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The Monotonicity of the Reliability Measures of the Beta Distribution

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Abstract—In this paper, we investigate the monotonic properties of the hazard (failure) rate and mean residual life function (life expectancy) of the beta distribution. The monotonic properties are sometimes very useful in identifying an appropriate model. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

The family of beta distributions is composed of all distributions with probability density function (PDF) of the form

$$f_Y(y) = \frac{1}{\beta(p,q)} \frac{(y-a)^{p-1}(b-y)^{q-1}}{(b-a)^{p+q-1}}, \qquad a \le y \le b \tag{1.1}$$

with $p > 0,\ q > 0$, and

$$\beta(p,q) = \int_0^1 y^{p-1}(1-y)^{q-1}\,dy.$$

If we make the transformation $X = (Y-a)/(b-a)$, we obtain the PDF of X as

$$f_X(x) = \frac{1}{\beta(p,q)} x^{p-1}(1-x)^{q-1}, \qquad 0 \le x \le 1. \tag{1.2}$$

This is the standard form of the beta distribution with parameters p and q.

Beta distributions are very versatile and a variety of uncertainties can be usefully modeled by them. Many of the finite range distributions encountered in practice can be easily transformed into the standard distribution. In reliability and life testing experiments, many times the data are modeled by finite range distributions, see for example [1]. For some applications of the beta distribution in reliability, see [2,3].

In this paper, we shall investigate the monotonic properties of the reliability measures namely the hazard (failure) rate and the mean residual life function (life expectancy). The monotonic properties are, sometimes, very useful in choosing the appropriate model.

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2. THE BACKGROUND AND DEFINITION

Let X be a nonnegative random variable denoting the life of a component having distribution function $F(x)$ and PDF $f(x)$ which is twice differentiable. Then the failure rate is defined as $\lambda(t) = f(t)/S(t)$, where $S(t) = 1 - F(t)$ is called the survival (reliability) function. The mean residual life function (MRLF), $\mu(t)$, assuming it exists, is defined as

$$\mu(t) = E(X - t \mid X > t) = \frac{\int_t^\infty S(x)\,dx}{S(t)}, \qquad \mu = E(X) < \infty.$$

It can be easily verified that

(1)
$$\lambda(t) = \frac{1 + \mu'(t)}{\mu(t)},$$

(2)
$$S(t) = \frac{\mu(0)}{\mu(t)} \exp\left[-\int_0^t \frac{dx}{\mu(x)}\right].$$

Thus, $S(t)$, $\lambda(t)$, and $\mu(t)$ are equivalent in the sense that given one of them, the other two can be determined. Hence, in the analysis of survival data, one sometimes estimates $\lambda(t)$ or $\mu(t)$ instead of $S(t)$ acording to the convenience of the procedure available. It is easily seen that the constancy of $\lambda(t)$ or $\mu(t)$ characterizes the exponential distribution. Note that both $\lambda(t)$ and $\mu(t)$ are conditional concepts. The failure rate at t provides information about the immediate future after time t while the MRLF provides information about the whole future after t. Guess and Proschan [4] remark that it is possible for the MRLF to exist but the failure rate function not to exist. Likewise, it is possible for the failure rate function to exist while the MRLF does not exist. For further discussion of these and related measures, see [5].

We now present the definitions of monotonic and nonmonotonic failure rates.

The failure rate is said to be

(i) increasing if $\lambda'(t) > 0$, for all t and is denoted by I,
(ii) decreasing if $\lambda'(t) < 0$, for all t and is denoted by D,
(iii) bathtub shaped if $\lambda'(t) < 0$, for $t \in (0, t_0), \lambda'(t_0) = 0, \lambda'(t) > 0$ for $t > t_0$ and is denoted by B,
(iv) upside down bathtub shaped if $\lambda'(t) > 0$, for $t \in (0, t_0), \lambda'(t_0) = 0, \lambda'(t) < 0$ for $t > t_0$ and is denoted by U.

3. MONOTONICITY OF FAILURE RATE

The failure rate of the beta distribution is given by

$$\lambda(t) = \frac{t^{p-1}(1-t)}{\beta(p,q) - \beta_t(p,q)}, \tag{3.1}$$

where

$$\beta_t(p,q) = \int_0^t x^{p-1}(1-x)^{q-1}\,dx$$

is the incomplete beta function.

Since the failure rate has a complex expression because of the integral in the denominator, the determination of the monotonicity is not straight forward. To alleviate this difficulty, Glaser [6] presented a method to determine the monotonicity of the failure rate having one turning point. The extension to more than one turning point has been studied by Gupta and Warren [7].

In the following, we presnt Glaser's procedure to determine the monotonicity of $\lambda(t)$.

Define $\eta(t) = -f'(t)/f(t)$.

This function contains useful information about $\lambda(t)$, but it is simpler because it does not involve $S(t)$. In particular, the shape of η (I, D, B, U), often determines the shape of the failure rate. The relation between $\lambda(t)$ and $\eta(t)$ is given by $\frac{d}{dt}\ln\lambda(t) = \lambda(t) - \eta(t)$. We now present the following result due to [6], which helps us to determine the shape of the failure rates of the four types described earlier.

THEOREM 1.

(a) *If $\eta(t)$ is I, then $\lambda(t)$ is I (IFR).*
(b) *If $\eta(t)$ is D, then $\lambda(t)$ is D.*
(c) *If $\eta(t)$ is B and*
  (i) *if there exists a y_0 such that $\lambda'(y_0) = 0$, then $\lambda(t)$ is B,*
  (ii) *otherwise $\lambda(t)$ is I.*
(d) *If $\eta(t)$ is U and*
  (i) *if there exists a y_0 such that $\lambda'(y_0) = 0$, then $\lambda(t)$ is U,*
  (ii) *otherwise $\lambda(t)$ is D.*

In the last two cases, determining the existence of y_0 leaves us with the original difficulty of evaluating the derivative of $\lambda(t)$. However, we can simplify this problem in many situations with the following lemma.

LEMMA 2. *Let $\varepsilon = \lim_{t\longrightarrow 0} f(t)$ and $\delta = \lim_{t\longrightarrow 0} g(t)\eta(t)$, where $g(t) = 1/\lambda(t)$.*

1. *Suppose $\eta(t)$ is B, then*
   (a) *if either $\varepsilon = 0$ or $\delta < 1$, then $\lambda(t)$ is I, and*
   (b) *if either $\varepsilon = \infty$ or $\delta > 1$, then $\lambda(t)$ is B.*
2. *Suppose $\eta(t)$ is U, then*
   (a) *if either $\varepsilon = 0$ or $\delta < 1$, then $\lambda(t)$ is U, and*
   (b) *if either $\varepsilon = 0$ or $\delta > 1$, then $\lambda(t)$ is D.*

3.1. Failure Rate of Beta Distribution

For the PDF (1.2), it can be verified that

$$\eta(t) = \frac{t(p+q-2) - (p-1)}{t - t^2}$$

and

$$\eta'(t) = \frac{p-1}{t^2} + \frac{q-1}{(1-t)^2}.$$

CASE 1. $p \geq 1,\ q \geq 1$.

In this case, $\eta'(t) > 0$ for all t and hence, X has IFR.

CASE 2. $p \leq 1,\ q \leq 1$.

In this case, $\eta'(t) < 0$ for all t and hence, X has DFR.

In order to discuss the other cases, we consider the critical points of $\eta(t)$, which are given by solving $\eta'(t) = 0$.

Or

$$\left(\frac{1-t}{t}\right)^2 = \frac{1-q}{p-1}. \tag{3.2}$$

In the following cases, we will see if the critical points are points of maxima or minima and investigate the shape of the failure rate of X.

CASE 3. $0 < q < 1$ and $p > 1$.

In this case, $1-t/t = \mp\sqrt{1-q/p-1}$. Let $t_1 = 1/1+\sqrt{1-q/p-1}$ and $t_2 = 1/1 - \sqrt{1-q/p-1}$. It is clear that $0 < t_1 < 1$ and $t_2 < 0$ if $\sqrt{1-q/p-1} > 1$ and $t_2 > 1$ if $\sqrt{1-q/p-1} < 1$.

Therefore, t_1 is a critical point of $\eta(t)$. Now we want to test if t_1 is a point of maxima or minima. For that consider

$$\eta''(t) = -2\left[\frac{p-1}{t^3} + \frac{1-q}{1-t^3}\right] < 0, \qquad \text{for } t_1 = \frac{1}{1+\sqrt{(1-q)/(p-1)}}.$$

Hence, t_1 is a point of maxima for $\eta(t)$. Since $\lim_{t \longrightarrow o} f(t) = 0$, X has U-shaped failure rate.

CASE 4. $q > 1$ and $0 < p < 1$.

As before, $1-t/t = \mp\sqrt{(q-1)/(1-p)}$. Let $t_1^* = 1/(1+\sqrt{(q-1)/(1-p)})$ and $t_2^* = 1/(1-\sqrt{(q-1)/(1-p)})$. By the same arguments as in Case 3, we conclude that $0 < t_1^* < 1$ and $t_2^* < 0$ or $t_2^* > 1$. Therefore, $t_1^* = 1/(1+\sqrt{(q-1)/(1-p)})$ is a critical point of $\eta(t)$.

Proceeding as before, it can be verified that t_1^* is a point of minima for $\eta(t)$. Also in this case, $\lim_{t \longrightarrow 0} f(t) = \infty$. Thus, X has a B shaped failure rate.

Thus, we have shown the following.

(1) If $p \geq 1$ and $q \geq 1$, then X has IFR.
(2) If $p \leq 1$ and $q \leq 1$, then X has DFR.
(3) If $0 < q < 1$ and $p > 1$, then X has U shaped failure rate.
(4) If $0 < p < 1$ and $q > 1$, then X has B shaped failure rate.

4. MONOTONICITY OF MEAN RESIDUAL LIFE FUNCTION

The MRLF of the beta distribution (1.2) is given by

$$\mu(t) = \frac{\int_t^\infty (\beta(p,q) - \beta_x(p,q))\, dx}{\beta(p,q) - \beta_t(p,q)}. \tag{4.1}$$

It is clear that $\mu(t)$ is a complicated function of t and the parameters. In order to investigate the monotonicity of $\mu(t)$, we consider the four cases as before.

CASE 1. $p \geq 1$ and q≥ 1.

Since X has IFR, X has DMRL (decreasing mean residual life), see [8].

CASE 2. $p \leq 1$ and $q \leq 1$.

Since X has DFR, X has IMRL (increasing mean residual life).

To discuss the other two cases, we use the following results due to [9].

LEMMA 3. *Suppose $\lambda(t)$ is of the type B, then*

(1) *$\mu(t)$ is decreasing if $\lambda(0) \leq 1/\mu$,*
(2) *$\mu(t)$ is of the type U if $\lambda(0) < 1/\mu$.*

LEMMA 4. *Suppose $\lambda(t)$, is of the type U, then*

(1) *$\mu(t)$ is decreasing if $\lambda(0) \geq 1/\mu$,*
(2) *$\mu(t)$ is of the type B if $\lambda(0) < 1/\mu$.*

CASE 3. $0 < q < 1$ and $p > 1$.

In this case, X has a U-shaped failure rate and $\lambda(0) < 1/\mu$. Hence, $\mu(t)$ is of the type B.

CASE 4. $0 < p < 1$ and $q > 1$.

In this case, X has B-shaped failure rate and $\lambda(0) > 1/\mu$. Hence, $\mu(t)$ is of the type U.

REMARK. For the location of the turning points in Cases 3 and 4, see [9].

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