Relational Equations in Totally Ordered Lattices and Their Complete Resolution

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The set of all solutions of a composite fuzzy relation equation of Sanchez (Inform. and Control 30 (1976)), defined on finite spaces, is studied by determining and characterizing all the lower solutions of such an equation.

1. INTRODUCTION

In [5], Sanchez introduced in a Brouwerian lattice the concept of max–min fuzzy relation equation in order to give a generalization of well known Boolean equations.

Let $L$ be a Brouwerian lattice and $F(X) = \{A : X \rightarrow L\}$ the set of all fuzzy sets of a non-empty set $X$ in the sense of Goguen [4]. Let $Y, Z$ be other non-empty sets, $Q \in F(X \times Y)$, $R \in F(Y \times Z)$ and $T \in F(X \times Z)$ fuzzy relations. A max–min fuzzy relation equation is an equation of the type

$$R \circ Q = T \quad (1)$$

where "\(\circ\)" is the max–min composition and $R$ is unknown. Let $\mathcal{R}$ be the set of all solutions $R$ of Eq. (1). In [5], Sanchez gives a method of resolution of (1), by determining if $\mathcal{R}$ is not empty, the greatest element of $\mathcal{R}$, whereas in [2], Di Nola and Pedrycz establish a necessary and sufficient condition for the existence of a Boolean solution $R \in \mathcal{R}$, viz., $R : Y \times Z \rightarrow \{0, 1\}$.

If $L$ is a totally ordered lattice and $Q \in F(Y)$, $T \in F(Z)$ are fuzzy sets, Eq. (1) is called \(\oplus\)-fuzzy relation equation. For these equations, Sanchez [6] determines all the lower solutions, viz., all the minimal elements of $\mathcal{R}$.

In this paper, by using the results of [6], we determine and characterize all the lower solutions of (1) in the hypothesis that $X, Y, Z$ are finite spaces and $L$ is a totally ordered lattice. Furthermore, we find a solution of $\mathcal{R}$ which has special properties.

We represent fuzzy sets and fuzzy relations as real matrices and we identify them with their membership functions.
2. Preliminaries

Let $L$ be a fixed totally ordered lattice with universal bounds 0 and 1. Let us denote by "\(\land\)" and "\(\lor\)," the meet and join operations, by "\(\leq\)" the natural ordering of $L$. Let $X$ be a non-empty set and $F(X) = \{A: X \to L\}$ the set of all fuzzy sets of $X$ in the sense of Goguen [4].

If $L = [0, 1]$, $F(X)$ is the Zadeh's membership functions set [7]. It is well known that $F(X)$ is a complete lattice with respect to the ordering and operations defined for any $x \in X$ as

\[
A \leq B \iff A(x) \leq B(x),
\]
\[
(A \land B)(x) = A(x) \land B(x),
\]
\[
(A \lor B)(x) = A(x) \lor B(x),
\]
where $A, B \in F(X)$.

From now on, we suppose that $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_m\}$, and $Z = \{z_1, z_2, \ldots, z_p\}$ are finite sets, and we put $I_n = \{1, 2, \ldots, n\}$ as the set of first $n$ natural positive numbers.

Following Sanchez [5, 6], we now recall the following definitions.

**Definition 1.** A fuzzy relation $Q$ between $X$ and $Y$ is an element of $F(X \times Y)$. We say inverse of $Q$, the fuzzy relation $Q^{-1} \in F(Y \times X)$ defined as $Q^{-1}(y_j, x_i) = Q(x_i, y_j)$ for any $i \in I_n$, $j \in I_m$.

Let $Q \in F(X \times Y)$, $R \in F(Y \times Z)$, $T \in F(X \times Z)$ be fuzzy relations. For brevity, we put

\[
Q(x_i, y_j) = q_{ij}, \quad R(y_j, z_k) = r_{jk}, \quad T(x_i, z_k) = t_{ik}
\]
for any $i \in I_n$, $j \in I_m$, $k \in I_p$.

**Definition 2.** We define $T = R \circ Q$, $T \in F(X \times Z)$, the max-min composite fuzzy relation of $R$ and $Q$ as

\[
t_{ik} = \bigvee_{j \in I_n} (q_{ij} \land r_{jk})
\]
for any $i \in I_n$, $k \in I_p$.

**Definition 3.** We define $S = Q^{-1} \circ T$, $S \in F(Y \times Z)$, the \(\odot\)-composite fuzzy relation of $Q^{-1}$ and $T$, as

\[
s_{jk} = S(y_j, z_k) = \bigwedge_{i=1}^{n} (q_{ij} \land t_{ik})
\]
for any \( j \in I_m, k \in I_p \), where
\[
q_{ij} t_{ik} = 1 \quad \text{if} \quad q_{ij} \leq t_{ik},
\]
\[
= t_{ik} \quad \text{if} \quad q_{ij} > t_{ik}.
\]

**Definition 4.** \( \mathcal{L} \) is called the lower solution of \( \mathcal{R} \) if \( \mathcal{L} \) is a minimal element of \( \mathcal{R} \), viz., for any \( R \in \mathcal{R} \) such that \( R \leq \mathcal{L} \) we have \( R = \mathcal{L} \).

Let us define, for any \( i \in I_n \), the fuzzy sets \( Q_i \in F(\{x_i\} \times Y) \) and \( T_i \in F(\{x_i\} \times Z) \) as
\[
Q_i(x_i, y_j) = q_{ij}, \quad T_i(x_i, z_k) = t_{ik}
\]
for any \( j \in I_m, k \in I_p \).

**Definition 5.** We define \( \ominus \) relation equation the max-min fuzzy equation
\[
R \ominus Q_i = T_i
\]
where \( R \in F(Y \times Z) \) is unknown.

**Definition 6.** For any \( i \in I_n \), we define \( W_i = Q_i \ominus T_i \), \( W_i \in F(Y \times Z) \), the \( \ominus \)-composite fuzzy relation of \( Q_i \) and \( T_i \), as
\[
W_i(y_j, z_k) = q_{ij} \sigma t_{ik}
\]
for any \( j \in I_m, k \in I_p \), where
\[
q_{ij} \sigma t_{ik} = 0 \quad \text{if} \quad q_{ij} < t_{ik},
\]
\[
= t_{ik} \quad \text{if} \quad q_{ij} \geq t_{ik}.
\]

Throughout this paper, we suppose that \( \mathcal{R} \neq \emptyset \).

### 3. THEORETICAL RESULTS

For any \( i \in I_n \), let \( \mathcal{R}_i \) be the set of solutions \( R \in F(Y \times Z) \) of the \( \ominus \)-fuzzy relation equation (2). Obviously we have
\[
\mathcal{R} = \bigcap_{i=1}^{n} \mathcal{R}_i
\]
and therefore the study of the fuzzy equation (1) is equivalent to the study of the fuzzy system constituted by \( n \) equations of type (2).

Now we recall some results of Sanchez:
THEOREM 1 (see [6]). If $\mathcal{R} = \emptyset$, then $\mathcal{R}$ has a non-zero lower solutions $L_i \in F(Y \times Z)$ such that the fuzzy union is just $W_i$. Such relations $L_i$ can be determined by the choice of a non-zero element in each column of $W_i$.

This theorem is clarified by the example of Section 5.

THEOREM 2 (see [5]). $\mathcal{R} \neq \emptyset$ iff $S$ is the greatest element of $\mathcal{R}$.

In [3], Di Nola and Sessa established some results about the lower solutions of the set $\mathcal{R}_i$. For the sake of completeness, here we point out such results. First, we need this lemma:

**Lemma.** Let be $R', R'' \in R$ (resp. $\mathcal{R}_i$) and $R \in F(Y \times Z)$ such that $R' \leq R \leq R''$. Then $R \in R$ (resp. $\mathcal{R}_i$).

**Proof.** By definition 2, we have $R' \circ Q \leq R \circ Q \leq R'' \circ Q$ and then $R \in R$ because $R' \circ Q = R'' \circ Q = T$.

**Theorem 3.** If $\mathcal{R}_i \neq \emptyset$, $i \in I_n$, then there exists at least lower solution $L_i$ such that $L_i \leq R$ for any $R \in \mathcal{R}_i$.

**Proof.** Since for any $R \in \mathcal{R}_i$ and $k \in I_p$, we have

$$\bigvee_{j=1}^m (q_{ij} \wedge r_{jk}) = t_{ik},$$

an index $j^* \in I_m$ must exist such that for any $k \in I_p$, $q_{ij^*} \geq q_{jk} \wedge r_{jk} = t_{ik}$ holds. Then $W_i(y_{j^*}, z_k) = q_{j^*} \sigma t_{ik} - t_{ik} < r_{j^*k}$, and if we define for any $k \in I_p$

$$L_i(y_{j^*}, z_k) = 0 \quad \text{if} \quad j \in I_m - \{j^*\}$$

$$= t_{ik} \quad \text{if} \quad j = j^*$$

we clearly obtain a lower solution $L_i \in \mathcal{R}_i$ such that $L_i \leq R$.

**Theorem 4.** If $\mathcal{R} \neq \emptyset$, then $\mathcal{A}_i = \{L_i \in \mathcal{R}_i \text{ such that } L_i \leq S\}$ is non-empty for any $i \in I_n$.

**Proof.** By Theorem 2, $S \in \mathcal{R}$ and therefore $S \in \mathcal{R}_i$ for any $i \in I_n$ because (3) holds. The thesis then follows by Theorem 3.

**Theorem 5.** If $\mathcal{R} \neq \emptyset$, the fuzzy relation $L = \bigvee_{i=1}^n L_i$, where $L_i \in \mathcal{A}_i$ and $i \in I_n$, belongs to $\mathcal{R}$.

**Proof.** By Theorem 4, $\mathcal{A}_i \neq \emptyset$, and furthermore we have for any $i \in I_n$

$$L_i \leq L \leq S.$$

By lemma, $L$ belongs to $\mathcal{R}_i$ for any $i \in I_n$ and then $L$ belongs to $\mathcal{R}$. 
In accord with this theorem, we define the set

\[ A = \left\{ \mathcal{L} \in \mathcal{R} \text{ such that } \mathcal{L} = \bigvee_{i=1}^{n} \mathcal{L}_i, \mathcal{L}_i \in A_i \quad \forall i \in I_n \right\}. \]

Now we define the fuzzy relation \( W \in F(Y \times Z) \) as

\[ W = \left( \bigvee_{i=1}^{n} W_i \right) \wedge S. \]

**THEOREM 6.** If \( \mathcal{R} \neq \emptyset \), \( W \in \mathcal{R} \).

*Proof.* By Theorems 1 and 4, for any \( i \in I_n \), every lower solution \( \mathcal{L}_i \in A_i \) is such that \( \mathcal{L}_i \leq W_i \). Hence, by Theorem 5, the fuzzy relation \( \mathcal{L} \) is an element of \( \mathcal{R} \) such that \( \mathcal{L} \leq W \) for any \( \mathcal{L} \in A \). On the other hand, we have \( W \leq S \) and lemma we deduce \( W \in \mathcal{R} \).

The fuzzy relation \( W \) enjoys some properties illustrated by the following

**THEOREM 7.** For any \( R \in \mathcal{R} \), \( R \wedge W \in \mathcal{R} \).

*Proof.* From (3), \( R \in \mathcal{R}_i \) for any \( i \in I_n \) and by Theorem 3, then there exists a lower solution \( \mathcal{L}_i \in A_i \) such that \( \mathcal{L}_i \leq R \).

Since \( R \leq S \), \( \mathcal{L}_i \in A_i \), for any \( i \in I_n \) and therefore \( \mathcal{L} = \bigvee_{i=1}^{n} \mathcal{L}_i \) is an element of \( A \) such that \( \mathcal{L} \leq R \). Since \( \mathcal{L} \leq W \), we have \( \mathcal{L} \leq R \wedge W \leq W \). By lemma, the thesis follows.

4. **Characterization of Lower Solutions of \( \mathcal{R} \)

We quote the following from Chap. I, Theorem 3, of Birkhoff's book [1].

**THEOREM 8.** Any finite non-empty subset \( X \) of a poset has minimal and maximal elements.

Then we can show the following result which characterizes all the lower solutions of a composite fuzzy relation equation (1):

**THEOREM 9.** If \( \mathcal{R} \neq \emptyset \), the minimal elements of \( A \) are the lower solutions of \( \mathcal{R} \) and vice versa.

*Proof.* Since \( \mathcal{R} \neq \emptyset \), by Theorems 4 and 5 the set \( A \) is non-empty. Since \( \mathcal{R} \) is a poset and \( A \) is a finite set, \( A \) has at least a minimal element \( M \) by Theorem 8. We claim that \( M \) is a lower solution of \( \mathcal{R} \). Indeed, let be \( R \in \mathcal{R} \)
such that \( R \leq M \). Since \( R \in \mathcal{R} \) for any \( i \in I_n \), then there exists, by Theorem 3, a lower solution \( \mathcal{L}_i \in \mathcal{R}_i \) such that \( \mathcal{L}_i \leq R \) for any \( i \in I_n \). Since \( R \leq S \), we have \( \mathcal{L}_i \in \mathcal{A}_i \) for any \( i \in I_n \). Consequently the fuzzy relation \( \mathcal{L} = \bigvee_{i=1}^{n} \mathcal{L}_i \) is an element of \( \mathcal{A} \) such that

\[
\mathcal{L} \leq R \leq M.
\]

Since \( M \) is minimal in \( \mathcal{A} \), we have \( \mathcal{L} = M \) and therefore by (4) we obtain \( R = M \).

Vice versa, let \( R \) be a lower solution of \( \mathcal{R} \). By reasoning as above, we find an element \( \mathcal{L} \in \mathcal{A} \) such that \( \mathcal{L} \leq R \). Then \( \mathcal{L} = R \) because \( R \) is minimal in \( \mathcal{R} \). So \( R \in \mathcal{A} \) and it is clearly minimal in \( \mathcal{A} \) too.

5. A Numeral Example

Here we suppose \( L = [0, 1] \) and \( X = \{x_1, x_2, x_3\} \), \( Y = \{y_1, y_2, y_3\} \), \( Z = \{z_1, z_2, z_3\} \). Let be \( Q \in F(X \times F) \) and \( T \in F(X \times Z) \) given by

\[
Q = x_2, \quad T = x_2,
\]

\[
x_1, \quad x_2, \quad x_3
\]

\[
y_1 \quad y_2 \quad y_3
\]

\[
z_1 \quad y_2 \quad y_3
\]

We have \( S = Q^{-1} \ominus T \) and \( W_i = Q_i \ominus T_i \), \( i \in I_3 \), as follows:

\[
S = y_2, \quad W_1 = y_2
\]

\[
y_1 \quad y_2 \quad y_3
\]

\[
z_1 \quad z_2 \quad z_3
\]

\[
y_1 \quad y_2 \quad y_3
\]

\[
z_1 \quad z_2 \quad z_3
\]

\[
y_1 \quad y_2 \quad y_3
\]

\[
z_1 \quad z_2 \quad z_3
\]

\[
y_1 \quad y_2 \quad y_3
\]

\[
z_1 \quad z_2 \quad z_3
\]
Consequently we have $W$ and the lower solutions of $\mathcal{R}_i$, $i \in I_3$, given by

$$
\begin{array}{c|ccc}
 z_1 & z_2 & z_3 & y_1 \\
 W = y_2 & 0.0 & 0.5 & 0.0 \\
y_3 & 0.8 & 0.3 & 0.0 \\
y_3 & 0.6 & 0.3 & 1.0 \\
\end{array} \quad \begin{array}{c|ccc}
 z_1 & z_2 & z_3 & y_1 \\
 \mathcal{L}_1' = y_2 & 0.0 & 0.0 & 0.0 \\
y_3 & 0.8 & 0.3 & 0.0 \\
y_3 & 0.0 & 0.0 & 1.0 \\
\end{array}
$$

$$
\begin{array}{c|ccc}
 z_1 & z_2 & z_3 & y_1 \\
 \mathcal{L}_1'' = y_2 & 0.0 & 0.0 & 0.0 \\
y_3 & 0.8 & 0.0 & 0.0 \\
y_3 & 0.0 & 0.3 & 0.0 \\
\end{array} \quad \begin{array}{c|ccc}
 z_1 & z_2 & z_3 & y_1 \\
 \mathcal{L}_1''' = y_2 & 0.0 & 0.0 & 0.0 \\
y_3 & 0.8 & 0.0 & 1.0 \\
y_3 & 0.0 & 0.3 & 0.0 \\
\end{array}
$$

Then

$$
A_1 = \{ \mathcal{L}_1', \mathcal{L}_1'' \}, \quad A_2 = \{ \mathcal{L}_2', \mathcal{L}_1''' \}, \quad A_3 = \{ \mathcal{L}_3' \}
$$

and

$$
A = \{ \mathcal{L} = \mathcal{L}_1' \lor \mathcal{L}_2' \lor \mathcal{L}_3', M = \mathcal{L}_1'' \lor \mathcal{L}_2' \lor \mathcal{L}_3' \},
$$

where $\mathcal{L}$ and $M$ are the relations

$$
\begin{array}{c|ccc}
 z_1 & z_2 & z_3 & y_1 \\
 \mathcal{L} = y_2 & 0.0 & 0.5 & 0.0 \\
y_3 & 0.8 & 0.3 & 0.0 \\
y_3 & 0.6 & 0.0 & 1.0 \\
\end{array} \quad \begin{array}{c|ccc}
 z_1 & z_2 & z_3 & y_1 \\
 M = y_2 & 0.0 & 0.5 & 0.0 \\
y_3 & 0.8 & 0.0 & 0.0 \\
y_3 & 0.6 & 0.3 & 1.0 \\
\end{array}
$$
REFERENCES

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