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NOTE

A Certain Functional–Differential Equation

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We resolve some conjectures concerning the zeros $t_n$ of the solution $f$ of the initial value problem $f'(t) = -f(qt)$, $0 < q < 1$, $f(0) = 1$. In particular,

$$\lim_{n \to \infty} t_{n+1}/t_n = 1/q.$$ 

Let $0 < q < 1$. Then the initial value problem

$$y'(t) = -y(qt), \quad y(0) = 1, \quad (1)$$

has the unique solution [1–3]

$$f(z) = \prod_{n=0}^{\infty} \left(1 - z/t_n\right), \quad 0 < t_0 < t_1 < t_2 < \cdots, \quad (2)$$

with positive zeros $t_i$. The equation (1) is the simplest of a class of functional–differential equations which have been subject to widespread recent study (see [2] for references), and the following conjectures, labelled as in [2], have been raised concerning the $t_i$: 

(A) $t_{n+1}/t_n$ is monotone decreasing [3];

(B) $\lim_{n \to \infty} t_{n+1}/t_n = 1/q$ [3];

(D) $t_n = \mu q^{-n} + \nu + o(1)$ as $n \to \infty$, with real constants $\mu, \nu$ [1].

It was proved in [2] that $t_{n+1} > q^{-1}t_n$ for all $n$ and that $\limsup_{n \to \infty} t_{n+1}/t_n \leq q^{-2}$. 

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Theorem. We have, as \( n \to \infty \),
\[
\frac{t_{n+1}}{t_n} = \frac{1}{q}\left(1 + \frac{1}{n + 1}\right) + o(n^{-2})
\]
(3)
and, with \( \gamma \) a positive constant,
\[
t_n = q^{-n}(\gamma + o(1)).
\]
(4)
Further, the conjecture (C) is true but (D) is false, and \( t_{n+1}/t_n \) is decreasing for large \( n \).

Proof of the Theorem

For convenience, set \( \lambda = 1/q \). As noted above, it was shown in [2] that
\[
t_{n+1} > \lambda t_n, \quad n \geq 0.
\]
(5)
In fact, (5) may be proved by the following method, different to that of [2]:
Since \( f' \) has a zero \( s \) in \( (t_0, t_1) \) and \( 0 = f(qs) \), by (1), it follows at once that \( t_1 > s = \lambda t_0 \). Repeating this gives (5).
Again, since \( f'(s) = 0 \) implies that \( f(qs) = 0 \), it now follows that \( f' \) has precisely one zero \( s_n = \lambda t_n \) in the interval \( (t_n, t_{n+1}) \), for each \( n \geq 0 \). But
\[
\frac{f'(z)}{f(z)} = \sum_{n=0}^{\infty} \frac{1}{z - t_n}
\]
vanishes at \( s_n \), and so we get
\[
\frac{s_n}{t_{n+1} - s_n} = \sum_{\mu \geq 0, \mu \neq n+1} \frac{1}{1 - t_{\mu}/s_n}.
\]
(6)
Next, (5) gives
\[
t_{n+j}/s_n = qt_{n+j}/t_n > \lambda^{-j}, \quad j \geq 2,
\]
and so
\[
\sum_{\mu \geq n+2} \left|\frac{1}{1 - t_{\mu}/s_n}\right| \leq \sum_{k=1}^{\infty} \frac{1}{\lambda^k - 1} = c_0 < \infty.
\]
(7)
On the other hand, for \( m \leq n \) we have, using (5),

\[
0 < \frac{1}{1 - t_m/s_n} - 1 = \frac{t_m}{(1 - t_m/s_n)s_n} \leq \frac{t_m}{(1 - q)s_n} \leq \frac{1}{(1 - q)\lambda^{n-m+1}}
\]

(8)

and hence

\[
\sum_{\mu=0}^{n} \frac{1}{1 - t_\mu/s_n} = n + 1 + h_n, \quad 0 < h_n < \sum_{k=1}^{\infty} \frac{1}{(1 - q)\lambda^k} < \infty.
\]

(9)

Thus (6), (7), and (9) give, for large \( n \),

\[
x_n = \frac{s_n}{t_{n+1} - s_n} = n + O(1)
\]

(10)

and so

\[
\frac{t_{n+1}}{\lambda t_n} - 1 = \frac{1}{n} + O(n^{-2}).
\]

(11)

From (11) we see at once that the conjecture (C) is true, but (D) is false. Further, (11) gives

\[
\log \frac{t_{n+1}}{t_n} = \log \lambda + \frac{1}{n} + O(n^{-2}),
\]

which leads at once to (4). Note also that if \( \lambda \) is large enough then \( c_0 + h_n < 1/4 \), so that \( x_n \) is increasing and (A) is true.

To prove (3), from which it follows that \( t_{n+1}/t_n \) is eventually decreasing, for every \( q \) in \((0, 1)\), we need to refine (10) somewhat. Since, for fixed \( m \), we now have \( t_{n+m}/t_n \to \lambda^m \), the dominated convergence theorem and (7) give

\[
\sum_{\mu \geq n+2} \frac{1}{1 - t_\mu/s_n} = \sum_{m=2}^{\infty} \frac{1}{1 - t_{n+m}/\lambda t_n} = c_1 + o(1), \quad c_1 = \sum_{k=1}^{\infty} \frac{1}{1 - \lambda^k},
\]

(12)
as $n \to \infty$. Let $0 < \delta < \frac{1}{4}$ and fix $Q$, chosen so large that

$$\sum_{k=Q+2}^{\infty} \frac{1}{\lambda^k - 1} < \frac{\delta}{8}, \quad \sum_{k=Q+2}^{\infty} \frac{1}{(1-q)\lambda^k} < \frac{\delta}{8}. \quad (13)$$

For large $n$ we then have

$$\sum_{\mu=n-Q}^{n} \frac{t_\mu}{(1-t_\mu/s_n)s_n} = \sum_{k=0}^{Q} \frac{t_{n-k}}{(1-t_{n-k}/s_n)s_n} = \sigma_Q + o(1), \quad (14)$$
in which

$$\sigma_Q = \sum_{k=0}^{Q} \frac{1}{(1-q^{k+1})\lambda^{k+1}} = \sum_{k=1}^{Q+1} \frac{1}{\lambda^k - 1} = -c_1 - \rho_Q,$$

$$0 < \rho_Q < \frac{\delta}{8}. \quad (15)$$

and, using (8) and (13),

$$\left| \sum_{\mu=0}^{n-Q-1} \frac{t_\mu}{(1-t_\mu/s_n)s_n} \right| \leq \sum_{k=Q+1}^{\infty} \frac{1}{(1-q)\lambda^{k+1}} < \frac{\delta}{8}. \quad (16)$$

Using (6), (12), (14), (15), and (16), the estimate (10) now becomes, for large $n$,

$$x_n = \frac{s_n}{t_{n+1}/s_n} = n + 1 + c_1 + \sigma_Q + d_n$$

$$= n + 1 + L_n, \quad |d_n| < \delta/2, |L_n| < \delta,$$

which proves (3).

REFERENCES

