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On the Characters and the Plancherel Formula of Nilpotent Groups*

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1. Let G be a connected and simply connected nilpotent group with the Lie algebra $\mathscr{L}$. We denote by T an irreducible unitary representation of G on a Hilbert space. Let da be a left-invariant Haar measure on G, and φ a C^∞ function of compact support. Then the operator defined by

$$T_\varphi = \int_G \varphi(a)\, T(a)\, da$$

is of the trace class (cf. [4], p. 108), and we have Eq. (1) below for its trace in terms of φ. We denote by $\mathscr{L}'$ the dual of the underlying space of $\mathscr{L}$, and by (ℓ, ℓ') the canonical bilinear form on $\mathscr{L} \times \mathscr{L}'$. Let us form the Fourier transform of the C^∞ function of compact support defined on $\mathscr{L}$ by $\varphi(\ell) \equiv \varphi\,(\exp \ell)\ (\ell \in \mathscr{L})$, through the formula

$$\tilde{\varphi}(\ell') = \int_{\mathscr{L}} \varphi(\ell)\, e^{i(\ell,\ell')}\, d\ell \qquad (\ell' \in \mathscr{L}'),$$

$d\ell$ denoting a positive translation-invariant measure on $\mathscr{L}$. Then we have

$$\operatorname{Tr}(T_\varphi) = \int_O \tilde{\varphi}(\ell')\, dv. \tag{1}$$

Here O stands for an orbit of the representation, contragredient to the adjoint representation, of G on $\mathscr{L}'$, and dv denotes an invariant measure on O. The orbit O is well-determined by the equivalence class of T and conversely, given any orbit in $\mathscr{L}'$, there exists a unique equivalence class of irreducible representations of G, which corre-

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sponds to it by virtue of the formula (1) (cf. Théorème, p. 145 in [4]). The measures da and $d\ell$ having been fixed, the measure dv is uniquely determined on each orbit; it will be referred to as the canonical measure in the sequel. Up to now, no direct characterization of the canonical measure has been known; this question is of interest, in particular, for the derivation of the Plancherel formula for G (cf. [4], Chap. III, Sections 4-6).

Let us consider now an arbitrary connected Lie group G with the Lie algebra $\mathscr{L}$. If O is an orbit of the kind considered above, and of a positive dimension, in $\mathscr{L}'$, we can assign to it an invariant measure as follows. We write σ for the adjoint representation of G, and ρ for the representation, contragredient to σ. Choosing an arbitrary element p of O, let us consider the map α_p from G onto O defined by $\alpha_p(a) = \rho(a)p$ $(a \in G)$. Its differential $\varphi_p = d\alpha_p$ is a map of $\mathscr{L}$ onto the tangent space T_p of O at p, and its kernel, identifiable to the Lie algebra of the stable group of p, coincides with the radical of the skew-symmetric bilinear form $B_p(\ell_1, \ell_2) = ([\ell_1, \ell_2], p)$ on $\mathscr{L} \times \mathscr{L}$. Therefore there exist a well-determined nondegenerate skew-symmetric bilinear form ω_p on $T_p \times T_p$, such that $\delta\alpha_p(\omega_p) = B_p$. Varying p on O we obtain in this fashion a 2-form ω, which is invariant under the action of G. In fact, writing $s(a)x = axa^{-1}$ $(a, x \in G)$ and $q = \rho(a)p$ $(p \in O)$ we have $\alpha_q \circ s(a) = \rho(a) \circ \alpha_p$ whence, by taking differentials, it follows that $\varphi_q \circ \sigma(a) = d\rho(a)|_p \circ \varphi_p$. But then, setting $\delta\rho(a)|_q\, \omega_q = \omega_p$, we obtain for all ℓ_1 and ℓ_2 in $\mathscr{L}$:

$$\omega_p'(\varphi_p(\ell_1), \varphi_p(\ell_2)) = \omega_q(\varphi_q(\sigma(a)\ell_1),$$

$$\varphi_q(\sigma(a)\ell_2)) = B_q(\sigma(a)\ell_1, \sigma(a)\ell_2) = B_p(\ell_1, \ell_2) = \omega_p(\varphi_p(\ell_1), \varphi_p(\ell_2)),$$

and therefore $\omega_p' = \omega_p$, proving the statement. Let us denote by d the dimension of O. Since this is necessarily an even number, we can form the exterior power $(\omega)^{d/2}$ of ω. We shall call the positive invariant measure, corresponding to it on O, the K-measure of O.

The author is indebted to B. Kostant for the conjecture that, in the nilpotent case, the canonical measure and the K-measure are essentially the same. This is strongly suggested by his construction (unpublished), making possible, in particular, to set up a one-to-one correspondence between certain orbits and the set of all equivalence classes of irreducible representations of a compact semisimple group (cf. Section 2 for more details), together with the observation, that the ratio of the (necessarily finite) total K-volume of an orbit and of the dimension of the corresponding irreducible representation is a constant depending only on the group.

The main purpose of the present paper is to verify this conjecture for the nilpotent groups (cf. Section 3), though its domain of validity seems to include all cases, where a formula of the type (1) can be set up at all. We shall return to the discussion of the case of the exponential groups[1] in a further publication.

The plan of this paper is as follows. Though not directly related to the main result, because of its great heuristic value, we thought it useful to begin with a self-contained discussion of the situation, just quoted, offered by a compact semisimple Lie group. The sole purpose of Section 2 is to place the subsequent considerations concerning the nilpotent case in a more general context. We show, how the 1-1 correspondence, pointed out by Kostant, between the set of all equivalence classes of irreducible representations of a connected and simply connected compact semisimple Lie group G and certain orbits in the corresponding algebra $\mathscr{L}$ (identified now with its dual through the Killing form) can be established, just as in the nilpotent case, by aid of a formula like (1). The only difference comes from the fact that, as suggested by Kirillov in the case of $SU(2)$ (cf. [3], Section 5 in §8), prior to forming the Fourier transform of the function φ on $\mathscr{L}$, we have to multiply it by a certain expression, depending only on $\mathscr{L}$. All this follows easily from the Weyl character formula along with a theorem of Harish-Chandra (cf. [2], Theorem 2) implying a simple relationship between the character of a given representation and the Fourier transform of a positive measure on $\mathscr{L}$, which is invariant with respect to the adjoint representation and is concentrated on the corresponding orbit. Let us observe, incidentally, that in the nilpotent case, the Fourier transform (formed with $\mathscr{L}'$), which has just been referred to, does not admit any manageable expression on $\mathscr{L}$. In Section 3 we prove the main result, according to which for nilpotent groups the ratio of the canonical measure and of the K-measure of an orbit depends only on the dimension of the latter. Taking into account the picture found in the compact semisimple case in Section 2, one can interpret formula (1), along with the precise expression, obtained in Section 3, for the canonical measure, as the analog for nilpotent groups of the Weyl character formula. Finally, in Section 4 we apply this result to obtain an explicit algorithm for the computation of the Plancherel measure. In fact, to obtain this, by virtue of Section 3 it suffices to represent a translation-invariant measure on $\mathscr{L}'$ as a continuous sum of the K-measures of the orbits

[1] A connected and simply connected solvable Lie group is called exponential, if the exponential map is onto.

of the representation ρ. We show, that the result implies the theorem of J. Dixmier ([*4*], p. 171), according to which the Plancherel measure is representable by a rational differential on the quotient space, according to ρ, of the union of orbits of maximal dimension in $\mathscr{L}'$. There is good reason to believe, that the same algorithm remains in force for exponential unimodular groups too.

As indicated above, the reading of Section 2 is not necessary for the understanding of the main results of this paper.

2. As mentioned above, the purpose of this Section is to discuss, following the main idea of the presentation of Kirillov's theory in [*4*], Part II, Chap. II, Sections 6-8, certain facts concerning irreducible representations of compact semisimple groups, the main objective being the demonstration of the constancy of the ratio of the dimension of an irreducible representation and of the total K-volume of the corresponding orbit.

In what follows throughout this Section, $\mathscr{L}$ will denote a fixed compact semisimple Lie algebra, and $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathscr{L}$. Let P be the collection of all nonzero roots of $\mathscr{L}$ with respect to $\mathfrak{h}$ We assume the elements of P under the form $i\alpha(h)$ $(h \in \mathfrak{h})$, where i is a fixed square root of -1. Writing for x and y in $\mathscr{L} : (x, y) = -\operatorname{Tr}(adx\, ady)$, for any root α there exists a well-determined element α' in $\mathfrak{h}$, such that $\alpha(h) = (h, \alpha')$ $(h \in \mathfrak{h})$; in the sequel α', too, will be referred to as a root, and we omit the prime. Let $\{\alpha_j; j = 1, 2,..., r\}$ $(r = \dim)$ be a system of simple roots and P_+ the collection of the corresponding positive roots. We write $h_j = 2\alpha_j/(\alpha_j, \alpha_j)$ $(j = 1, 2,..., r)$. Then, as it is known (cf. [*1*], Exposés 17-21), there is a 1-1 correspondence between the set of all equivalence classes of irreducible representations of the connected and simply connected group G determined by $\mathscr{L}$, and the set Λ of all elements λ of $\mathfrak{h}$, for which (λ, h_j) $(j = 1, 2,..., r)$ is nonnegative and integral. We denote by ρ the half sum of all positive roots, and by W the Weyl group corresponding to $\mathfrak{h}$; for s in W and h in $\mathfrak{h}$, sh will stand for the action of s on h. Then the function χ_λ on $\mathfrak{h}$, which corresponds to the restriction to $\exp \mathfrak{h}$ of the character of an irreducible representation of class λ $(\lambda \in \Lambda)$ is given by

$$\chi_\lambda(h) = \frac{Q_\lambda(h)}{D(h)},$$

where

$$Q_\lambda(h) = \sum_{s \in W} \epsilon(s) \exp[i(h, \rho + \lambda)]$$

and

$$D(h) = \prod_{\alpha\in P_+} (\exp [i \tfrac{1}{2} (h, \alpha)] - \exp [- i \tfrac{1}{2} (h, \alpha)])$$
$$= \sum_{s\in W} \epsilon(s) \exp [i(sh, \rho)] \qquad [\epsilon(s) = \det (s) \text{ for } s \text{ in } W].$$

In what follows, $d\ell$ and dh will denote volume elements on $\mathscr{L}$ and $\mathfrak{h}$ respectively, corresponding to the Euclidean metric determined by the negative Killing form. We denote again by $\sigma(a)$ the adjoint representation of G and by da the element of the Haar measure on G, such that $\int_G da = 1$. We put $\pi(h) = (i)^m \prod_{\alpha\in P_+} \alpha(h)$ $(h \in \mathfrak{h})$, where m is the number of positive roots. Then ([2], p. 105) there exists a constant C_0, such that we have for all functions f, which are continuous and of a compact support on $\mathscr{L}$:

$$\int_{\mathscr{L}} f(\ell)\, d\ell = C_0 \int |\pi(h)|^2 \left(\int_G f(\sigma(a)\, h)\, da\right) dh.$$

We shall also use the following formula, due to Harish-Chandra (cf. [2], Theorem 2, p. 104)[2]:

$$\pi(h)\, \pi(-h') \int_G \exp (\sigma(a)\, h, h')\, da = d_0 \left(\sum_{s\in W} \epsilon(s) \exp (sh, h')\right),$$

valid for any two elements h and h' in $\mathfrak{h}$; d_0 stands for $\prod_{\alpha\in P_+} (\alpha, \rho)$.

Given any function, invariant with respect to the Weyl group, on $\mathfrak{h}$, we denote with the same letter the unique σ-invariant function on $\mathscr{L}$ determined by it. We write $a(\ell)$ for $D(\ell)/\pi(\ell)$ $(\ell \in \mathscr{L})$ and recall, that in a neighborhood of the neutral element in $\mathscr{L}$, where the exponential mapping is 1-1, $|a(\ell)|^2\, d\ell$ is the expression for a Haar measure on G. Finally, for any λ in Λ we write $O(\lambda)$ for the orbit, with respect to the adjoint representation, of $\lambda + \rho$.

With these notations, we have the following

Proposition 1. *For any λ in Λ and $f \in C^\infty$ of a compact support on $\mathscr{L}$ we have*

$$\int_{\mathscr{L}} f(\ell)\, \chi_\lambda(\ell)\, |a(\ell)|^2\, d\ell = \int_{O(\lambda)} \tilde{f}_1(\ell)\, dv; \tag{2}$$

Here dv is an invariant measure on $O(\lambda)$; $f_1(\ell) \equiv f(\ell)\, \overline{a(\ell)}$ and

$$\tilde{f}(\ell') = \int_{\mathscr{L}} f(\ell) \exp [i(\ell, \ell')]\, d\ell \qquad (\ell' \in \mathscr{L}).$$

[2] Substitute $-H$ for the H used in [2], and observe that $\langle\pi, \pi\rangle/w = d_0 = \prod_{\alpha\in P_+}(a,\rho)$.

Moreover, the orbit $O(\lambda)$ and the measure dv are uniquely determined by (2).

Proof. We start by observing, that

$$\chi_\lambda(\ell) \mid a(\ell) \mid^2 = Q_\lambda(\ell) \overline{D(\ell)} \mid \pi(\ell) \mid^{-2},$$

and therefore, we have for the left-hand side of (2)

$$\int_{\mathscr{L}} f(\ell) Q_\lambda(\ell) \overline{D(\ell)} \mid \pi(\ell) \mid^{-2} d\ell = C_0 \int Q_\lambda(h) \overline{D(h)} F(h)\, dh,$$

where we have put

$$F(h) = \int_G f(\sigma(a)\, h)\, da.$$

On the other hand, writing d_λ for the volume of $O(\lambda)$ with respect to dv, we have for any $g(\ell)$ continuous on $\mathscr{L}$, putting $\tilde{\lambda} = \lambda + \rho$,

$$\int_{O(\lambda)} g(\ell)\, dv = d_\lambda \int_G g(\sigma(a)\, \tilde{\lambda})\, da$$

and therefore the right-hand side of (2) gives

$$d_\lambda \int_G \tilde{f}_1(\sigma(a)\, \tilde{\lambda})\, da = d_\lambda \int_G \left(\int_{\mathscr{L}} f_1(\ell) \exp\, [i(\ell, \sigma(a)\, \tilde{\lambda})]\, d\ell \right) da$$

$$= d_\lambda \int_{\mathscr{L}} \overline{a(\ell)} f(\ell)\, G(\ell)\, d\ell$$

where

$$G(\ell) = \int_G \exp\, [i(\sigma(a)\, \ell, \tilde{\lambda})]\, da.$$

Since this function is clearly σ-invariant, we can conclude that

$$\int_{O(\lambda)} \tilde{f}_1(\ell)\, dv = d_\lambda C_0 \int \pi(h)\, G(h)\, \overline{D(h)} F(h)\, dh.$$

Summing up, to establish the equality appearing in the Proposition, it suffices to show, that $Q_\lambda(h) \equiv d_\lambda \pi(h)\, G(h)$ $(h \in \mathfrak{h})$, or that

$$d_\lambda \pi(h) \int_G \exp\, [i(\sigma(a)\, h, \tilde{\lambda})]\, da = \sum_{s \in W} \epsilon(s) \exp\, [i(sh, \tilde{\lambda})].$$

But for this it suffices to substitute $i\lambda$ in place of h' in the formula of Harish-Chandra, and to choose for d_λ

$$\frac{\pi(-i\lambda)}{d_0} = \frac{\prod_{\alpha>0}(\alpha, \lambda+\rho)}{\prod_{\alpha>0}(\alpha, \rho)}.$$

Observe, that this is the dimension of the representation, corresponding to λ in Λ.

To prove, that the orbit $O(\lambda)$ in the right-hand side of (2) is uniquely determined, since $a(0) = 1$, it is enough to show, that if O_1 and O_2 are two orbits of σ and dv_1 and dv_2 are nontrivial invariant measures on them, and if we have

$$\int_{O_1} \tilde{f}(\ell)\, dv_1 = \int_{O_2} \tilde{f}(\ell)\, dv_2$$

for all C^∞ functions, vanishing outside a given neighborhood of the neutral element in $\mathscr{L}$, then $O_1 = O_2$ and $dv_1 = dv_2$. Let L be a translation-invariant differential operator on $\mathscr{L}$. Replacing f first by Lf, then by an approximate identity in the above equation, we conclude, that

$$\int_{O_1} P(\ell)\, dv_1 = \int_{O_2} P(\ell)\, dv_2$$

for any polynomial function $P(\ell)$ on $\mathscr{L}$. If O_1 and O_2 are different, then, since they are compact and without common point, by an appropriate choice of P, we can arrange that the left-hand side should be arbitrarily small, while the right-hand side remains close to the volume of O_2 with respect to dv_2 which, according to assumption, is nonzero. Hence $O_1 = O_2$; but then dv_1 and dv_2, too, coincide, and this finishes the proof of Proposition 1.

Summing up once more, Proposition 1 establishes a 1-1 correspondence between the family of orbits, determined by elements of the form $\lambda + \rho$ ($\lambda \in \Lambda$), in $\mathscr{L}$ and the set of all equivalence classes of irreducible representations of G in the following fashion: The character $\chi_\lambda(h)$ equals $D(h)/\pi(h)$ times the Fourier transform of the positive, σ-invariant measure, concentrated on the orbit of $\lambda + \rho$, in $\mathscr{L}$, the total mass of which equals $d_\lambda = \chi_\lambda(0)$. Our next Proposition shows, that the K-volume (cf. Section 1) and d_λ are proportional.

Proposition 2. *For a fixed λ in Λ, let us denote by $V(K)$ the*

K-volume of the orbit of $\lambda + \rho$. Then, with the notations used above, we have

$$V(K) = C_0 2^m \cdot m! \prod_{\alpha \in P_+} (\alpha, \lambda + \rho).$$

In particular,

$$\frac{V(K)}{d_\lambda} = C_0 2^m \cdot m! \prod_{\alpha \in P_+} (\alpha, \rho)$$

does not depend on λ.

Proof. As it is known, we can choose an orthonormal basis $\{E_\alpha, E'_\beta; \alpha, \beta \in P_+\}$ in the orthogonal complement $\mathfrak{h}^\perp$ of $\mathfrak{h}$ in $\mathscr{L}$ in such a fashion, that $adhE_\alpha = \alpha(h) E'_\alpha$ and $adhE'_\alpha = -\alpha(h) E_\alpha$ for all h in $\mathfrak{h}$. Next we observe, that the Lie algebra of the stabilizer of $\lambda + \rho$ in $\mathscr{L}$ coincides with $\mathfrak{h}$. To this end we remark first, that the subalgebra of $\mathscr{L}$, which has just been referred to, coincides with the centralizer of the element $\bar{\lambda} = \lambda + \rho$, of $\mathfrak{h}$, in $\mathscr{L}$. On the other hand, if

$$\ell = h + \sum_{\alpha \in P_+} (c_\alpha E_\alpha + c'_\alpha E'_\alpha) \qquad (h \in \mathfrak{h})$$

is some element of $\mathscr{L}$, we have

$$\| [\ell, \bar{\lambda}] \|^2 = \| ad\bar{\lambda}\ell \|^2 = \sum_{\alpha \in P_+} [\alpha(\bar{\lambda})]^2 (c_\alpha^2 + c'^2_\alpha).$$

But since (α, ρ) is positive for all α in P_+ (cf. [1], Lemma 2 in Exposé 19), and since λ is in Λ, $\alpha(\bar{\lambda})$ is always positive. Therefore, if ℓ is in the centralizer of $\bar{\lambda}$, we must have $c_\alpha = c'_\alpha = 0$ for all α, and thus ℓ belongs to $\mathfrak{h}$. We are going to compute the K-volume of the orbit of $\lambda + \rho$, to be denoted again by $O(\lambda)$, in two steps. First we determine the volume $V(E)$ of $O(\lambda)$ with respect to the metric induced on $O(\lambda)$ by the Euclidean metric of $\mathscr{L}$, and then we compute the ratio $V(K)/V(E)$. As regards the first point, we start by observing, that the tangent space to $O(\lambda)$ at $\bar{\lambda}$, since it is spanned by the vectors $\{ad\ell\bar{\lambda}; \ell \in \mathscr{L}\}$, by virtue of what we saw above, coincides with $\mathfrak{h}^\perp$. Therefore, since $\sigma(a)$ is an orthogonal transformation of $\mathscr{L}$ into itself for any a in G, at the point $\sigma(a)\,\bar{\lambda}$ of $O(\lambda)$ the orthogonal complement of the tangent space can be identified with $\sigma(a)\,\mathfrak{h}$. Let us set again $r = \dim \mathfrak{h}$. For a positive ϵ, we denote by S_ϵ the r-dimensional solid sphere of radius ϵ in $\mathfrak{h}$, and we write $v(\epsilon)$ for its volume. Also, we write

$f_\epsilon(\ell)$ for the characteristic function of the set $\cup_{a \in G} \sigma(a)(\bar{\lambda} + S_\epsilon)$ in $\mathscr{L}$. Then we have the following relation:

$$V(E) = \lim_{\epsilon \to 0} \left(\frac{1}{v(\epsilon)}\right) \int_{\mathscr{L}} f_\epsilon(\ell)\, d\ell$$

$$= C_0 \lim_{\epsilon \to 0} \left(\frac{1}{v(\epsilon)}\right) \int_{S_\epsilon} | \pi(\bar{\lambda} + h) |^2\, dh = C_0 \,| \pi(\bar{\lambda}) |^2;$$

in other words, we have $V(E) = C_0 \prod_{\alpha \in P_+} (\alpha, \lambda + \rho)^2$. Let us denote by d the dimension of $O(\lambda)$; we have $d = 2m$. Next we observe, that $V(K)/V(E)$ is the same as the ratio of the two d-forms, representing the volume element in the K-metric (cf. Section 1) and in the Euclidean metric of $O(\lambda)$, at $\bar{\lambda}$, say. We denote by $\{e_j; j = 1, 2,..., d\}$ the vectors $\{E_\alpha, E'_\alpha; \alpha \in P_+\}$ in some order. Then, by virtue of a known formula, the coefficient of $\prod_{j=1}^{d} \wedge e_j$ in the $(d/2)$th power of

$$\sum_{i,j=1}^{d} ([e_i, e_j], \bar{\lambda})\, e_i \wedge e_j$$

equals $2^m \cdot m!$ times the square root of det $(([e_i, e_j], \bar{\lambda}); i, j = 1,..., d)$; one verifies by an easy computation, that the latter is the same as $\prod_{\alpha \in P_+} (\alpha, \lambda + \rho)$. On the other hand, the factor of $\prod_{j=1}^{d} \wedge e_j$ in the d form, corresponding to the Euclidean volume element of $O(\lambda)$, at $\bar{\lambda}$ is equal to the square root of det $((ade_i\, \bar{\lambda}, ade_j\, \bar{\lambda}); i, j = 1,..., d)$; one verifies easily, that this is the same as $| \pi(\bar{\lambda}) |^2$. Therefore the ratio $V(K)/V(E)$ is given by $2^m \cdot m! \prod_{\alpha \in P_+} (\alpha, \lambda + \rho)/| \pi(\bar{\lambda}) |^2$, and thus finally we obtain that $V(K) = C_0 2^m \cdot m! \prod_{\alpha \in P_+} (\alpha, \lambda + \rho)$, finishing the proof of Proposition 2.

We close this section with the following observation. Let $\varphi(a)$ be a C^∞ function which vanishes outside a neighborhood, where canonical coordinates of the first kind can be introduced, of the unity of G. We write $\varphi(\ell)$ for the corresponding function on G. For a λ in Λ, we denote by $T^{(\lambda)}$ an irreducible representation, of the class λ, of G. Then, by virtue of the previous considerations, we have

$$\mathrm{Tr}\,(T_\varphi^{(\lambda)}) = \int_{O(\lambda)} \tilde{\varphi}(\ell)\, dv,$$

where dv is the element of the K-measure for $O(\lambda)$, provided the Haar measure da on the left, and the volume element $d\ell$ (by aid of which the Fourier transform in the right-hand side is formed) of

$\mathscr{L}$ are chosen in such a fashion, that the ratio $da/d\ell$ at the neutral element of $\mathscr{L}$ should be the same as $V(K)/d_\lambda = C_0 2^m \cdot m! \prod_{\alpha\in P_+} (\alpha, \rho)$. We shall see in the next section that, in the nilpotent case the value of the analogously defined constant is $4^m \cdot m! \, \pi^m$.

3. In this Section we consider again a nilpotent Lie algebra $\mathscr{L}$ of dimension $n > 0$, with the corresponding connected and simply-connected group G. We denote by σ the adjoint representation of G, and by ρ the representation, which is contragredient to σ, on $\mathscr{L}'$. In other words, denoting by (ℓ, ℓ') the canonical bilinear form on $\mathscr{L} \times \mathscr{L}'$, we have, for all a in G,

$$(\sigma(a)\ell, \ell') = (\ell, \rho(a^{-1})\ell') \qquad (\ell \in \mathscr{L}, \ell' \in \mathscr{L}').$$

Let H be a subalgebra of $\mathscr{L}$, of a dimension $h < n$. We shall say, that the ordered $(n-h)$-tuple of elements $\{\ell_1, \ell_2, ..., \ell_{n-h}\}$ of $\mathscr{L}$ is a supplementary basis of H, provided for each $j = 1, 2, ..., n-h$ the subspace, spanned by H and $\{\ell_1, ..., \ell_j\}$, of $\mathscr{L}$ is a subalgebra of dimension $h + j$. Let us denote by exp H the subgroup, which is the image of H through the exponential map, of G and let us set $g_j(t) = \exp(\ell_j t)$ $(j = 1, 2, ..., n-h)$. The map of $\exp H \times R^{n-h}$ into G, which assigns to $(h, (t_1, t_2, ..., t_{n-h}))$ $(h \in \exp H)$ the element

$$hg_1(t_1)\, g_q(t_2) \cdots g_{n-h}(t_{n-h})$$

of G, is a homeomorphism between $\exp H \times R^{n-h}$ and G ([*4*], p. 96, Remarque 2), and $dt_1\, dt_2 \cdots dt_{n-h}$ defines an invariant measure on the homogeneous space of right-classes of G according to exp H ([*4*], p. 121, Remarque).

An element ℓ_0' of $\mathscr{L}'$ having been fixed, we shall denote by $B(\ell_1, \ell_2)$ the skew-symmetric bilinear form on $\mathscr{L} \times \mathscr{L}$ defined by $([\ell_1, \ell_2], \ell_0')$ $(\ell_1, \ell_2 \in \mathscr{L})$. Given a subspace H of $\mathscr{L}$, we shall write $(H)_B^\perp$ for its orthogonal complement, with respect to B, in $\mathscr{L}$, and $H^\perp$ for its orthogonal complement, with respect to the canonical bilinear form in $\mathscr{L}'$.

We start with a new proof for a statement, announced first by Kirillov ([*4*], Théorème on *p*. 50, and also the Remark below).

LEMMA 1. *Let ℓ_0' be a nonzero element in $\mathscr{L}'$, and let O be its orbit with respect to ρ; we set $d = \dim O$, and suppose $d > 0$. Let*

$$\mathscr{L}_n = \mathscr{L} \supset \mathscr{L}_{n-1} \supset \cdots \supset \mathscr{L}_1 \supset \mathscr{L}_0 = (O)$$

be a Jordan-Hölder sequence in $\mathscr{L}$, and $\{\ell'_j; j = 1, 2,..., n\}$ a basis in $\mathscr{L}'$ such that $\ell'_j \in \mathscr{L}^{\perp}_{j-1} - \mathscr{L}^{\perp}_j$ $(j = 1, 2,..., n)$. Then there exist n polynomials $\{P_j; j = 1, 2,..., n\}$ of the d variables $\{z_k; k = 1, 2,..., d\}$ and d indices $0 < j_1 < j_2 < \cdots < j_d \leqslant n$ in the following fashion:

(1) *The orbit O coincides with the collection of all elements of $\mathscr{L}'$ which can be written as $\sum_{j=1}^{n} P_j(z)\, \ell'_j$ $(z \in R^d)$;*

(2) $P_{j_k}(z) \equiv z_k$ $(k = 1, 2,..., d)$;

(3) *for any $j = 1, 2,..., n$ $P_j(z)$ depends only on $\{z_1, z_2, ..., z_k\}$, where k is the largest integer with which $j_k \leqslant j$.*

Proof. We denote by R the radical of $B(\ell_1, \ell_2)$ (formed with ℓ'_0); in other words, $R = (\mathscr{L})^{\perp}_B$. We write $\mathscr{L}_j = R + \mathscr{L}_j$ and observe, that there exist d integers $0 < j_1 < j_2 < \cdots < j_d \leqslant n$, such that if $J = \{j_k; k = 1, 2,..., d\}$, we have $\mathscr{L}_j \supsetneq \mathscr{L}_{j-1}$ if and only if j belongs to J. We set $E_k = \mathscr{L}_j$ if $j = j_k$, $E_0 = R$ and observe, that for $k = 0, 1,..., d$ $F_k = (E_k)^{\perp}_B$ is a subalgebra in $\mathscr{L}$. To this end it suffices to show, that if I is an ideal in $\mathscr{L}$, then $(I)^{\perp}_B$ is a subalgebra. Suppose, that r_1 and r_2 are orthogonal to I. Then, since we have, by virtue of the identity of Jacobi, for any ℓ in $\mathscr{L}$

$$B([r_1, r_2], \ell) = - B([r_2, \ell], r_1) - B([\ell, r_1], r_2),$$

we see that $[r_1, r_2]$ too is orthogonal to I, since for ℓ in I the summands on the right-hand side vanish. Let us choose a basis $\{\ell_j\}$ in $\mathscr{L}$ such that $(\ell_i, \ell'_j) = \delta_{ij}$; we write $e_k = \ell_j$ if $j = j_k$, and determine elements $\{f_j; j = 1, 2,..., d\}$ in $\mathscr{L}$ satisfying $B(e_i, f_j) = \delta_{ij}$. The subspace, spanned by R and the elements $f_{k+1}, f_{k+2}, ..., f_d$ of $\mathscr{L}$ coincides with E_k, and thus the system $\{f_d, f_{d-1}, ..., f_1\}$ is a supplementary basis to R. Therefore, setting $g_k(t) = \exp(f_k t)$, $T = (t_1, t_2, ..., t_d)$ and $g(T) = g_1(t_1)\, g_2(t_2) \cdots g_d(t_d)$, we may conclude that the map of R^d into O, which sends T into $\rho(g(T))\, \ell'_0$, is surjective. Let us write $\rho(g(T))\, \ell'_0 = \sum_{j=1}^{n} Q_j(T)\, \ell'_j$. We are going to prove, that the polynomials $\{Q_j(T)\}$ possess the following two properties

(1) $Q_{j_k}(T) \equiv t_k \bmod (t_1, t_2, ..., t_{k-1})$,

(2) For any $j = 1, 2,..., n$, $Q_j(t)$ depends only on $(t_1, t_2, ..., t_k)$, where k is the largest integer, such that $j_k \leqslant j$.

Assuming this, we may conclude first, that the map, considered above, of R^d onto O is a bijection. Next, setting $z_k = Q_j(T)$ for $j = j_k$, $(k = 1, 2,..., d)$, we can express stepwise the variables $\{t_k\}$ through the variables $\{z_j\}$. Substituting finally the result into the polynomials $\{Q_j(T); j \in J\}$, and writing $P_j(z) \equiv Q_j(T)$ $(j = 1, 2,..., n)$, we obtain a system of polynomials $\{P_j\}$, having the properties claimed in the Lemma.

Since $Q_j(T) \equiv (\ell_j, \rho(g(T))\,\ell_0')$, to prove (2) it is clearly enough to show, that

$$((adf_d)^{a_d}(adf_{d-1})^{a_{d-1}}\cdots(adf_1)^{a_1}\ell_j, \ell_0') = 0$$

($a_i \geqslant 0$), if, for some k with $j_k > j$, we have $a_k > 0$. But in this case the left hand side can be written as $B(f_\ell, \bar{\ell}_j)$, where $\bar{\ell}_j$ is some element in $\mathscr{L}_j$ and $j_\ell > j$. But this is zero, since f_ℓ lies in $F_{\ell-1} \subset (\mathscr{L}_j)_B^{\perp}$. To establish (1) it suffices to prove that, for $j = j_k$,

$$((adf_k)^{a_k}(adf_{k-1})^{a_{k-1}}\cdots((adf_1)^{a_1}\ell_j, \ell_0') = 0$$

if $a_k > 1$ or $a_k = 1$ and $a_\ell > 0$ for some $\ell < k$, and -1 if $a_k = 1$ and $a_\ell = 0$ for $\ell < k$. The second statement being implied by $B(e_i, f_j) = \delta_{ij}$, we consider only the first. In this case, since $[\mathscr{L}, \mathscr{L}_j] \subseteq \mathscr{L}_{j-1}$, the left hand side has the form $B(f_k, \bar{\ell}_{j-1})$, where $\bar{\ell}_{j-1}$ lies in $\mathscr{L}_{j-1}$ ($j = j_k$). But as above, we can show, that this is zero, finishing thus the proof of the Lemma.

Remark. Let G as above; we recall, that a continuous representation U, of G, on a finite-dimensional vector space V is called unipotent, provided the differential dU of this representation contains nilpotent operators only. We wish to show, that in the previous Lemma $\mathscr{L}'$ and ρ can be replaced by V and U, respectively. In fact, considering the dual V' of V as an Abelian Lie algebra, and setting $\tau(\ell) = -$ [transpose of $dU(\ell)$] ($\ell \in \mathscr{L}$), let us form the semidirect product $\mathscr{P}$ of $\mathscr{L}$ and V' by defining the bracket of ℓ in $\mathscr{L}$ and v' in V' as $\tau(\ell)\,v'$. Then the dual V of the ideal V' of $\mathscr{P}$ can be identified with the quotient space of the dual of $\mathscr{P}$ according to $\mathscr{L}'$, and the representation induced in it by $d\rho$, formed for $\mathscr{P}$, is identifiable with dU. Therefore, to obtain the desired parametrization for the orbit of an element V_0 of V, with respect to U, it suffices to choose a Jordan-Hölder sequence of $\mathscr{P}$, containing V' and then to consider the parametrization (in the sense of the previous Lemma) of the orbit, with respect to ρ for exp $\mathscr{P}$, of an element of $\mathscr{P}'$, lying over v_0, and finally to take the result mod $\mathscr{L}'$.

Let again ℓ_0' be a nonzero element of the dual of $\mathscr{L}$. We recall (cf. [*4*], pp. 153-154), that a subalgebra H, of $\mathscr{L}$, is said to be subordinated to ℓ_0', if this is perpendicular to the first derived algebra of H. In this case we can form a character χ of $K = \exp H$ by setting $\chi(\exp h) = \exp[i(h, \ell_0')]$ ($h \in H$). Let us form the representation T, induced by χ, of G. We denote again by d the dimension of the orbit O of ℓ_0' with respect to ρ. Then, as it is known ([4], p. 154), T is irreducible if and only if we have $\dim H = \dim \mathscr{L} - \frac{1}{2} d$.

The content of the following lemma is a special case of known results quoted before. The purpose of the new proof given for it here is to obtain a suitable algorithm for the computation of the canonical measure (cf. the Remark directly following the proof).

LEMMA 2. *Let ℓ_0' a fixed element of $\mathscr{L}'$; we suppose that the dimension d of the corresponding orbit is positive. Assume that the subalgebra H, of dimension $\dim \mathscr{L} - \frac{1}{2} d$, is subordinated to ℓ_0', and form the unitary representation T as above. Let ψ be a C^∞ function of compact support on G. Then the operator T_ψ is of class Hilbert-Schmidt, and we have*

$$\mathrm{Tr}\,(T_\psi T_\psi^*) = \int_O \tilde{\varphi}(\ell')\, dv,$$

where O is the orbit of ℓ_0', dv an invariant measure on O, and $\tilde{\varphi}(\ell')$ the Fourier transform of the function $\varphi(\ell)$, which corresponds on $\mathscr{L}$, through the exponential map, to $\psi^\sim \times \psi$ $(\psi^\sim(a) \equiv [\overline{\psi(a^{-1})}])$. The integral on the right-hand side converges absolutely.

Proof. Let $\{\ell_1, \ell_2, \ldots, \ell_m\}$ be a supplementary basis of H in $\mathscr{L}$; we write $g_j(t) = \exp(\ell_j t)$ $(j = 1, 2, \ldots, m;\ 2m = d)$. We set $K = \exp H$, and using the observations made before Lemma 1, we identify the homogeneous space Γ of the right classes of G according to K with the closed subset $\{g_1(t_1) \cdots g_m(t_m);\ (t_1, t_2, \ldots, t_m) \in R^m\}$ of G. Then every element of G can uniquely be written in the form $k\gamma$ $(k \in K, \gamma \in \Gamma)$. In particular, for any γ in Γ and a in G, we set $\gamma a = k(\gamma a)\, \gamma\bar{a}$ $(k(\gamma a) \in K, \gamma\bar{a} \in \Gamma)$. Let $d\gamma$ be any positive invariant measure on Γ $(dt_1\, dt_2 \cdots dt_m$, say). Then, if $\chi(\exp h) = \exp[i(h, \ell_0')]$ $(h \in H)$, the representation T can be realized in the Hilbert space $L^2(\Gamma, d\gamma)$, such that $T(a) f(\gamma) = \chi(k(\gamma a)) f(\gamma\bar{a})$. Let us write $\varphi(a) = (\psi^\sim \times \psi)(a)$ $(a \in G)$. We divide the proof in several steps.

(a) First, we show, that T_φ is implemented in $L^2(\Gamma, d\gamma)$ by a kernel, thus leading to an expression of $\mathrm{Tr}\,(T_\varphi)$ in terms of φ. If $f(\gamma)$ is continuous, and of compact support on Γ, we have

$$T_\varphi f(\gamma) = \int_G \varphi(a)\, T(a) f(\gamma)\, da = \int_G \varphi(a)\, \chi(k(\gamma a)) f(\gamma\bar{a})\, da$$

$$= \int_G \varphi(\gamma^{-1} a)\, \chi(k(a)) f(\overline{\gamma\gamma^{-1} a})\, da.$$

We denote by dk the invariant measure on K, satisfying

$$\int_G g(a)\, da = \int_{K\times\Gamma} g(k\gamma)\, dk\, d\gamma$$

for any function g, which is continuous and of compact support, on G. Then the previous relation gives

$$T_\varphi f(\gamma) = \int_{K\times\Gamma} \varphi(\gamma^{-1}k\gamma')\, \chi(k) f(\gamma\bar{a})\, dk\, d\gamma' = \int_\Gamma K_\varphi(\gamma, \gamma') f(\gamma')\, d\gamma'$$

$$(a = \gamma^{-1}k\gamma'),$$

where we have

$$K_\varphi(\gamma, \gamma') = \int_K \varphi(\gamma^{-1}k\gamma')\, \chi(k)\, dk.$$

Since, on account of our choice of φ, the operator T_φ is positive, we conclude, that the continuous function K_φ on $\Gamma \times \Gamma$ is positive-definite and therefore

$$\operatorname{Tr}(T_\varphi) = \int_\Gamma K_\varphi(\gamma, \gamma)\, d\gamma \tag{3}$$

irrespective of whether the expressions on both sides of this equation are finite or not (cf. [*4*], pp. 116-119).

(b) Let us put $\varphi(\ell) \equiv \varphi(\exp \ell)$ $(\ell \in \mathscr{L})$; the left-hand side is C^∞ and has a compact support on $\mathscr{L}$. We write again

$$\tilde{\varphi}(\ell') = \int_{\mathscr{L}} \varphi(\ell) \exp[i(\ell,\ell')]\, d\ell \qquad (\ell' \in \mathscr{L}')$$

and express $K_\varphi(\gamma, \gamma)$ by aid of $\tilde{\varphi}(\ell')$ as follows. We observe first, that choosing a linear measure dh on H in a suitable fashion, we have

$$K_\varphi(\gamma, \gamma) = \int_H \varphi(\sigma(\gamma^{-1})\, h) \exp[i(h, \ell_0')]\, dh, \tag{4}$$

where σ is the adjoint representation of G. Next, by fixing the volume element $d\ell'$ on $\mathscr{L}'$ in an appropriate manner, we obtain

$$\varphi(\ell) = \int_{\mathscr{L}'} \tilde{\varphi}(\ell') \exp[-i(\ell,\ell')]\, d\ell'.$$

In this case we shall call $d\ell'$ the measure, dual to $d\ell$. Choosing elements

γ and h in Γ and H, respectively, and replacing ℓ by $\sigma(\gamma^{-1})\, h$ in the above equation, we conclude, that

$$\varphi(\sigma(\gamma^{-1})\, h) = \int_{\mathscr{L}'} \tilde{\varphi}(\ell') \exp\left[-i(\sigma(\gamma^{-1})\, h, \ell')\right] d\ell'$$

$$= \int_{\mathscr{L}'} \tilde{\varphi}(\rho(\gamma^{-1})\,\ell') \exp\left[-i(h, \ell')\right] d\ell'.$$

Let us take the measure dh', dual to dh, on H' and let us form the measure $dh^{\perp}$, uniquely determined by $d\ell'$ and dh', by aid of which we have

$$\int_{\mathscr{L}'} g(\ell')\, d\ell' = \int_{H'} \left(\int_{H^{\perp}} g(\ell' + h^{\perp})\, dh^{\perp}\right) dh'.$$

Then the previous equation gives

$$\varphi(\sigma(\gamma^{-1})\, h) = \int_{H'} \left(\int_{H^{\perp}} \tilde{\varphi}(\rho(\gamma^{-1})\,(\ell' + h^{\perp}))\, dh^{\perp}\right) \exp\left[-i(h, h')\right] dh'.$$

In the exponential on the right-hand side, (h, h') stands for the canonical bilinear form on $H \times H'$, and it is understood that the class, containing ℓ', on $\mathscr{L}'$ modulo $H^{\perp}$ is h'. Substituting this in (4), and applying the Fourier inversion formula on H, we obtain

$$K_{\varphi}(\gamma, \gamma) = \int_{H^{\perp}} \tilde{\varphi}(\rho(\gamma^{-1})\,(\ell_0' + h^{\perp}))\, dh^{\perp}. \tag{5}$$

(c) Next we observe, that the orbit U of ℓ_0', with respect to the restriction of ρ to $K = \exp H$, coincides with $\ell_0' + H^{\perp}$ (cf. [*4*], p. 158). In fact, one verifies easily, using $\dim H = \dim \mathscr{L} - \frac{1}{2} d$, that U is open in $\ell_0' + H^{\perp}$, but since it is the orbit of a unipotent representation of the nilpotent group $K = \exp H$ (by virtue of the Remark after Lemma 1), U necessarily coincides with $\ell_0' + H^{\perp}$. Let us denote by S the stabilizer, with respect to ρ, of ℓ_0' in G. Its Lie algebra R is the radical of $B(\ell_1, \ell_2) = ([\ell_1, \ell_2], \ell_0')$ $(\ell_1, \ell_2 \in \mathscr{L})$. By virtue of our choice, H is maximal self-orthogonal with respect to B, hence it contains R (cf. [*4*], p. 157). Since, by Lemma 1, S is connected,[3] we have $S = \exp R \subseteq K$. Let us denote by Λ the homogeneous space of right classes of K according to S. The map ω, of Λ into $\ell_0' + H^{\perp}$, defined for the class $\bar{k}$ of k in K mod S by $\omega(\bar{k}) = \rho(k^{-1})\,\ell_0'$, is a bijection between Λ and $\ell_0' + H^{\perp}$. We denote by $d\lambda$ the inverse

[3] In fact, the orbit O in Lemma 1 is simply connected.

image, through ω, of $dh^{\perp}$. Then $d\lambda$ is an invariant measure on Λ. To see this, it suffices to observe, that, for any k in K, we have $\rho(k)\,\ell_0' = \ell_0' + h_0^{\perp}$, where $h_0^{\perp} \in H^{\perp}$, and therefore, if g is continuous of compact support on $\mathscr{L}'$, we obtain

$$\int_{H^{\perp}} g(\rho(k)\,(\ell_0' + h^{\perp}))\, dh^{\perp} = \int_{H} g(\ell_0' + h^{\perp} + \rho(k)\, h^{\perp})\, dh^{\perp}$$
$$= \int_{H^{\perp}} g(\ell_0' + h^{\perp})\, dh^{\perp}.$$

Using $d\lambda$, (5) can be rewritten as

$$K_{\varphi}(\gamma, \gamma) = \int_{\Lambda} \tilde{\varphi}(\rho((k\gamma)^{-1})\,\ell_0')\, d\lambda.$$

We recall, that $K_{\varphi}(\gamma, \gamma)$ is continuous and nonnegative on Γ. Using (3), we obtain finally

$$\mathrm{Tr}\,(T_{\varphi}) = \int_{\Gamma} \Big(\int_{\Lambda} \tilde{\varphi}(\rho(k\gamma)^{-1})\,\ell_0')\, d\lambda\Big)\, d\gamma.$$

(d) Observe, that $dv = d\lambda\, d\gamma$ is an invariant measure on G/S; we denote by the same symbol the corresponding measure on O. Using the notations of Lemma 1 we set $dz = dz_1\, dz_2 \cdots dz_d$. Taking into account the remarks made before Lemma 1 we see that dz, too, is an invariant measure on O, and therefore it differs only by a positive factor from dv. From this we conclude that, in order to prove Lemma 2, it suffices to show that the function $\tilde{\varphi}(z_1, z_2, \ldots, z_d)$, obtained from $\tilde{\varphi}(\ell')$ by replacing ℓ' through the parametrization of O, is summable with respect to the Lebesque measure on R^d. Since $\varphi(\ell)$ is C^{∞} and has a compact support, $\tilde{\varphi}(\ell')$ is rapidly decreasing, and therefore there exists a constant C, such that

$$|\,\tilde{\varphi}(z)\,| < C \Big(\sum_{j=1}^{n} (P_j(z))^2\Big)^{-d}$$

for all z in R^d. But by virtue of the properties of the polynomials $\{P_j\}$, described in Lemma 1, the right-hand side is certainly summable. We recall finally, that $\varphi(a) = (\psi^{\sim} \times \psi)\,(a)\ (a \in G)$, and therefore $T_{\varphi} = T_{\psi}^{*} T_{\psi}$.

Remark. Observe, that given a Haar measure da on G, and a translation-invariant measure $d\ell$ on $\mathscr{L}$, Lemma 2 yields the following algorithm to determine the canonical measure dv on O. Take first invariant measures dk and $d\gamma$ on K and Γ, respectively, such that $da = dk\, d\gamma$ holds. Next, take the measure dh, which is the inverse image of dk through the exponential map from H onto K, on H, and forming the dual measure $d\ell'$ of $d\ell$, determine the measure $dh^{\perp}$ on $H^{\perp}$, such that $d\ell' = dh'\, dh^{\perp}$ holds; here dh' is dual to dh. Denoting finally by $d\lambda$ the measure on $\Lambda = K/S$, which is the inverse image of $dh^{\perp}$ through the map $k \to \rho(k^{-1})\, \ell_0'$ of Λ onto $\ell_0' + H^{\perp}$, the canonical measure on O is obtained by transferring to it the measure $d\lambda\, d\gamma$ of G/S.

It is easy to verify directly, that the result is independent of the factorization $da = dk\, d\gamma$ with which we started.

THEOREM. *Let G be a connected and simply connected nilpotent Lie group with the Lie algebra $\mathscr{L}$. Let da be a Haar measure on G, and $d\ell$ a translation-invariant volume element on $\mathscr{L}$. We choose an element ℓ_0' in the dual $\mathscr{L}'$ of $\mathscr{L}$, such that the corresponding orbit O has a positive dimension. Then the ratio of the K-measure and of the canonical measure belonging to O depends on the dimension d of O only, and is given by $(d/2)!\, \pi^{d/2}\, 2^d$ divided by the ratio of da and $d\ell$ at the neutral element of $\mathscr{L}$.*

Proof. We choose a subalgebra H, subordinated to ℓ_0', and having a dimension $h = \dim \mathscr{L} - \frac{1}{2} \dim O$, of $\mathscr{L}$ (cf. [4], p. 159, Remarque 2). We form again $B(\ell_1, \ell_2) = ([\ell_1, \ell_2], \ell_0')$ $(\ell_1, \ell_2 \in \mathscr{L})$, denote its radical by R, and set $r = \dim R$. Then, if $d = 2m$, we have $m = n - h = h - r$. Let us choose a decreasing sequence of subalgebras $\mathscr{L}_n = \mathscr{L} \supset \mathscr{L}_{n-1} \supset \mathscr{L}_{n-2} \supset \cdots \supset \mathscr{L}_1 \supset \mathscr{L}_0 = (0)$, such that $\dim \mathscr{L}_j = j$, $\mathscr{L}_{n-m} = H$ and $\mathscr{L}_{n-d} = R$. We select for each $j = 1, 2, \ldots, n$ a nonzero element ℓ_j in $\mathscr{L}_j - \mathscr{L}_{j-1}$, and we write $g_j(t) = \exp(t\ell_j)$. Since the system $\{\ell_1, \ell_2, \ldots, \ell_n\}$ is a supplementary basis to the trivial subalgebra of $\mathscr{L}$, the canonical coordinates of the second kind $\{t_1, t_2, \ldots, t_n\}$ in $g = g_1(t_1)\, g_2(t_2) \cdots g_n(t_n)$ are valid globally on G, and $dt_1\, dt_2 \cdots dt_n$ defines a Haar measure da for G. Let us put $\ell = \sum_{j=1}^{n} y_j \ell_j$ and $d\ell = dy_1\, dy_2 \cdots dy_n$. The measure $d\ell$, when transferred to G by means of the exponential map, gives rise to a Haar measure ([4], p. 90), which, in the present case, coincides with da. In fact, in order to see this it suffices to observe that the Jacobian of the variables $\{t_k\}$, connected with the variables $\{y_j\}$ through

$$g_1(t_1) \cdots g_n(t_n) = \exp\left[\sum_{j=1}^{n} y_j \ell_j\right],$$

according to the latter has the value 1 at the origin of R^n. Observe, that it is enough to prove the above theorem for this choice of da and $d\ell$ respectively; in fact, one sees at once from the character formula (1) (cf. Section 1) that, if we change the ratio of da and $d\ell$ from 1 to c, the canonical measure on each orbit gets multiplied with the same constant. By virtue of our choice, the system $\{\ell_{r+1}, ..., \ell_n\}$ is a supplementary basis to R. Therefore, putting $T = (t_{r+1}, t_{r+2}, ..., t_n) \in R^d$ and $g(T) = g_{r+1}(t_{r+1}) \cdots g_n(t_n)$, we can conclude, that the map sending T into $\rho([g(T)]^{-1}) \ell_0'$, of R^d into O is a diffeomorphism, and that $dt = dt_{r+1}\, dt_{r+2} \cdots dt_n$ defines an invariant measure on O. To prove the theorem it suffices to establish, that if at ℓ_0', say, $c_1\, dt$ and $c_2\, dt$ correspond to the d-forms representing the K-measure and the canonical measure, respectively, on O, then we have $c_1/c_2 = (d/2)!\, \pi^d\, 2^d$. We do this in two steps.

(a) First we compute c_2 using the Remark following Lemma 2. Using the canonical coordinates of the second kind $\{t_j; j = 1, 2, ..., n\}$ introduced above, we observe, that $dk = dt_1\, dt_2 \cdots dt_h$ and $d\gamma = dt_{k+1} \cdots dt_n$ are invariant measures on $K = \exp H$ and on $\Gamma = G/K$ respectively, and, in addition, $da = dk\, d\gamma$. The measure dh on H, corresponding to dk, is $dh = dy_1\, dy_2 \cdots dy_h$. Let us choose a basis $\{\ell_j'\}$ satisfying $(\ell_i, \ell_j') = \delta_{ij}$ $(i, j = 1, 2, ..., n)$ in $\mathscr{L}'$. We write $\ell' = \sum_{j=1}^n y_j' \ell_j'$ and observe, that the measures $d\ell'$ and dh', which are dual to $d\ell$ and dh, respectively, on $\mathscr{L}'$ and H' are given by

$$d\ell' = (2\pi)^{-n}\, dy_1'\, dy_2' \cdots dy_n' \qquad \text{and} \qquad dh' = (2\pi)^{-h}\, dy_1'\, dy_2' \cdots dy_h'.$$

Therefore, the measure $dh^\perp$ on $H^\perp$ satisfying $d\ell' = dh'\, dh^\perp$ is the same as

$$dh^\perp = (2\pi)^{-m}\, dy_{h+1}'\, dy_{h+2}' \cdots dy_n'.$$

Let us write now

$$T = (t_{r+1}, t_{r+2}, ..., t_h) \in R^m \qquad \text{and} \qquad h(T) = g_{r+1}(t_{r+1}) \cdots g_h(t_h).$$

If

$$\rho([h(T)]^{-1}) \ell_0' = \sum_{j=h+1}^{n} y_j' \ell_j',$$

then c_2 is the same as $(2\pi)^{-m}$ times the value D of the Jacobian of the variables $\{y_j';\ j = h + 1, ..., n\}$ according to the variables

$\{t_j; j = r + 1,..., h\}$ at $T = 0$. Let us denote by a_{jk} the partial derivative of y'_j according to t_k at $T = 0$. Then we have

$$(ad\ell_k)'\ell'_0 = \sum_{j=h+1}^{n} a_{jk}\ell'_j,$$

and thus

$$a_{jk} = (\ell_j, (ad\ell_k)'\ell'_0) = B(\ell_k, \ell_j)$$

and

$$D = \det(B(\ell_{r+i}, \ell_{h+k}); i, k = 1, 2,..., m).$$

(b) To determine c_1, by virtue of the definition of the K-measure (cf. 1), we have to proceed as follows. We consider the 2-form

$$\omega = \sum_{i,j>0} B(\ell_{r+i}, \ell_{r+j})\, dt_{r+i} \wedge dt_{r+j}$$

at ℓ'_0, and form its mth exterior power; then this is the same as $c_1\, dt$. Thus, for the value of c_1, we obtain

$$m!\, 2^m[\det(B(\ell_{r+i}, \ell_{r+j}); i, , = 1, 2,..., d]^{1/2}.$$

Taking into account, that ℓ_{r+i} lies in H for $i = 1, 2,..., m$, and that H, being subordinated to ℓ'_0, is self-orthogonal with respect to B, we conclude, that $B(\ell_{r+i}, \ell_{r+j}) = 0$ for $i, j = 1, 2,..., m$. Therefore, the above determinant has the value D^2, and thus we obtain finally, that $c_1/c_2 = (2\pi)^m\, m!\, 2^m = (d/2)!\, \pi^{d/2}\, 2^d$, finishing the proof of the theorem.

4. The purpose of this concluding section is to obtain, by aid of the expression of the canonical measure found in the preceding section, an algorithm for the Plancherel measure.

In what follows we continue to denote by $\mathscr{L}$ a non-Abelian nilpotent Lie algebra and by G the corresponding connected and simply-connected group. For an element x in the dual $\mathscr{L}'$ of $\mathscr{L}$, we write $B_x(\ell_1, \ell_2) = ([\ell_1, \ell_2], x)$ $(\ell_1, \ell_2 \in \mathscr{L})$, and denote by $R(x)$ the radical of B_x.

Let $\mathscr{L} = \mathscr{L}_n \supset \mathscr{L}_{n-1} \supset \cdots \supset \mathscr{L}_0 = (0)$ be a Jordan-Hölder sequence for $\mathscr{L}$, such that $\dim \mathscr{L}_j = j$ $(j = 0, 1,..., n)$. We write $\mathscr{L}_j(x) = \mathscr{L}_j + R(x)$ and observe (cf. the proof of Lemma 1) that there exists a well-determined system of $d = \dim O(x)$ integers $1 \leqslant j_1 < j_2 < \cdots < j_d \leqslant n$ (we assume $d > 0$), such that $\dim \mathscr{L}_j(x) = \dim \mathscr{L}_{j-1}(x) + 1$ if and only if $j = j_k$ for some $k = 1, 2,..., d$; we write $f(x) = \{j_k\}$. The purpose of the following lemma is to prepare the proof of Lemma 4, which brings a proof, adopted to the needs of the present context, of Theorem, Section 5, Chap. 1, Part II of [*4*].

LEMMA. 3. *There exists a uniquely determined system f of integers between* 1 *and* $n = \dim \mathscr{L}$, *such that the set* $\mathcal{O} = \{x; f(x) = f\}$ *in* $\mathscr{L}'$ *is open in the Zariski topology.*

Proof. For each $j = 1, 2,..., n$ let ℓ_j be a nonzero element in $\mathscr{L}_j - \mathscr{L}_{j-1}$. We denote by $M_j(x)$ the matrix

$$\{B_x(\ell_i, \ell_k); 1 \leqslant i \leqslant j, k = 1, 2,..., n\}.$$

Then we have $\dim R(x) = n - \operatorname{rank} M_n(x)$, and therefore the set $\mathcal{O}_0 = \{x; \dim R(x) = r\}$ ($r = \min \dim R(x)$, $x \in \mathscr{L}'$) is open in the Zariski topology. Next we observe that, since

$$\dim \mathscr{L}_j(x) = \dim R(x) + \operatorname{rank} M_j(x),$$

writing

$$d_j = \sup \dim \mathscr{L}_j(x) \qquad (x \in \mathcal{O}_0)$$

the set

$$\mathcal{O}_j = \mathcal{O}_0 \cap \{x; \dim \mathscr{L}_j(x) = d_j\}$$

is open in the Zariski topology. Obviously $r = d_0 \leqslant d_1 \leqslant \cdots \leqslant d_n = n$; we denote by f the set of integers $1 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_d \leqslant n$, such that $d_j = d_{j-1} + 1$ if and only if $j \in f$. Let us write $\mathcal{O} = \bigcap_{j=1}^n \mathcal{O}_j$; we claim, that putting $E = \{x; f(x) = f\}$ we have $E = \mathcal{O}$. In fact, we observe first, that evidently $\mathcal{O} \subseteq E \subseteq \mathcal{O}_0$. Let p be an element in $E - \mathcal{O}$. Then there exists an index j such that $\dim \mathscr{L}_j(p) < d_j$. But this is impossible, since [by virtue of $f(p) = f$] we have

$$\dim \mathscr{L}_j(p) = \sum_{j_k \leqslant j} 1 + r = d_j$$

Thus $\mathcal{O} = \{x; f(x) = f\}$ is open in the Zariski topology and evidently, f is uniquely determined by this property.

Remark. 1. We observe that in general $\mathcal{O}$ is strictly contained in $\mathcal{O}_0$. In fact, let us consider the four dimensional nilpotent algebra $\mathscr{L}$

defined by a basis $\{e_j; j = 1, 2, 3, 4\}$ with the commutation relations $[e_4, e_3] = e_2$, $[e_4, e_2] = e_1$, all other brackets being zero. Then $\mathscr{L}_j = \{e_j, ..., e_1\}$ ($j = 1, 2, 3, 4$) is a Jordan-Hölder sequence. One can verify through a simple computation, that in coordinates with respect to a basis, dual to $\{e_i\}$, in $\mathscr{L}'$ we have $\mathscr{O}_0 = \{x; x_1^2 + x_2^2 \neq 0\}$ but $\mathscr{O} = \{x; x_1 \neq 0\}$.

Remark. 2. For an x in $\mathscr{L}'$ with $\dim O(x) > 0$ let us denote by T_p the tangent space, to the orbit $O(x)$, at p, and let us consider again the non-degenerate 2-form ω_p defined by

$$\omega_p(\varphi_p(\ell_1), \varphi_p(\ell_2)) = B_p(\ell_1, \ell_2) \qquad (p \in O(x); \ell_1, \ell_2 \in \mathscr{L})$$

(cf. Section 1). For each element t in T_p there exists a unique element ψt in its dual, such that $\omega_p(s, t) = (s, \psi t)$ for all s in T_p. We identify the exterior algebra $\Lambda(T_p)$ over T_p with that over its dual through ψ. Then ω_p gives rise to a scalar product on $\Lambda(T_p)$ such that, if

$$u = \prod_{k=1}^{r} \wedge u_k \qquad \text{and} \qquad v = \prod_{j=1}^{s} \wedge v_j \qquad (u_k, v_j \in T_p),$$

the scalar product $\omega_p(u, v)$ is 0 if $r \neq s$ and $\det\{\omega_p(u_i, v_k); 1 \leqslant i, k \leqslant r\}$ otherwise. We consider again the system f determined in Lemma 3, and for each $k = 1, 2, ..., d$ we select a nonzero element e_k in $\mathscr{L}_j - \mathscr{L}_{j-1}$ ($j = j_k$). Let us denote by E_p the d-vector

$$\prod_{k=1}^{d} \wedge \varphi_p(e_k) = \varphi_p\left(\prod_{k=1}^{d} \wedge e_k\right).$$

Then the function $Q(p)$ defined as $\omega_p(E_p, E_p)$ if $\dim O(p) > 0$ and zero otherwise is the square of a ρ-invariant polynomial function,[4] homogeneous of degree $d/2$, on $\mathscr{L}'$, and $Q(p) = 0$ if and only if p lies in $\mathscr{O}$, where $\mathscr{O}$ has the same meaning as in Lemma 3. In fact, we observe first, that we have $Q(p) = \det\{([e_i, e_j], p); 1 \leqslant i, j \leqslant d\}$ for all p in $\mathscr{L}'$ implying that $Q(p)$ is the square of a homogeneous polynomial function of degree $d/2$. As far as the invariance of $Q(p)$ is concerned, by virtue of the invariance of ω_p, it suffices to establish the ρ-invariance of E_p. We observe first that, if $q = \rho(a) p$, we have $R(q) = \sigma(a) R(p)$ and thus

$$\sigma(a) \mathscr{L}_j(p) = \sigma(a) (\mathscr{L}_j + R(p)) = \mathscr{L}_j + R(q) = \mathscr{L}_j(q),$$

[4] The ρ invariance of Q will not be used in the sequel.

which implies at once that $f(\rho(a)p) = f(p)$. Hence, in particular, $\mathcal{O}$ is invariant under ρ. Therefore it is enough to show, that $E_q = d\rho(a)|_p E_p$ if $q = \rho(a)p$ on $p \in \mathcal{O}$. But since

$$\varphi_q(\sigma(a)\ell) = d\rho(a)|_p \varphi_p(\ell)$$

for all ℓ in $\mathcal{L}$ (cf. Section 1), it suffices to verify, that φ_q maps $\prod_{k=1}^d \wedge e_k$ and $\prod_{k=1}^d \wedge \sigma(a) e_k$ onto the same element in $\Lambda^d(T_q)$. We have $\sigma(a) e_k - e_k \in \mathcal{L}_j \subset \mathcal{L}_j(q)$ $(j = j_k - 1)$, and therefore to obtain the desired conclusion it suffices to note, that by virtue of the choice of the system $\{e_k\}$, $e_i \in \mathcal{L}_j(q) - \mathcal{L}_{j-1}(q)$ $(j = j_i\,,\ i = 1, 2,..., d)$, and thus $\mathcal{L}_{j-1}$ is spanned by $R(q)$ and $\{e_i;\ 1 \leqslant i \leqslant k-1\}$. Since the vectors $\{e_k\}$ are independent mod $R(p)$ if $p \in \mathcal{O}$, we have in this case $Q(p) \neq 0$. On the other hand, if p lies outside the set

$$\mathcal{O}_0 = \{p;\ \dim O(p) = d\},$$

then clearly $Q(p) = 0$. Hence, to finish the proof of the statement made above, it is enough to show that we have $Q(p) = 0$ for all p in $\mathcal{O}_0 - \mathcal{O}$. With the notations of the previous lemma, let j be the smallest integer, for which $\dim \mathcal{L}_j(p) < d_j$. Then evidently $j = j_k$ for some $k = 1, 2,..., d$ and $\mathcal{L}_{j-1}(q) = \mathcal{L}_j(q)$ $(j = j_k)$, and thus the system $\{e_i;\ 1 \leqslant i \leqslant k\}$ is dependent modulo $R(p)$, implying $Q(p) = 0$.

Q.E.D.

LEMMA. 4. *Let $\mathcal{L}$ be a non-Abelian nilpotent Lie algebra, $\mathcal{L}_n = \mathcal{L} \supset \mathcal{L}_{n-1} \supset \cdots \supset \mathcal{L}_0 = (0)$ a Jordan-Hölder sequence in $\mathcal{L}$* (dim $\mathcal{L}_j = j$ *for* $j = 0, 1, 2,... n$) *and $\{\ell_j';\ j = 1, 2,..., n\}$ a basis in $\mathcal{L}'$ such that ℓ_j' is in $\mathcal{L}_{j-1}^{\perp} - \mathcal{L}_j^{\perp}$. Then there exists a nonconstant ρ-invariant polynomial function $Q(x)$ on $\mathcal{L}'$, a positive integer d, d indices $0 < j_1 < j_2 < \cdots < j_d \leqslant n$, and n functions $\{P_j(z, x);\ j = 1, 2,..., n\}$ on $R^d \times \mathcal{L}'$ $[z = (z_1, z_2,..., z_d) \in R^d]$ with the following properties:*

(1) *For each fixed x with $Q(x) \neq 0$, the functions $\{P_j(z; x)\}$ are polynomials in z, giving a parametrization, in the sense of Lemma 1, with $\{\ell_j'\}$ and $\{j_k\}$ (Lemma 1) as above, of the orbit $O(x)$ [hence, in particular,* $\dim O(x) = d$].

(2) *For each fixed z in R^d and $j = 1, 2,..., n$ the function $[Q(x)]^N P(z, x)$ is a polynomial function on $\mathcal{L}'$, where N is a sufficiently large positive integer.*

Proof. Let us choose a basis $\{\ell_i\}$ in $\mathcal{L}$, such that $(\ell_i, \ell_j') = \delta_{ij}$. Let B_x, $\mathcal{O}$ and f be as in Lemma 3. We write $e_k = \ell_j$ $(j = j_k;\ k = 1, 2,..., d)$, and for x in $\mathcal{O}$ we define $f_i(x) = \sum_{k=1}^d a_{ik}(x)\, e_k$ such that

$B_x(e_j, f_i(x)) = \delta_{ji}$. Let us write again, as in Remark 2 above, $Q(x) = \det\{B_x(e_i, e_j)\}$; then for any ℓ' in $\mathscr{L}'$, $Q(x)(f(x), \ell')$ is the restriction of a polynomial function in x to $\mathcal{O}$. To obtain Lemma 4, it suffices to repeat the reasoning of Lemma 1 for each x in $\mathcal{O}$, replacing $\{f_i\}$ in the lemma with $\{f_i(x)\}$ as determined above. Then $Q(x) = \det\{B_x(e_i, e_j)\}$ will satisfy the requirements since, by virtue of Remark 2, we have $x \in \mathcal{O}$ if and only if $Q(x) \neq 0$. Q.E.D.

Remark. 3. Let us denote by K the complement of f in the set of all integers between 1 and n. Then for any fixed z and k in K, $P_k(z, x)$ is of the form $x_k + Q_k(x_1, x_2, ..., x_{k-1})$ ($Q_1 \equiv$ const.), where $x_i = (\ell_i, x)$ ($i = 1, 2, ..., n$). In fact, let us denote by π the canonical projection of $\mathscr{L}'$ onto $\mathscr{L}'_k = \mathscr{L}'/\mathscr{L}_k^\perp$. Writing $O(\pi(x))$ for the orbit of $\pi(x)$ under $\pi \circ \rho$, we have $O(\pi(x)) = \pi(O(x))$. In order to be able to conclude, that $P_k(z, x)$ depends only on the class of x modulo $\mathscr{L}_k^\perp$, it suffices to take into account the following easily verifiable fact. Let τ be a unipotent representation of G on a real vector space V of dimension n [τ is said to be unipotent if $d\tau(\ell)$ is nilpotent for all ℓ in $\mathscr{L}$]. Suppose, that $V = V_n \supset V_{n-1} \supset \cdots \supset V_0 = (0)$ is a Jordan-Hölder sequence with respect to τ, and let us choose a basis $\{v_j\}$ in V such that $v_j \in V_j - V_{j-1}$ ($j = 1, 2, ..., n$). Then a parametrization, of the type as described in Lemma 1 with $\{v_j\}$ instead of $\{\ell_k'\}$, of any orbit of τ is uniquely determined. Finally we observe that, since $\pi(\ell_k')$ is invariant under $\pi \circ \rho$, we have, for any real a,

$$\pi(O(x + a\ell_k')) = O(\pi(x)) + a\pi(\ell_k'),$$

and therefore also $P_k(z, x + a\ell_k') \equiv P_k(z, x) + a$, finishing the proof of our statement.

Remark. 4. Let us write $\lambda_k(x) = P_k(0, x)$ ($k \in K$); then the system $\{\lambda_k(x)\}$ so obtained is functionally independent by virtue of Remark 3. Furthermore, it generates the field of all rational invariants of ρ. To see this, we observe first, that any rational invariant is the quotient of two invariant polynomial functions (cf. [*4*], p. 61). Let $P(x)$ be such a polynomial function, and let $Q(x_k)$ ($x_k = (\ell_k, x)$, $k \in K$) be its restriction to the hyperplane $\{x; (\ell_j, x) = 0, j \in f\}$. Then we have evidently $P(x) \equiv Q(\lambda_k(x))$ for all x.

Before passing to the description of the Plancherel measure, we make several preliminary observations. Let p be an element of $\mathcal{O}$; we continue to identify the tangent space T_p, to $O(p)$ at p, with its dual through ω_p, as this was done in Remark 2. We denote by ι the identity map from $O(p)$ into $\mathcal{O}$. Identifying the tangent space to the open submanifold $\mathcal{O}$ of $\mathscr{L}'$, at p with $\mathscr{L}'$, the differential $d\iota$ (at p) of ι maps T_p into $\mathscr{L}'$ and the transposed map, $\delta\iota$, coincides with $-\varphi_p$. In fact, we have, for all ℓ in $\mathscr{L}$, $d\iota\varphi_p(\ell) = -(ad\ell)'p$, and therefore

$$\begin{aligned}\omega_p(\varphi_p(k), \delta\iota(\ell)) = (\ell, d\iota\varphi_p(k)) &= -(\ell, (adk)'p) = -([k,\ell], p)\\ &= -\omega_p(\varphi_p(k), \varphi_p(\ell)) \qquad (k \in \mathscr{L}).\end{aligned}$$

Let us consider again the system $\{e_k; k = 1, 2,..., d\}$ and the function $Q(p)$ determined in Remark 2. We denote by $(Q(p))^{1/2}$ a polynomial function on $\mathscr{L}'$, of which $Q(p)$ is the square, and with which, putting $\eta_p = m!\,2^m(Q(p))^{-1/2} \prod_{k=1}^{d} \wedge e_k$ $(2m = d)$, we have $\delta\iota(\eta_p) = \omega_p{}^m$ $(p \in \mathcal{O})$. For each p, η_p is uniquely determined by this property modulo the ideal generated in $\Lambda(\mathscr{L})$ by $R(p)$.

Next we consider the quotient space $\Lambda = \mathcal{O}/\rho$. We identify it by means of the system $\{\lambda_k(x); k \in K\}$ of Remark 4 with the (Zariski) open set, which is the image of $\mathcal{O}$ in R^d. Thus, in particular, Λ gets equipped with the structure of a differentiable manifold of dimension $r = n - d$. Let π be the canonical map from $\mathcal{O}$ onto Λ. If p is a point of $\mathcal{O}$, $T_{\pi(p)}$ the tangent space of Λ at $\pi(p)$, and $T^*_{\pi(p)}$ its dual, we have $\delta\pi(T^*_{\pi(p)}) = R(p)$ and

$$0 \to T^*_{\pi(p)} \xrightarrow{\delta\varphi} \mathscr{L} \xrightarrow{\varphi_p} T_p \to 0.$$

Let e' be a nonzero element in $\Lambda^n(\mathscr{L}')$; we denote by $d\ell$ the corresponding volume element on $\mathscr{L}$. We choose a Haar measure da on G, such that the ratio of da and $d\ell$ at 0 should be 1. For λ in Λ we denote by O_λ, $T^{(\lambda)}$ and dv_λ the corresponding orbit, irreducible representation and canonical measure, respectively. Concerning dv_λ, it is understood, that it corresponds to the choice, just made, of $d\ell$ and da (cf. Section 1). Thus we have, according to (1) in Section 1, for all function φ, which is C^∞ and of compact support, on G:

$$\operatorname{Tr}(T_\varphi^{(\lambda)}) = \int_{O_\lambda} \tilde{\varphi}(\ell')\, dv_\lambda$$

To determine the Plancherel measure means to find a positive measure $d\mu$ on Λ, such that

$$\varphi(e) = \int_\Lambda \operatorname{Tr}(T_\varphi^{(\lambda)})\, d\mu$$

Let e be such in $\Lambda^n(\mathscr{L})$, that $(e, e') = (2\pi)^{-n}$ according to the duality between $\Lambda^n(\mathscr{L})$ and $\Lambda^n(\mathscr{L}')$. Then the volume element $d\ell'$, corresponding to e on $\mathscr{L}'$, will be dual to $d\ell$ in the sense, that

$$\varphi(0) = \int_{\mathscr{L}'} \tilde{\varphi}(\ell')\, d\ell'$$

holds. We recall, that according to the Theorem of Section 3, the ratio on any O_λ ($\lambda \in \Lambda$) of the K-measure and of the canonical measure, for the normalization of the latter as above, equals to $(d/2)!\, \pi^{d/2} \cdot 2^d$. Let us put $E = \prod_{k=1}^{d} \wedge e_k$ and observe, that for each p in $\mathcal{O}$ there exists a well-determined element ϵ_p in $\Lambda^r(R(p))$, such that $(Q(p))^{1/2} (2\pi)^{d/2} e = \epsilon_p \wedge E$. With respect to any basis in $\Lambda^r(\mathscr{L})$, ϵ_p, multiplied with $(Q(p))^{r-1/2}$, will have polynomial coefficients. Furthermore, ϵ_p is invariant (that is $\epsilon_q = \sigma(a)\, \epsilon_p$ if $q = \rho(a)\, p$), since for any a in G $\sigma(a)\, R(p) = R(\rho(a)\, p)$, $\sigma(a)\, E \equiv E$ modulo the ideal generated by $R(\rho(a)\, p)$ in $\Lambda(\mathscr{L})$ (cf. Remark 2), and $\det \sigma(a) \equiv 1$. By virtue of Lemma 4 and Remark 4 $\mathcal{O}$ is diffeomorphic to $\Lambda \times R^d$. *Therefore to obtain $d\mu$ it suffices to take the positive measure which corresponds to the r-form ϑ satisfying $\delta\pi(\vartheta) = \epsilon$, on Λ.*

Remark. 5. In terms of the parameters $\{\lambda_k;\, k \in K\}$ (cf. Remark 4) ϑ has the form $R(\lambda)\, (\prod_{k \in K} \wedge d\lambda_k)$, where R is a rational function, as it can easily be seen by what preceeds. It is in this form that the Plancherel theorem for nilpotent groups is stated in [*4*], Part II, Chap. III, Section 6.

We sum up the result in the following

PROPOSITION 3. *Let G be a connected and simply connected nilpotent Lie group of dimension n, with the Lie algebra $\mathscr{L}$. We denote by d the maximum of the dimension of the orbits of $\rho(a) = (Ad(a^{-1}))'$, $(a \in G)$ in $\mathscr{L}'$ and assume, that $d > 0$. Then we can select d elements $\{e_k;\, k = 1, 2,\ldots, d\}$ in $\mathscr{L}$ in the following fashion:* (1) *if* $Q(x) = \det \{([e_i, e_j], x);\, 1 \leqslant i, j \leqslant d\}$, *$Q$ is ρ invariant and $Q \not\equiv 0$;* (2) *setting $\mathcal{O} = \{x; Q(x) \neq 0\}$, $E = \prod_{k=1}^{d} \wedge e_k$ and fixing appropriately an n-vector $e \in \Lambda^n(\mathscr{L})$, we have, with the positive measure $d\mu$ corresponding on $\Lambda = \mathcal{O}/\rho$ to the unique $r = n - d$ form ϵ on $\mathcal{O}$ satisfying $(Q(p))^{1/2}\, e = \epsilon_p \wedge E$ and $\epsilon_p \in \Lambda^r(R(p))$ ($R(p) =$ radical of $B_p(\ell_1, \ell_2) = ([\ell_1, \ell_2], p)$, $\ell_1, \ell_2 \in \mathscr{L}$):*

$$\varphi(e) = \int_{\Lambda} \mathrm{Tr}\,(T_\varphi^{(\lambda)})\, d\mu$$

for any function φ, which is C^∞ and of a compact support on G.

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