SOLUBLE NORMAL SUBGROUPS OF SKEW LINEAR GROUPS

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A skew linear group is a subgroup of GL(n,D) for some positive integer n and division ring D. In [13] and [14] we studied a locally finite normal subgroup H of a skew linear group G and found in particular that the structure of G/C_G(H) is very restricted. Here we consider corresponding questions concerning soluble normal subgroups of G. The results are similar to those of [13] and [14], but the conclusions are slightly weaker, necessarily so as we show with examples.

Let F be the centre of the division ring D. We say that the subgroup G of GL(n,D) is absolutely irreducible if the subring F[G] of the matrix ring D^{n×n} generated by F and G is the full matrix ring. See [13] for the background concerning this concept.

The locally finite results of [13] and the soluble results here can be combined, and it is in this form that we state our conclusions. Thus we consider the class P(\mathfrak{A} U L^\infty) of all groups with a series of finite length whose factors are abelian or locally finite. (In general we use Hall’s calculus of group classes as expounded in [4, Chapter 1] except that we use Q in place of H.) Our main theorem is the following.

Theorem A. Let G be an absolutely irreducible skew linear group and H a normal P(\mathfrak{A} U L^\infty)-subgroup of G.

(i) H contains an abelian normal subgroup A of G with H/A locally finite and G/C_G(H) is abelian by periodic.

(ii) If H modulo its centre is periodic, then G/C_G(H) is periodic.

We do not consider whether or not any of the periodic images of G in the conclusions of Theorem A are locally finite (cf. [13]). We do show at least that each of them (and also H/A in part (i), a fact pointed out by Snider [5] in a somewhat different context) can be any locally finite group. We also show that the group G/HC_G(H) in part (i) need not be periodic and indeed can contain free abelian subgroups of uncountable rank. This again contrasts with the results of [13]. The involved example which shows this is 3.12 below.

If G is merely an irreducible subgroup of GL(n,D) meaning that row n-space over D is irreducible as D-G-bimodule, then no theorem like Theorem A exists. Many
suitable examples exist in the literature in other contexts. For example, by [10, 1.2], if $K$ is any $(L,P)\mathbb{Z}$-group, for example if $K$ is poly (torsion-free abelian), and if $G$ is the split extension $H \cdot K$ of the group ring $H=\mathbb{Z}K$ by $K$, then $G$ is isomorphic to a (necessarily irreducible) subgroup of $D^*$ for some division ring $D$. Of course, $H$ here is abelian and $G/C_G(H)$ is isomorphic to $K$. With minor modifications to the argument one can replace $K$ by an ordered group or by a group containing $K$ with an arbitrarily locally finite image.

Part (ii) of Theorem A is an important step in the proof of part (i) and in one case part (ii) reduces to the consideration of locally finite-dimensional algebras. Here, not surprisingly, we can weaken substantially the hypotheses. As in [12] let $\mathcal{X}$ denote the class of all groups with a local system of finitely generated subgroups, all of whose finite homomorphic images are soluble. Also $L_1\mathcal{X}$ denotes the class of periodic groups and $\mathcal{P}$ the ascendent series operator.

Whenever a skew linear group $G$ has a unique maximal unipotent normal subgroup, as it does if the ground division ring is locally finite-dimensional, we denote it by $u(G)$. For any group $G$ set

$$\Delta(G) = \{ g \in G : (G : C_G(g)) < \infty \}.$$

**Theorem B.** Let $G$ be a subgroup of $GL(n,D)$, where $D$ is a locally finite-dimensional division algebra and let $H$ be a normal $\mathcal{P}(\mathcal{X} \cup L_1\mathcal{X})$-subgroup of $G$ with $u(H) = \langle 1 \rangle$.

(i) $H$ contains an abelian normal subgroup $A$ of $G$ such that $H/A$ and $G/AC_G(H)$ are locally finite. In particular, $G/C_G(H)$ is abelian by locally-finite.

(ii) If $H$ modulo its centre $Z$ is periodic, then $G/C_G(H)$ is locally finite.

(iii) If $H$ is locally nilpotent, then $H$ is centre by locally-finite.

(iv) If $H$ is a Baer group (e.g., if $G$ is a Fitting group), then $H \leq \Delta(G)$.

(v) If $H$ is periodic, then $G/C_G(H)$ is locally finite.

In part (v) much more information about $G/C_G(H)$ is available, but it is complicated to state, see the proof 2.3(v) below; but for example, if char $D = 0$, then $G/C_G(H)$ is metabelian by finite. We have not included the case where $H$ is locally finite in Theorem A since this is fully covered in [13] and [14]. For the connection between the hypotheses of Theorems A and B see 2.4 below.

In the situation of Theorem A we need not have $H \leq \Delta(G)$, even if $H$ is abelian. This follows from Example 3.12 below. Also we need not have $H \leq \Delta(G)$ in Theorem B even if $H$ is locally nilpotent, locally finite and metabelian and $G$ is linear. For example if $q$ is a prime let $G$ be the wreath product $W$ of a Prüfer $q^{\infty}$-group by a cyclic group of order $q$ and set $H=G$. Note that $W$ has a faithful absolutely irreducible linear representation of degree $q$ in any characteristic except $q$.

This leads us to the linear case upon which, of course, Theorem B depends. The following pieces together results from various sources. We state it mainly for comparison with the above. The phrase 'bounded by a function of $n$ only' we shorten to 'n-bounded'.
Theorem C. Let $G$ be a linear group of degree $n$ over the field $F$ and let $H$ be a normal $P(\mathfrak{U}L_1\mathfrak{F})$-subgroup of $G$ with $u(H) = \langle 1 \rangle$.

(i) $H$ contains an abelian normal subgroup $A$ of $G$ with $H/A$ and $G/AC_G(H)$ locally finite such that $G/HC_G(H)$ has an abelian normal subgroup with finite $n$-bounded index and exponent dividing $n!$.

(ii) If the centre $Z$ of $H$ has finite index $m$ in $H$, then $(G: C_G(H))$ divides $m!n!m^n$.

(iii) If $H$ is nilpotent of class $c$, then $(G: C_G(H))$ is finite and $(n,c)$-bounded.

(iv) If $H$ is periodic, then $(G: HC_G(H))$ is finite and $n$-bounded.

We have already remarked above that in Theorem C, parts (i) and (iv) we need not have $H \triangleleft A(G)$. The same example, $G = H = W$, has $G/C_G(H)$ infinite. Also $G/HC_G(H)$ can be infinite. For let $A$ be any non-trivial torsion-free abelian group, let $i$ be the inversion automorphism of $A$, let $G$ be the split extension $\langle i \rangle[A$ and set $H = \langle i \rangle A^2$. Since $i^a = ia^2$ for all $a \in A$ the subgroup $H$ is normal in $G$. Also $C_G(A^2) = A$ and $C_A(i) = \langle 1 \rangle$, so $C_G(H) = \langle 1 \rangle$ and $G/HC_G(H)$ is isomorphic to $A/A^2$. Finally, $G$ has a faithful absolutely irreducible linear representation of degree 2 in any characteristic.

In Theorem C, part (i) we clearly cannot in general choose $A$ so that $H/A$ has finite exponent, for $H$ could be say $\text{PSL}(2, k)$ for any infinite locally infinite field $k$. However, if $H$ is soluble or if the characteristic is zero, then $A$ can be chosen with $(H: A)$ finite and $n$-bounded. This is essentially Mal'cev's and Jordan's theorems [6, 3.5 and 9.4]. No such conclusion is possible in the situation of Theorem B (or of Theorem A). For a start it is well known (see [15, Section 2]) that there exist nilpotent multiplicative subgroups $H$ of class 2 of a locally finite-dimensional division algebra such that $H^m$ is not abelian for every positive integer $m$; for example, $H$ could be the direct product over $p$ of groups $H_p$, where $p$ ranges over infinitely many primes and $H_p$ is the split extension of a cyclic $p$-group of order at least $p^2$ by an infinite cyclic group acting as an automorphism of order $p$. Using the techniques of [8, Section 3] a whole range of examples $G$ can be constructed with $G/HC_G(H)$ not of finite exponent, even ones with $H$ abelian and $G$ soluble.

Our notation is mostly standard. If $G$ is a group acting on a set $X$ and $Y$ is a subset of $X$, then

$$C_X(G) = \{x \in X: x^g = x \text{ for all } g \in G\},$$

$$C_G(Y) = \{g \in G: y^g = y \text{ for all } y \in Y\},$$

$$N_G(Y) = \{g \in G: Y^g = Y\},$$

$$\Delta_X(G) = \{x \in X: (G: C_G(x)) < \infty\}.$$ 

If $R$ is a ring, $S$ a subring of $R$ and $A$ a subset of $R$, then $S[A]$ denotes the subring of $R$ generated by $S$ and $A$,

$$r_R(A) = \{r \in R: Ar = \{0\}\} \text{ and } l_R(A) = \{r \in R: rA = \{0\}\}.$$ 

We use the latter notation even if $R$ is commutative.
1. The proof of Theorem C

(i) $H$ has a unique maximal soluble normal subgroup $S$ say. By Theorem 3 of [12] the group $H/S$ is locally finite. Now $S$ contains a triangularizable normal subgroup $A$ of $G$ such that $(S:A)$ is $n$-bounded (cf. [6, exercise 3.1]). Since $u(A) \leq u(H) = \langle 1 \rangle$, the group $A$ is abelian.

Now $u(G) \cap H = u(H) = \langle 1 \rangle$ and $u(G) \leq C_G(H)$. Using [6, p. 2] we may pass to $G/u(G)$ and assume that $G$ is absolutely completely reducible. Set $K = C_H(A)$. Note that $(G : C_G(A))$ and hence $(H : K)$ divides $n!$ by [6, 1.12 and 1.8]. Clearly, we may assume that $A$ is the centre of $K$. Then $G/A$ has a linear representation $\rho$ of degree $n^2$ that if faithful on $K/A$ (let $G$ act on $F[K] \leq F^{n \times n}$ by conjugation). If char $F = 0$, then $(K : A)$ is $n$-bounded by Jordan's theorem (applied to $K/A$) and the choice of $A$. Consequently [13, 1.10] applied to $G/K$ yields that $(G : KC_G(K/A))$ is $n$-bounded in all cases. Therefore so is $(G : KC_G(A \cup K/A))$. By standard stability theory $C_G(A \cup K/A)/C_G(K)$ is isomorphic to a subgroup of $\text{Hom}(K/A, A)$, and the torsion subgroup of $A$ has rank at most $n$. If char $F = 0$, it follows that $(G : KC_G(K))$ is $n$-bounded.

Suppose char $F > 0$. By 1.2 and 1.7 of [13] there are characteristic subgroups $A \leq B \leq C$ of $K$ with $(B : A)$ and $(K : C)$ $n$-bounded and $C/B$ a direct product of perfect simple groups. Thus

$$|\text{Hom}(K/A, A)| \leq (K : C)^n(B : A)^n$$

and so in this case too $(G : KC_G(K))$ is $n$-bounded. Since $(H : K)$ divides $n!$ it follows that $(G : KC_G(KUH/K))$ is $n$-bounded. By stability theory there is an embedding of $C_G(KUH/K) / AC_G(H)$ into the abelian group $H^1(H/K, A)$, which has finite exponent dividing $(H : K)$. Part (i) follows.

(ii) Here $(G : C_G(Z))$ divides $n!$ and so $(G : C_G(ZUH/Z))$ divides $m!n!$. Finally $C_G(ZUH/Z)/C_G(H)$ embeds into $\text{Hom}(H/Z, Z)$ and the latter has order dividing $m^n$. Part (ii) follows.

(iii) This follows at once from [11, 3.3]. It also follows from part (ii) and [6, 3.13].

(iv) This is the proposition of [13].

2. The locally finite-dimensional case

2.1. Let $G$ be a subgroup of $\text{GL}(n, R)$, where $R$ is a finitely generated integral domain, such that $u(G) = \langle 1 \rangle$ and suppose that $G$ has an abelian normal subgroup $A$ with $G/A$ periodic. Then $G$ is abelian by finite. If $A$ is central in $G$, then the centre $Z$ of $G$ has finite index in $G$.

Proof. Firstly $G/A$ is locally finite by [6, 4.9 (and 6.4, 5.9 and 5.11)]. Let $C = C_G(A)$. Then $C'$ is locally finite by Schur's theorem (e.g. [6, lemma p. 213]) and by [6, 4.8] there is a normal subgroup $N$ of $G$ of finite index whose only torsion
elements are unipotent. Let \( B = C \cap N \). Then \([B, C] \leq C' \cap N \leq u(G) = \langle 1 \rangle\) and so \( B \) is abelian. It follows from [6, p. 2 and 1.12] that \((G : C)\) divides \( n!\) and therefore \( B \) has finite index in \( G \). If \( A \leq Z \), then \( C = G \) and \([B, G] = \langle 1 \rangle\). That is \( B \leq Z \).

2.2. Let \( G \) be a periodic group and let \( A \) be a \( G \)-module that is finitely generated as a \( Z \)-module. Then \( H^1(G, A) \) is periodic.

**Proof.** Periodic subgroups of \( GL(n, Z) \) are finite [6, 4.8]. Hence \( C = C_G(A) \) has finite index \( m \) say in \( G \). Now \( H^1(C, A) \equiv \text{Hom}(C, A) \) is certainly periodic. Also the image of corestriction of \( H^1(C, A) \) in \( H^1(G, A) \) contains \( mH^1(G, A) \). The result follows.

2.3. The Proof of Theorem B. (i) Let \( F \) be the centre of \( D \). If \( X \) is any finitely generated subgroup of \( G \), then \( \dim_F F[X] \) is finite and \( X \) carries a Zariski topology over \( F \). If \( Y \) is a subgroup of \( X \), then \( Y^0 \) denotes the connected component of the identity of \( Y \) in the induced topology. It does not depend on the choice of \( X \). As in [7] we set \( G^+ = \bigcup X^0 \), where \( X \) runs over all finitely generated subgroups of \( G \). Then \( G^+ \) is a normal subgroup of \( G \) with \( G/G^+ \) locally finite. We need a relativised version of this construction. Set \( A = \bigcup (H \cap X)^0 \) where \( X \) is as before. Then \( A \) is a normal subgroup of \( G \) with \( A \leq H \cap G^+ \) and \( H/A \) locally finite. (Always \( H^+ = A \) but we need not have equality.)

Let \( K \) be any \( \mathcal{P}(X \cup L_1 \mathcal{R}) \)-subgroup of \( GL(n, D) \). By the linear case and a simple transfinite induction \( K \) is locally soluble-by-finite and so \( K^+ \) is locally soluble by [7, 5.4]. Thus \( H \) is abelian by locally-finite by [7, 2.6]. Let \( g \in A' \). There exists a finitely generated subgroup \( X \) of \( G \) such that \( g \) lies in the derived group of \( Y = (H \cap X)^0 \). By Theorem C, part (i) and 2.1 the group \( Y/u(Y) \) is abelian. Thus \( g \) is unipotent and \( A' \leq u(H) = \langle 1 \rangle \). That is \( A \) is abelian.

Let \( a \in A \), \( h \in H \) and \( g \in G^+ \). There exists a finitely generated subgroup \( X \) of \( G \) with \( a \) and \( h \) in \( X \) and \( g \) in \( X^0 \). Now \( u(A \cap X) \leq u(A) = \langle 1 \rangle \). Hence by part (iii) of Theorem C we have that \( [A \cap X, X^0] = \langle 1 \rangle \). Thus \([a, g] = 1\) and so \([A, G^+] = \langle 1 \rangle \). Also \((H \cap X)^0 \leq A \cap X \) by the definition of \( A \) and consequently \( A \cap X \) is closed in \( H \cap X \). Therefore \([H \cap X, X^0] \leq A \). It follows that \([h, g] \in A \) and that \([H, G^+] \leq A \).

Again let \( g \in G^+ \). Then \([H, g] \leq A \) and \( A \langle g \rangle \) is an abelian normal subgroup of \( H \langle g \rangle \). Let \( U = u(A \langle g \rangle) \). By [8, 1.1] and [7, 3.1] we have that \( gU \) lies in \( A(H \langle g \rangle)/u \) and so \([H, g]U/U \) is finitely generated. But \( H \cap U = \langle 1 \rangle \) and therefore \( A_g = [H, g] \) is a finitely generated subgroup of \( A \). Also \( A_g \) is normal in \( H \). By 2.2 the group \( H^1(H/A, A_g) \) is periodic. Now \( hA \rightarrow [h, g] \) is a derivation of \( H/A \) into \( A_g \) and so its \( r \)-th power for some positive integer \( r \) is inner. Thus far some \( a \in A_g \) we have

\[
[h, g^r] = [h, g]^r = [h, a]
\]

for every \( h \) in \( H \).

Thus \( g' \in aC_G(H) \) and so \( G^+/AC_{G^*}(H) \) is periodic. Since \( G^+ \) stabilizes the series \( \langle 1 \rangle \leq A \leq H \), the group \( G^+/C_{G^*}(H) \) is abelian. It follows that \( G/AC_{G^*}(H) \) is locally finite.
(ii) We prove that $A \leq Z$. It will then follow from (i) that $G/C_G(H)$ is locally finite. Let $a \in A$ and $h \in H$. There is a finitely generated subgroup $X$ of $G$ with $a \in (H \cap X)^0$ and $h \in X$. Now $(H \cap X)/u(H \cap X)$ is centre by finite by 2.1 and so $[a, h]$ is unipotent. Thus $[A, H] \leq u(H) = \langle 1 \rangle$ and $A \leq Z$ as claimed.

(iii) This follows from [7, 2.7].

(iv) By [7, 1.2d] the group $H$ is a central product of nilpotent groups $H_i$. A glance at the proof [7, 4.2] shows that the $H_i$ can be chosen to be characteristic in $H$ and so normal in $G$. (In this proof the $H_i$ are the primary components of $H$ modulo its centre.) By [11, 3.1] we have that $H_i \leq A(G)$ for each $i$. Part (iv) follows.

(v) We may pass to $G/u(G)$ via [7, 3.1] and assume that $G$ is completely reducible subgroup of $\text{GL}(n, D)$. By point 4 of [9] the $F$-algebra $R = F[G] \leq D^{n \times n}$ is semisimple Artinian. Let $R = \bigoplus R_i$, where each $R_i$ is simple and let $\pi_i : R \to R_i$ be the natural projection. Then the structure of $G/C_G(H \pi_i)$ is given by the results of [13] and [14]. In particular each $G/C_G(H \pi_i)$ is locally finite. Clearly $C_G(H) = \bigcap_i C_G(H \pi_i)$ and the proof of Theorem B is complete.

Part of the proof of (v) above shows the following, which gives the connection between the basic hypotheses of Theorems A and B.

2.4. Let $G$ be a skew linear group over a locally finite-dimensional division algebra over the field $F$ such that $u(G) = \langle 1 \rangle$ (e.g., if $G$ is completely reducible). Then $G$ is a subdirect product of a finite number of absolutely irreducible skew linear groups over locally finite-dimensional division $F$-algebras.

3. Absolutely irreducible groups

Throughout this section $n$ is a positive integer and $D$ is a division ring with centre $F$.

3.1. Let $A$ be an abelian normal subgroup of the absolutely irreducible subgroup $G$ of $\text{GL}(n, D)$. Then $G/C_G(A)$ is periodic.

Proof. Choose if possible a counterexample $(D, G, A)$ with $n$ minimal. Set $J = F[A] \leq D^{n \times n}$. By [13, 2.5] the ring $J$ is semiprime and certainly $J$ has the maximal condition on annihilators. Thus $J$ has only a finite number $r$ of minimal prime ideals $p_i$ and $\bigcap p_i = \{0\}$, see [1, 1.16]. Also each $p_i$ is an annihilator ideal of $J$. Clearly $G$ permutes the $p_i$; let $Y = \bigcap N_G(p_i)$. Then $(G : Y)$ is finite and $T = F[Y]$ is Artinian by [3, point 4] and semiprime by [13, 2.5]. Thus $T$ is semisimple, say $T = \bigoplus_{i=1}^r T_i$ where the $T_i$ are simple Artinian of degree $n_i$ say. Since $D^{n \times n}$ can contain at most $n$ pairwise orthogonal idempotents $\sum n_i \leq n$. Let $\pi_i : T \to T_i$ be the natural projection.
Suppose that \( s > 1 \). Then each \( n_i < n \) and by the choice of \( n \) each \( Y/C_y(A \pi_i) \) is periodic. But \( C_y(A) = \bigcap_i C_y(A \pi_i) \). Therefore \( Y/C_y(A) \) and hence \( G/C_G(A) \) is periodic. This contradiction shows that \( s = 1 \) and \( T \) is simple. Now \( Y \) is an ideal of \( T \) with \( l_i(p_i) \cdot p_i Y = \{ 0 \} \) and \( p_i = r_l l_j(p_i) \). Hence \( r = 1 \) and \( J \) is domain. Then \( C = J \setminus \{ 0 \} \) is a right divisor set of \( J \) and so also of \( T \) and \( \bigcup_{c \in C} r_T(c) \) is an ideal of \( T \). As \( T \) is simple the ideal is \( \{ 0 \} \); that is, \( T \) is torsion-free as right \( J \)-module. By [13, 2.1] the ring \( T \) is a crossed product of \( C_T(A) \) by \( G/C_G(A) \) and consequently \( G/C_G(A) \) is periodic by [13, 2.2]. This contradicts the assumption that we chose a counterexample and completes the proof of 3.1.

Let \( S_f \) be the subgroup-of-finite-index operator and let \( \mathcal{S} \) be a class of groups satisfying \( QS_f \mathcal{S} = \mathcal{S} \). The example we have in mind is \( \mathcal{S} = P(\mathfrak{U} \cup L \mathfrak{F}) \).

3.2. Let \( H \) be a normal \( \mathcal{S} \)-subgroup of the absolutely irreducible subgroup \( G \) of \( GL(n, D) \). Suppose that \( H \) does not contain an abelian normal subgroup of \( G \) with \( H/A \) locally finite and that \( D, G \) and \( H \) have been chosen so that \( n \) is minimal. If \( N \) is any normal subgroup of \( G \), then the subring \( F[N] \) of \( D^{n \times n} \) is prime.

**Proof.** The ring \( S = F[N] \) is certainly semiprime [13, 2.5], so assume that \( S \) is not prime. \( S \) has the maximal condition on annihilators and by [1, 1.16] again has only a finite number \( r > 1 \) of minimal prime ideals \( p_i \), \( \bigcap p_i = \{ 0 \} \) and \( r_5 l_5(p_i) = p_i \). Set \( Y = \bigcap_i N_G(p_i) \). Then \( Y \) is a normal subgroup of \( G \) of finite index containing \( N \); also \( L_i = l_5(p_i) \neq \{ 0 \} \) and \( l_i p_i Y = \{ 0 \} \), so \( p_i Y \) is a proper ideal of \( T = F[Y] \) and \( T \) is not simple.

\( T \) is semisimple Artinian ([3] and [13, 2.5] again). Let \( T = \bigoplus_{i=1}^s T_i \) where \( T_i \) is simple Artinian of degree \( n_i \). Then \( s > 1 \) and \( \sum n_i \leq n \) and so each \( n_i \leq n \). Let \( \pi_i : T \to T_i \) be the natural projection.

Now \( (H \cap Y) \pi_i \in QS_f \mathcal{S} = \mathcal{S} \), so by the minimality of \( n \) there are normal subgroups \( A_i \) of \( Y \) with \( (H \cap Y)/A_i \) locally finite and \( A_i \pi_i \) abelian. Then \( A_0 = \bigcap_i A_i \) is an abelian normal subgroup of \( Y \) with \( A_0 \leq H \cap Y \) and \( (H \cap Y)/A_0 \) locally finite. Since \( (G : Y) \) is finite, \( A = \bigcap_{g \in G} A_0^g \) is an abelian normal subgroup of \( G \) with \( A \leq H \) and \( H/A \) locally finite. This contradiction proves that \( S \) is prime.

3.3. Let \( H \) be a normal subgroup of the absolutely irreducible subgroup \( G \) of \( GL(n, D) \) such that \( H \) is centre by locally-finite and \( G/C_G(H) \) is not periodic. Suppose that \( D, G \) and \( H \) have been chosen so that \( n \) is minimal. If \( N \) is any normal subgroup of \( G \), then the subring \( F[N] \) is prime.

**Proof.** Repeat the first two paragraphs of the proof of 3.2. Let \( K = H \cap Y \). Then by the minimality of \( n \) each \( Y/C_y(K \pi_i) \) is periodic. Hence \( Y/C_y(K) \) is periodic and therefore so too is \( Y/C_y(K \cup H/K) \).

Let \( A \) be the centre of \( H \). By hypothesis \( H/A \) is locally finite and \( G/C_G(A) \) is periodic by 3.1. Schur's theorem yields that \( H' \) is locally finite, so \( H \) has a character-
istic locally finite subgroup $L$ with $H/L$ torsion-free abelian. The isolator of $AL/L$ in $H/L$ is $H/L$. Thus $[H, C_G(A)] \leq L$. Set $C = C_G(KUH/KUA)$ and let $B$ denote the centre of $K$. Then $G/C$ is periodic and by stability theory $C/C_G(H)$ is isomorphic to a subgroup of $\text{Der}(H/K, B \cap L)$. The latter is periodic abelian since $H/K$ is finite and $B \cap L$ periodic. Consequently $G/C_G(H)$ is periodic, a contradiction that completes the proof.

3.4. Let $G$ be a $P(\mathfrak{A} \cup L \mathfrak{B})$ group. Then $G$ has a characteristic series of finite length whose factors are abelian or locally finite.

Proof. Certainly $G$ has a characteristic series of finite length whose factors are either torsion-free locally nilpotent or locally finite. Hence we may assume that $G$ is torsion-free and locally nilpotent.

$G$ has a series $1 = G_0 \leq G_1 \leq \cdots \leq G_r = G$ where the $G_{2i+1}/G_{2i}$ are abelian and the $G_{2i}/G_{2i-1}$ are locally finite. We prove by induction on $r$ that $G$ is soluble of derived length at most $r$. The result will then follow.

By induction applied to $G_{2r-2}$ the subgroup $H = G_{2r-1}$ is soluble of derived length at most $r$. Let $X$ be any finitely generated subgroup of $G$. Regard $X$ as a unipotent linear group over $\mathbb{C}$. The Zariski closure $Y$ of $H \cap X$ in $X$ is soluble of derived length at most $r$ and $X/Y$ is finite and isomorphic to a unipotent linear group over $\mathbb{C}$ (use [6, 5.11, 5.9 and 6.6]). Therefore $X = Y$ and $G$ is soluble of derived length at most $r$ as required.

3.5. Let $H$ be a normal $P(\mathfrak{A} \cup L \mathfrak{B})$-subgroup of the absolutely irreducible subgroup $G$ of $\text{GL}(n, \mathbb{D})$. Then $H$ contains an abelian normal subgroup $A$ of $G$ with $H/A$ locally finite.

Proof. Choose if possible a counterexample $D, G, H$ with $n$ minimal. By 3.4 there is a characteristic series of $H$ of finite length whose factors are abelian or locally finite. Choose $D, G, H$ and $n$ as above and with this series of minimal length. Then $H$ contains normal subgroups.

$\langle 1 \rangle \leq C \leq K \leq B \leq H$

of $G$ with $C$ and $B/K$ abelian and $K/C$ and $H/B$ locally finite.

By 3.1 the group $G/C_G(C)$ is periodic. Replace $K$ and $B$ by $C_K(C)$ and $C_B(C)$. Thus we may assume that $C$ is central in $B$. By Schur’s theorem $K'$ is locally finite, so by the theorem of [13] the group $G/C_G(K')$ is locally finite. Replace $B$ by $C_B(K')$. We have now reduced the problem to the case where $B$ is soluble.

By 3.2 and [12, 4.3.3] there is a characteristic subgroup $E$ of $B$ such that $G/\Delta_G(E)$ is periodic and $\Delta_B(E) \leq E$. In particular, $H/\Delta_B(E)$ is locally finite. But $\Delta_B(E) \leq E$ is an $FC$-group and so is locally finite modulo its centre $A$. Then $A$ is an abelian normal subgroup of $G$ with $A \leq H$ and $H/A$ locally finite.
3.6. Let $G$ and $H$ be as in 3.5. Then $S = F[H]$ is a semiprime Goldie subring of $R = D^{n \times n}$ whose (classical) ring of quotients embeds naturally into $R$.

**Proof.** Let $A$ be as in 3.5 and set $J = F[A]$. Then $J$ and $S$ are semiprime. Let $C$ be the set of regular elements of $J$. Then $C$ is a right divisor set in both $S$ and $R$, since $C$ is normalized by $G$, and $T = \bigcup_{c \in C} I_R(c)$ is an ideal of the simple ring $R$. Thus $T = \{0\}$. Any left regular element of $R$ is a unit of $R$ and therefore $K = JC^{-1} \leq SC^{-1} \leq R$.

Now $K$ is a direct sum of finitely many fields and is normalized by $H$ ([1, 1.16] again). Then $SC^{-1} = K[H]$ is locally Artinian (meaning that every finite subset lies in an Artinian subring) and so $SC^{-1}$ is semisimple Artinian by [13, 2.6]. Right regular elements of $S$ are right regular in $SC^{-1}$ and so are units. Therefore $SC^{-1}$ is the ring of quotients of $S$ and by Goldie’s theorem [1, 1.27] the ring $S$ is right Goldie. Similarly it is left Goldie.

In a private communication R.L. Snider has provided the author with a proof of the following generalization of the proposition of [12].

3.7. (R.L. Snider). Let $H$ be a normal subgroup of the absolutely irreducible subgroup of $G$ of $GL(n, F)$ with $G/H$ locally finite. Then the subring $F[H]$ of $D^{n \times n}$ is semisimple Artinian.

In fact the only use we make here of 3.7 is covered by (b) of the proposition of [12].

3.8. Let $H$ be a normal subgroup of the absolutely irreducible subgroup $G$ of $GL(n, D)$ and suppose that $H$ modulo its centre $A$ is locally finite. Then $G/C_G(H)$ is periodic.

**Proof.** Choose if possible a counterexample $D, G, H$ with $n$ minimal. By 3.3 the subring $F[N]$ is prime for every normal subgroup $N$ of $G$. In particular, $F[A]$ is a domain and its field $J$ of fractions embeds naturally into $R = F[G]$, for example by 3.6. Of course $H$ centralizes $J$ and $G$ normalizes $J$. Replace $H$ and $G$ by $HJ^*$ and $GJ^*$. Thus we may assume that $J^* = A$ and $J = F[A]$.

By Schur’s theorem $H'$ is locally finite. Let $T$ be the maximal locally finite subgroup of $H$ and let $Z$ denote the centre of $T$. Then $K = J[Z]$ is a field, being commutative, prime and [13, 2.4] semisimple. Let $C_0 = C_G(T)$ and $C = C_G(A \cup T)$. By the theorem of [13] the group $G/C_0$ is locally finite and hence $F[C_0]$ is Artinian by 3.7. Now $C$ centralizes $AT/T$ and hence also $H/T$. By elementary stability theory $[H, C] \leq Z \subseteq K$. Hence by [13, 3.1] applied with $C_0$ in place of $G$ the group $K*C_0/(K*C_{K*C}(H))$ is periodic. As remarked above $G/C_0$ is locally finite, and $G/C_G(A)$ is periodic by 3.1. Therefore, $G/(C \cap K*C_{K*C}(H))$ is periodic. Now

$$C \cap K*C_{K*C}(H) \leq N_{K*C}(H) \cap K*C_{K*C}(H) = N_{K*}(H)C_{K*C}(H).$$
Also $J$ is a central subfield of the prime ring, $S = F[H]$ and $S$ is locally finite-dimensional over $J$. By [13, 2.6] again, $S$ is simple Artinian. Apply Theorem B to the normal subgroup $H$ of $L = N_{K^*}(H)H \subseteq S$. Then $L/C_L(H)$ is locally finite. But then so too is $N_{K^*}(H)/C_{K^*}(H)$ and hence $G/C_G(H)$ is periodic. This contradiction completes the proof of 3.8.

3.9. Let $A \leq H$ be normal subgroups of the absolutely irreducible subgroup $G$ of $GL(n, D)$ with $A$ abelian with $H/A$ locally finite. Then $G/C_G(H)$ is abelian by periodic.

Proof. Let $K = C_H(A)$ and denote the centre of $K$ by $Z$. Now $G/C_G(K)$ is periodic by 3.8 and $[H, C_G(K)] \leq C_H(K) = Z$. Thus by stability theory $C_G(K)/C_G(H)$ is isomorphic to a subgroup of the abelian group $\text{Der}(H/K, Z)$. The point is proved.

3.10. The proof of Theorem A. Part (i) of Theorem A follows from 3.5 and 3.9 and part (ii) follows from 3.8.

3.11. Let $p$ be a prime and $C$ a Prüfer $p^{\infty}$-group. Then $\text{Der}(C, ZC)$, for $ZC$ the group ring, is uncountable.

Proof. Let $\langle a \rangle$ be a cyclic group of order $pq$ and set $V = \mathbb{Z}\langle a \rangle$. Then $\text{Der}(\langle a \rangle, V) \cong (a - 1)V$ and $\text{Der}(\langle a^p \rangle, V) \cong (a^p - 1)V$. Further with the obvious isomorphisms

$$
\text{Der}(\langle a \rangle, V) \xrightarrow{\text{Res}} \text{Der}(\langle a^p \rangle, V)
$$

commutes and $\text{rank}_\mathbb{Z}(a - 1)V = pq - 1$ while $\text{rank}_\mathbb{Z}(a^p - 1)V = pq - p$. Thus the restriction map of $\text{Der}(\langle a \rangle, V)$ to $\text{Der}(\langle a^p \rangle, V)$ is onto but not one-to-one. Let $C_i$ be the subgroup of $C$ of order $p^i$. Then $\text{Der}(C, ZC) \cong \lim_{\leftarrow} \text{Der}(C_i, ZC)$, the maps being restrictions. The result follows.

3.12. The Main Counterexample. Let $S$ be a locally finite group and $V$ a $\mathbb{Z}$-torsion-free faithful right $S$-module, written multiplicatively. Let $C \leq \text{Der}(S, V)$. There exists a torsion-free group $T$ with a homomorphism $\pi$ of $T$ onto $S$ such that $B = \ker \pi$ is abelian (e.g., set $T = F/R'$ for $R \rightarrow F \rightarrow S$ a free presentation of $S$). Make $V \times C$ into a $T$ module via

$$(vc)' = v^\ell t^n (tn)'^c \quad \text{for} \quad v \in V, \ c \in C \text{ and } t \in T.$$ 

The split extension $W = T[ VC$ is torsion-free and locally abelian-by-finite. Note that the action of $c \in C$ as a derivation is given by $t\pi \mapsto [c, t]$. 

Let $F$ be a field. By [2] the group ring $FW$ is a domain and thus by Goldie's theorem [1, 1.28] is an Ore domain. Let $D$ be its division ring of quotients and set $K = F(BVC)$, the quotient field of $F[BVC]$ in $D$. Then $T \leq D^*$ normalizes $A_0 = K^*$. Set $G = TA_0 \leq D^*$. Then $A_0$ is abelian and since $S$ acts faithfully on $V \leq A_0$, we have $T \cap A_0 = B$ and $G/A_0 \cong S$. As $S$ is locally finite, so $K[T] = F[G]$ is a division ring and so $F[G] = D$. Therefore, $G$ is absolutely irreducible.

Suppose that $A_1$ is an abelian normal subgroup of $G$ with $G/A_1$ periodic. If $t \in T \setminus B$ there exists $v \in V$ with $v^t \neq v$. There is a positive integer $r$ with $v^r \in A_1$. Since $v^r \neq v$, it follows that $A_0 \geq C_G(A_0 \cap A_1) = A_1$. This has two immediate consequences. Firstly, $C_G(A_0) = A_0$, so in Theorem A, part (ii) the group $\langle G/C_G(H) \rangle$ there can be any locally finite group. Secondly, since $A_1 \leq A_0$, nothing more can be said about the group $\langle H/A \rangle$ of Theorem A, part (i).

Let $L = F(BV) \subseteq K$. Since $V$ is $\mathbb{Z}$-torsion-free so too is $\text{Der}(S, V)$. Let $C_0$ be a maximal $\mathbb{Z}$-independent subset of $\text{Der}(S, V)$ and set $C = \langle C_0 \rangle$. Then $R = L[C_0] \leq K$ is a polynomial ring over the field $L$ in the elements of $C_0$ and as such is a unique factorization domain [16, p. 38]. Also $[C, T] \leq V \leq L$, so $T$ normalizes $R$ and hence permutes its atoms (=irreducible elements). The elements of $C_0$ are non-associate atoms. Let $C_0 \cap Y_0$ be a full set of non-associate atoms of $R$, where $C_0 \cap Y_0 = 0$, and set $Y = \langle Y_0 \rangle$. By the unique factorization theorem $K^* = L^* \times C \times Y$. Now $T$ normalizes the set $L^*C_0 \cup L^*Y_0$ of all atoms and the set $L^*C_0$. Also $L^*C_0 \cap L^*Y_0 = \emptyset$. Therefore $T$ normalizes $L^*Y_0$ and consequently $A = L^*Y = \langle L^*Y_0 \rangle$ is a $T$-submodule of $K$ satisfying $K^* = A \times C$ and $L^* \leq A$.

Set $H = TA$. Then $G = TK^* = HC$ and $H \cap C = H \cap \langle K^* \cap C = (T \cap K^*)A \cap C = BA \cap C = \langle 1 \rangle$. Also

$$[C, H] = [C, T] \leq V \leq H,$$

so $H$ is normal in $G$. Clearly $C \leq C_G(A \cup H/A)$ and since $S$ acts faithfully on $V$, we have that $C_H(A \cup H/A) = A$. Let $I$ be the set of all $c \in C$ for which there exists $a \in A$ such that $[a, t] = [c, t]$ for all $t \in T$. Then $I$ is a subgroup of $C$. If $x \in C_G(H)$, then $x = hc$ for some $h \in H$ and $c \in C$. Then $c \in C_G(A \cup H/A)$, so $h \in C_H(A \cup H/A) = A$ and $[c, t] = [h^{-1}, t]$ for all $t \in T$. Hence $AC_G(H) \leq AI$ and clearly $I \leq AC_G(H)$. Therefore

$$G/AC_G(H) = HC/AI \cong S \times (C/I) \quad \text{and} \quad G/HC_G(H) \cong C/I.$$

We show now that $S$ and $V$ can be chosen so that $C/I$ contains a free abelian group of uncountable rank. Assume that $S$ is infinite with no non-trivial finite homomorphic images, set $V = ZS$ and suppose that $C \leq \text{Der}(S, V)$ has cardinal greater than the cardinal of $S$. For example, $S$ could be any Prüfer group by 3.11.

Let $J$ be the subgroup of $C$ of inner derivations; that is, $J$ is the set of all $c \in C$ for which there exist $v \in V$ with $[v, t] = [c, t]$ for all $t \in T$. Then $J \leq I$ and $|J| \leq |S| < |C|$. Therefore $C$ has a $Z$-basis $C_0 = C_1 \cup C_2$ with $C_1 \cap C_2 = \emptyset$, $J \leq \langle C_1 \rangle$ and $|C_2| = |C|$. We prove that $I \leq \langle C_1 \rangle$. The claim above about $C/I$ will then be proved. Let $a \in A$ and $c \in C$ be such that $[a, t] = [c, t]$ for all $t \in T$. 


Consider first the case where \( a \in L \). By hypothesis there is now a \( \mathbb{Z} \)-basis \( X \) of \( V \) normalized by \( S \). Then \( T \) normalizes the polynomial ring over the field \( F(B) \) in the elements of \( X \). Just as with \( R \) and \( K^* \) above, the unique factorization theorem yields that \( L^* = V \times W \) for some \( T \) submodule \( W \supseteq F(B)^* \) of \( L^* \). Then \( a = uv \) for some \( u \in V \) and \( w \in W \). If \( t \in T \), then

\[
[v, t][w, t] = [a, t] = [c, t] \in V.
\]

Hence \([w, t] \in V \cap W = \langle 1 \rangle \) and \([v, t] = [c, t] \). Therefore \( c \in J \).

Now consider the case where \( a \notin L^* \). Then \( a = ly \) for some \( l \in L^* \) and \( y \in Y \setminus \langle 1 \rangle \). Also \([a, T] \cup [l, T] \subseteq L^* \), so \([y, T] \subseteq L^* \) and \( T \) normalizes \( L^*y \). Now \( y = \prod y_i^{m_i} \) for some non-zero integers \( m_i \) and atoms \( y_i \in Y_0 \) of \( R \). By the uniqueness of factorization again \( T \) permutes the \( L^*y_i \). But \( S \) has no non-trivial finite homomorphic images and therefore \( T \) normalizes each \( L^*y_i \).

Consider for example \( y_1 \). We have \( y_1 = \alpha_1 c_1 + \cdots + \alpha_s c_s \) say, where the \( \alpha_i \in L^* \) and the \( c_i \) are distinct monomials on \( C_0 \). If \( t \in T \), then for some \( \beta \in L^* \) we have \( y_1' = \beta y_1 \). Hence \( \alpha_i' [c_i, t] = \alpha_i \beta \) for all \( i \) and so \( [c_i, t] = [\alpha_i^{-1}, t] \) for all \( t \in T \). By the case \( 'a \in L' \) this implies that each \( c_i^{-1} c_i \in J \). It also follows that \( T \) centralizes \( \alpha_i^{-1} c_i \) for each \( i \). Thus \( y_1 = \alpha_1 c_1 d_1 \) where

\[
d_1 = 1 + \alpha_1^{-1} \alpha_2 c_1^{-1} c_2 + \cdots + \alpha_1^{-1} \alpha_s c_1^{-1} c_s \in C_{L^*}(T).
\]

Since \( y_1 \in Y_0 \) we have that \( s > 1 \) and \( d_1 \notin L^* \). Also \( y_1 \) is an atom and the denominators of the \( c_i^{-1} c_i \) lie in \( \langle C_1 \rangle \) since \( J \subseteq \langle C_1 \rangle \). Therefore \( c_1 \in \langle C_1 \rangle \).

We have now shown that \( a = ly = l'c'd' \) where \( l' \in L^*, c' \in \langle C_1 \rangle \) and \( d' \in C_{K^*}(T) \). Then \([c, t] = [a, t] = [l', t][c', t] \) and \([c(c')^{-1}, t] = [l', t] \) for all \( t \in T \). By the case \( 'a \in L' \) again \( c(c')^{-1} \in J \) and therefore \( c \in Jc' \subseteq \langle C_1 \rangle \). Thus \( J \subseteq \langle C_1 \rangle \) as claimed.

The construction is complete. That is we have built a metabelian normal subgroup \( H \) of an absolutely irreducible skew-linear group \( G \) such that \( G/HC_G(H) \) contains a free abelian subgroup of uncountable rank. There is no restriction on the characteristic. This shows that in Theorem A, part (i) the conclusion that \( G/HC_G(H) \) is abelian by periodic cannot strengthened to periodic.

References