defined by (3.15) and (3.16). Unfortunately, we are not able to establish the global bifurcation for (3.1) (as we did to (2.1)) since (3.1) is a delay *system*, and the application of the global bifurcation theory for such a system would be more demanding. In this subsection, we will study the bifurcation direction and the stability of the bifurcated periodic solution when b passes $b=b_0$. To achieve this goal, we will apply the center manifold theory and employ the algorithm for computing the normal forms for systems of delay differential equations developed in Hassard et al. [3]. To this end, we need to introduce some notations following Hassard et al. [3].

Let $x_1=I-I^*$, $x_2=N-N^*$ and $b=b_0+v$, then (3.1) becomes

$$\begin{cases} \dot{x_1}(t)=-\mu(1-Q_0)x_1(t)+\mu(1-Q_0)^2x_2(t)-\dfrac{\mu}{N^*}x_1^2(t)-\dfrac{\mu(1-Q_0)^2}{N^*}x_2^2(t)+\dfrac{2\mu(1-Q_0)}{N^*} \\ \qquad\times x_1(t)x_2(t)+\dfrac{\mu(1-Q_0)^2}{N^{*2}}x_2^3(t)+\dfrac{\mu}{N^{*2}}x_1^2(t)x_2(t)-\dfrac{2\mu(1-Q_0)}{N^{*2}}x_1(t)x_2^2(t)+\cdots, \\ \dot{x_2}(t)=-\varepsilon x_1(t)-dx_2(t)+(b_0+v)\mathrm{e}^{-(d_1\tau+aN^*)}((1-aN^*)x_2(t-\tau) \\ \qquad-\dfrac{a(2-aN^*)}{2}x_2^2(t-\tau)+\dfrac{a^2(3-aN^*)}{6}x_2^3(t-\tau))+\cdots. \end{cases} \tag{3.17}$$

Choosing the phase space as $C=C([-\tau,0],R^2)$. For $\varphi\in C$, let

$$L_v\varphi=B_1\varphi(0)+B_2\varphi(-\tau),$$

where

$$B_1=\begin{pmatrix} -\mu(1-Q_0) & \mu(1-Q_0)^2 \\ -\varepsilon & -d \end{pmatrix},\quad B_2=\begin{pmatrix} 0 & 0 \\ 0 & (b_0+v)(1-aN^*)\mathrm{e}^{-(d_1\tau+aN^*)} \end{pmatrix},$$

and

$$F(v,\varphi)=\begin{pmatrix} -\dfrac{\mu}{N^*}\varphi_1^2(0)-\dfrac{\mu(1-Q_0)^2}{N^*}\varphi_2^2(0)+\dfrac{2\mu(1-Q_0)}{N^*}\varphi_1(0)\varphi_2(0) \\ +\dfrac{\mu(1-Q_0)^2}{N^{*2}}\varphi_2^3(0)+\dfrac{\mu}{N^{*2}}\varphi_1^2(0)\varphi_2(0)-\dfrac{2\mu(1-Q_0)}{N^{*2}}\varphi_1(0)\varphi_2^2(0) \\ +\cdots-\dfrac{a(2-aN^*)}{2}\varphi_2^2(-\tau)+\dfrac{a^2(3-aN^*)}{6}\varphi_2^3(-\tau))+\cdots \end{pmatrix}.$$

By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\theta,v)$ in $\theta\in[-\tau,0]$ such that

$$L_v\varphi=\int_{-\tau}^{0}\mathrm{d}\eta(\theta,v)\varphi(\theta)\quad\text{for }\varphi\in C. \tag{3.18}$$

In fact, if we choose

$$\eta(\theta,v)=\begin{cases} B_1, & \theta=0 \\ -B_2\delta(\theta+\tau), & \theta\in[-\tau,0), \end{cases}$$

then (3.18) is satisfied.

For $\varphi \in C^1([-\tau, 0], R^2)$, define

$$A(v)\varphi = \begin{cases} \dfrac{d\varphi(\theta)}{d\theta}, & \theta \in [-\tau, 0), \\ \int_{-\tau}^{0} d\eta(t, v)\varphi(t), & \theta = 0 \end{cases}$$

and

$$R(v)\varphi = \begin{cases} 0, & \theta \in [-\tau, 0), \\ F(v, \varphi), & \theta = 0. \end{cases}$$

Hence, we can rewrite (3.17) as the following form:

$$\dot{x}_t = A(v)x_t + R(v)x_t, \tag{3.19}$$

where $x = (x_1, x_2)^{\mathrm{T}}$, $x_t = x(t + \theta)$, for $\theta \in [-\tau, 0]$.

For $\psi \in C^1[0, \tau]$, define

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0, \tau], \\ \int_{-\tau}^{0} d\eta(t, 0)\psi(-t), & s = 0. \end{cases}$$

For $\varphi \in C[-\tau, 0]$ and $\psi \in C[0, \tau]$, define the bilinear form

$$\langle \psi, \varphi \rangle = \bar{\psi}(0)\varphi(0) - \int_{-\tau}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta)\, d\eta(\theta)\varphi(\xi)\, d\xi, \tag{3.20}$$

where $\eta(\theta) = \eta(\theta, 0)$. Obviously, A^* and A are adjoint operators.

By the results in Section 3.1, we know that $\pm\omega_0$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^* (see, Hassard et al. [3]). Moreover, by direct computation, we can show that

$$q(\theta) = \begin{pmatrix} 1 \\ \dfrac{\mu(1 - Q_0) + i\omega_0}{\mu(1 - Q_0)^2} \end{pmatrix} e^{i\omega_0\theta}$$

is the eigenvector of $A(0)$ corresponding to $i\omega_0$; and

$$q^*(s) = D \begin{pmatrix} 1 \\ -\dfrac{\mu(1 - Q_0) - i\omega_0}{\varepsilon} \end{pmatrix}^{\mathrm{T}} e^{i\omega_0 s}$$

is the eigenvector of A^* corresponding to $-i\omega_0$, where

$$D = \left[1 - (1 + b_0(1 - aN^*)e^{-(d_1\tau + aN^*)}) \frac{(\mu(1 - Q_0) - i\omega_0)^2}{\mu\varepsilon(1 - Q_0)^2} \right]^{-1}.$$

Furthermore,

$$\langle q^*, q \rangle = 1 \quad \text{and} \quad \langle q^*, \bar{q} \rangle = 0.$$

Using the same notations as in Hassard et al. [3], we first compute the coordinates to describe the center manifold $\mathscr{C}_0$ at $v = 0$. Let x_t be the solution of Eq. (3.17) when $v = 0$.

Define

$$z(t) = \langle q^*, x_t \rangle, \quad W(t, \theta) = x_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\}.$$

On the center manifold $\mathscr{C}_0$ we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta),$$

where

$$W(z,\bar{z},\theta)=W_{20}(\theta)\frac{z^2}{2}+W_{11}z\bar{z}+W_{02}\frac{\bar{z}^2}{2}+W_{30}\frac{z^3}{6}+\cdots,$$

z and $\bar{z}$ are local coordinates for center manifold $\mathscr{C}_0$ in the direction of q^* and $\bar{q}^*$. Note that W is real if x_t is real. We only consider real solutions.

For the solution $x_t\in\mathscr{C}_0$ of (3.17), since $v=0$, we obtain

$$\dot{z}(t)=\mathrm{i}\omega_0 z+\bar{q}^*(0)F(W(z,\bar{z},0)+2\mathrm{Re}\{z(t)q(0)\})\stackrel{\text{def}}{=}\mathrm{i}\omega_0 z+\bar{q}^*(0)F_0(z,\bar{z}).$$

We rewrite this as

$$\dot{z}(t)=\mathrm{i}\omega_0 z(t)+g(z,\bar{z}),\tag{3.21}$$

where

$$g(z,\bar{z})=\bar{q}^*(0)F(W(z,\bar{z},0)+2\mathrm{Re}\{z(t)q(0)\})=g_{20}\frac{z^2}{2}+g_{11}z\bar{z}+g_{02}\frac{\bar{z}^2}{2}+g_{21}\frac{z^2\bar{z}}{2}+\cdots,\tag{3.22}$$

From (3.19) and (3.21), we have

$$\begin{aligned}\dot{W}&=\dot{u}_t-\dot{z}q-\dot{\bar{z}}\bar{q}\\&=\begin{cases}AW-2\mathrm{Re}\{\bar{q}^*(0)F_0q(\theta)\}, & \theta\in[-\tau,0),\\ AW-2\mathrm{Re}\{\bar{q}^*(0)F_0q(\theta)\}+F_0, & \theta=0,\end{cases}\\&\stackrel{\text{def}}{=}AW+H(z,\bar{z},\theta),\end{aligned}$$

where

$$H(z,\bar{z},\theta)=H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\bar{z}+H_{02}(\theta)\frac{\bar{z}^2}{2}+\cdots.\tag{3.23}$$

Expanding the above series and comparing the coefficients, we obtain

$$\begin{aligned}&(A-2\mathrm{i}\omega_0)W_{20}(\theta)=-H_{20}(\theta),\\&AW_{11}(\theta)=-H_{11}(\theta),\\&\cdots\cdots\cdots\end{aligned}\tag{3.24}$$

Notice that

$$x_1(t)=z+\bar{z}+W^{(1)}(t,0),\quad x_2(t)=\frac{\mu(1-Q_0)+\mathrm{i}\omega_0}{\mu(1-Q_0)^2}z+\frac{\mu(1-Q_0)-\mathrm{i}\omega_0}{\mu(1-Q_0)^2}\bar{z}+W^{(2)}(t,0),$$

and

$$x_2(t-\tau)=\frac{\mu(1-Q_0)+\mathrm{i}\omega_0}{\mu(1-Q_0)^2}\mathrm{e}^{-\mathrm{i}\omega_0\tau}z+\frac{\mu(1-Q_0)-\mathrm{i}\omega_0}{\mu(1-Q_0)^2}\mathrm{e}^{i\omega_0\tau}\bar{z}+W^{(2)}(t,-\tau),$$

where

$$W^{(j)}(t,0)=W_{20}^{(j)}(0)\frac{z^2}{2}+W_{11}^{(j)}(0)z\bar{z}+W_{02}^{(j)}(0)\frac{\bar{z}^2}{2}+\cdots,\quad j=1,2,$$

and

$$W^{(2)}(t,-\tau)=W_{20}^{(2)}(-\tau)\frac{z^2}{2}+W_{11}^{(2)}(-\tau)z\bar{z}+W_{02}^{(2)}(-\tau)\frac{\bar{z}^2}{2}+\cdots.$$

It follows that

$$F_0 = \begin{pmatrix} -\left[\frac{\mu}{N^*} + \frac{(\mu(1-Q_0)+\mathrm{i}\omega_0)^2}{N^*\mu(1-Q_0)^2} - 2\frac{\mu(1-Q_0)+\mathrm{i}\omega_0}{N^*(1-Q_0)}\right] z^2 + \left[\frac{2\mu}{N^*} - \frac{2\mu^2(1-Q_0)^2+\omega_0^2}{N^*\mu(1-Q_0)^2}\right] z\bar{z} \\ -\left[\frac{\mu}{N^*} + \frac{(\mu(1-Q_0)-\mathrm{i}\omega_0)^2}{N^*\mu(1-Q_0)^2} - 2\frac{\mu(1-Q_0)-\mathrm{i}\omega_0}{N^*(1-Q_0)}\right]\bar{z}^2 - \left[\frac{\mu}{N^*}\left(2W_{11}^{(1)} + W_{20}^{(1)}\right)\right. \\ +\frac{2(\mu(1-Q_0)+\mathrm{i}\omega_0)}{N^*} W_{11}^{(2)} + \frac{\mu(1-Q_0)-\mathrm{i}\omega_0}{N^*} W_{20}^{(2)} \\ -\frac{2\mu(1-Q_0)}{N^*}\left(W_{11}^{(2)} + \frac{W_{20}^{(2)}}{2} + \frac{\mu(1-Q_0)+\mathrm{i}\omega_0}{\mu(1-Q_0)^2} W_{11}^{(1)} + \frac{\mu(1-Q_0)-\mathrm{i}\omega_0}{2\mu(1-Q_0)^2} W_{20}^{(1)}\right) \\ \left. -\frac{3(\mu^2(1-Q_0)^2+\omega_0^2)(\mu(1-Q_0)+\mathrm{i}\omega_0)}{N^{*2}\mu^2(1-Q_0)^4} - \frac{3\mu(1-Q_0)+\mathrm{i}\omega_0}{N^{*2}(1-Q_0)^2} + \frac{2}{N^{*2}}\left(\frac{(\mu(1-Q_0)+i\omega_0)^2}{1-Q_0} + 2\frac{\mu^2(1-Q_0)^2+\omega_0^2}{\mu(1-Q_0)^3}\right)\right] z^2\bar{z} \\ +\cdots - \frac{a(2-aN^*)}{2}\left(\frac{\mu(1-Q_0)+\mathrm{i}\omega_0}{\mu(1-Q_0)^2}\right)^2 \mathrm{e}^{-2\mathrm{i}\omega_0\tau} z^2 - a(2-aN^*)\frac{\mu^2(1-Q_0)^2+\omega_0^2}{\mu^2(1-Q_0)^4} z\bar{z} \\ -\frac{a(2-aN^*)}{2}\left(\frac{\mu(1-Q_0)-\mathrm{i}\omega_0}{\mu(1-Q_0)^2}\right)^2 \mathrm{e}^{2\mathrm{i}\omega_0\tau}\bar{z}^2 + \left[-\frac{a(2-aN^*)}{2}\left(2\frac{\mu(1-Q_0)+\mathrm{i}\omega_0}{\mu(1-Q_0)^2}\mathrm{e}^{-\mathrm{i}\omega_0\tau} W_{11}^{(2)}(-\tau)\right.\right. \\ \left.\left. +\frac{\mu(1-Q_0)-\mathrm{i}\omega_0}{\mu(1-Q_0)^2}\mathrm{e}^{\mathrm{i}\omega_0\tau} W_{20}^{(2)}(-\tau)\right) + \frac{a^2(3-aN^*)}{2}\frac{(\mu^2(1-Q_0)^2+\omega_0^2)(\mu(1-Q_0)+\mathrm{i}\omega_0)}{\mu^3(1-Q_0)^6}\mathrm{e}^{-\mathrm{i}\omega_0\tau}\right] z^2\bar{z} + \cdots \end{pmatrix}.$$

Hence, we have

$$g_{20} = \bar{D}\left[-2\left(\frac{\mu}{N^*} + \frac{(\mu(1-Q_0)+\mathrm{i}\omega_0)^2}{N^*\mu(1-Q_0)^2} - 2\frac{\mu(1-Q_0)+\mathrm{i}\omega_0}{N^*(1-Q_0)}\right) + \frac{a(2-aN^*)(\mu(1-Q_0)+\mathrm{i}\omega_0)^3}{\varepsilon\mu^2(1-Q_0)^4}\mathrm{e}^{-2\mathrm{i}\omega_0\tau}\right],$$

$$g_{11} = \bar{D}\left[\frac{2\mu}{N^*} - \frac{2\mu^2(1-Q_0)^2+\omega_0^2}{N^*\mu(1-Q_0)^2} + a(2-aN^*)\frac{(\mu^2(1-Q_0)^2+\omega_0^2)(\mu(1-Q_0)+\mathrm{i}\omega_0)}{\varepsilon\mu^2(1-Q_0)^4}\right],$$

$$g_{02} = \bar{D}\left[-2\left(\frac{\mu}{N^*} + \frac{(\mu(1-Q_0)-\mathrm{i}\omega_0)^2}{N^*\mu(1-Q_0)^2} - 2\frac{\mu(1-Q_0)-\mathrm{i}\omega_0}{N^*(1-Q_0)}\right) + \frac{a(2-aN^*)(\mu^2(1-Q_0)^2+\omega_0^2)(\mu(1-Q_0)-\mathrm{i}\omega_0)}{\varepsilon\mu^2(1-Q_0)^4}\mathrm{e}^{2\mathrm{i}\omega_0\tau}\right],$$

$$\begin{aligned} g_{21} = 2\bar{D}\Bigg\{ & \frac{\mu}{N^*}(2W_{11}^{(1)} + W_{20}^{(1)}) + \frac{2(\mu(1-Q_0)+\mathrm{i}\omega_0)}{N^*} W_{11}^{(2)} + \frac{\mu(1-Q_0)-\mathrm{i}\omega_0}{N^*} W_{20}^{(2)} - \frac{2\mu(1-Q_0)}{N^*} \\ & \times\left(W_{11}^{(2)} + \frac{W_{20}^{(2)}}{2} + \frac{\mu(1-Q_0)+\mathrm{i}\omega_0}{\mu(1-Q_0)^2} W_{11}^{(1)} + \frac{\mu(1-Q_0)-\mathrm{i}\omega_0}{2\mu(1-Q_0)^2} W_{20}^{(1)}\right) \\ & -\frac{3(\mu^2(1-Q_0)^2+\omega_0^2)(\mu(1-Q_0)+\mathrm{i}\omega_0)}{N^{*2}\mu^2(1-Q_0)^4} - \frac{3\mu(1-Q_0)+\mathrm{i}\omega_0}{N^{*2}(1-Q_0)^2} \\ & +\frac{2}{N^{*2}}\left(\frac{(\mu(1-Q_0)+\mathrm{i}\omega_0)^2}{1-Q_0} + 2\frac{\mu^2(1-Q_0)^2+\omega_0^2}{\mu(1-Q_0)^3}\right) - \frac{\mu(1-Q_0)+\mathrm{i}\omega_0}{\varepsilon}\Bigg[-\frac{a(2-aN^*)}{2} \\ & \times\left(2\frac{\mu(1-Q_0)+\mathrm{i}\omega_0}{\mu(1-Q_0)^2}\mathrm{e}^{-\mathrm{i}\omega_0\tau} W_{11}^{(2)}(-\tau) + \frac{\mu(1-Q_0)-i\omega_0}{\mu(1-Q_0)^2}\mathrm{e}^{\mathrm{i}\omega_0\tau} W_{20}^{(2)}(-\tau)\right) \\ & +\frac{a^2(3-aN^*)}{2}\frac{(\mu^2(1-Q_0)^2+\omega_0^2)(\mu(1-Q_0)+\mathrm{i}\omega_0)}{\mu^3(1-Q_0)^6}\mathrm{e}^{-\mathrm{i}\omega_0\tau}\Bigg]\Bigg\}. \end{aligned}$$

We still need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$.

For $\theta \in [-\tau, 0)$, we have

$$H(z,\bar{z},\theta) = 2\mathrm{Re}\{\bar{q}^*(0)F_0 q(\theta)\} = -gq(\theta) - \bar{g}\bar{q}(\theta)$$
$$= -\left(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots\right)q(\theta) - \left(\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{\bar{z}^2}{2} + \cdots\right)\bar{q}(\theta).$$

Comparing the coefficients with (3.24) gives

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}(\theta) \quad \text{and} \quad H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta).$$

It follows from the definition of W that

$$\dot{W}_{20}(\theta) = 2\mathrm{i}\omega_0 W_{20}(\theta) - g_{20}q(0)\mathrm{e}^{\mathrm{i}\omega_0\theta} - \bar{g}_{02}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\omega_0\theta}.$$

Solving for this, we obtain

$$W_{20}(\theta) = \frac{g_{20}}{\mathrm{i}\omega_0}q(0)\mathrm{e}^{\mathrm{i}\omega_0\theta} - \frac{\bar{g}_{02}}{3\mathrm{i}\omega_0}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\omega_0\theta} + E_1\mathrm{e}^{2\mathrm{i}\omega_0\theta}, \tag{3.25}$$

and similarly

$$W_{11}(\theta) = \frac{g_{11}}{\mathrm{i}\omega_0}q(0)\mathrm{e}^{\mathrm{i}\omega_0\theta} - \frac{\bar{g}_{11}}{\mathrm{i}\omega_0}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\omega_0\theta} + E_2, \tag{3.26}$$

where E_1 and E_2 are both 2-dimensional vectors, and can be determined by setting $\theta = 0$ in H. In fact, since

$$H(z,\bar{z},0) = -2\mathrm{Re}\{\bar{q}^*(0)F_0q(0)\} + F_0,$$

we have

$$H_{20} = -g_{20}q(0) - \bar{g}_{20}\bar{q}(0) + 2\begin{pmatrix} -\left[\dfrac{\mu}{N^*} + \dfrac{(\mu(1-Q_0)+\mathrm{i}\omega_0)^2}{N^*\mu(1-Q_0)^2} - 2\dfrac{\mu(1-Q_0)+\mathrm{i}\omega_0}{N^*(1-Q_0)}\right] \\ -\dfrac{a(2-aN^*)}{2}\left(\dfrac{\mu(1-Q_0)+\mathrm{i}\omega_0}{\mu(1-Q_0)^2}\right)^2\mathrm{e}^{-2\mathrm{i}\omega_0\tau} \end{pmatrix},$$

$$H_{11} = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \begin{pmatrix} \dfrac{2\mu}{N^*} - \dfrac{2\mu^2(1-Q_0)^2+\omega_0^2}{N^*\mu(1-Q_0)^2} \\ -a(2-aN^*)\dfrac{\mu^2(1-Q_0)^2+\omega_0^2}{\mu^2(1-Q_0)^4} \end{pmatrix}.$$

Hence, combining the definition of A, we can get

$$\begin{pmatrix} -\mu(1-Q_0)-2\mathrm{i}\omega_0 & \mu(1-Q_0)^2 \\ -\varepsilon & -d+b_0(1-aN^*)\mathrm{e}^{-((d_1+2\mathrm{i}\omega_0)\tau+aN^*)}-2\mathrm{i}\omega_0 \end{pmatrix}E_1$$
$$= 2\begin{pmatrix} -\left[\dfrac{\mu}{N^*} + \dfrac{(\mu(1-Q_0)+\mathrm{i}\omega_0)^2}{N^*\mu(1-Q_0)^2} - 2\dfrac{\mu(1-Q_0)+\mathrm{i}\omega_0}{N^*(1-Q_0)}\right] \\ -\dfrac{a(2-aN^*)}{2}\left(\dfrac{\mu(1-Q_0)+\mathrm{i}\omega_0}{\mu(1-Q_0)^2}\right)^2\mathrm{e}^{-2\mathrm{i}\omega_0\tau} \end{pmatrix},$$

and

$$\begin{pmatrix} -\mu(1-Q_0) & \mu(1-Q_0)^2 \\ -\varepsilon & -d+b_0(1-aN^*)\mathrm{e}^{-(d_1\tau+aN^*)} \end{pmatrix}E_2 = \begin{pmatrix} \dfrac{2\mu}{N^*} - \dfrac{2\mu^2(1-Q_0)^2+\omega_0^2}{N^*\mu(1-Q_0)^2} \\ -a(2-aN^*)\dfrac{\mu^2(1-Q_0)^2+\omega_0^2}{\mu^2(1-Q_0)^4} \end{pmatrix}.$$

Solving the above equations to obtain E_1 and E_2, and substituting them into (3.25) and (3.26), respectively, we can get $W_{20}(\theta)$ and $W_{11}(\theta)$. Then g_{21} can be expressed by the parameters and delay in Eq. (3.1).

Based on the above analysis, we can see that each g_{ij} is determined by the parameters and delay in (3.1). Thus, we can compute the following quantities:

$$\begin{aligned} C_1(0) &= \frac{\mathrm{i}}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\operatorname{Re} C_1(0)}{\operatorname{Re} \lambda^{'}(b_0)}, \\ \beta_2 &= 2\operatorname{Re} C_1(0). \end{aligned} \tag{3.27}$$

We know that (see Hassard et al. [3]) (i) μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0 (< 0)$, then the Hopf bifurcation is supercritical (subcritical) meaning that bifurcated periodic solution exist for $\mu > \mu_0 (< \mu_0)$; (ii) β_2 determines the stability of the bifurcated periodic solution: the bifurcated periodic solution is orbitally stable (unstable) if $\beta_2 < 0 (> 0)$.

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