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ORIGINAL ARTICLE

An efficient numerical method for computation of the number of complex zeros of real polynomials inside the open unit disk

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Abstract In this paper, a simple and efficient numerical method is proposed for computing the number of complex zeros of a real polynomial lying inside the unit disk. The proposed protocol uses the Boubaker polynomial expansion scheme (BPES) to build sequence of polynomials based on the concept of Sturm sequences. The method is used in a direct way without using any restrictions in reference to other existing methods. The protocol is applied to some example polynomials of different orders and utility of the algorithm is noticed.

1. Introduction

Computation of the number of non-real zeros of real polynomials inside the open unit disk is very important in complex analysis and system control. For example, for a corrector of the form:

$$u_{n+1} = \sum_{i=0}^{k-1} A_i u_{n-i} + h \sum_{i=-1}^{k-1} a_i u'_{n-i} \tag{1}$$

Simpson's stability rule is ensured if the polynomial:

$$P(z) = z^k - \sum_{i=0}^{k-1} A_i z^{k-1-i} \tag{2}$$

has all of its zeros in the open unit disk.

The Newton mapping of non-zero polynomials is also based on this notion. In fact, for a given polynomial $P(z)$, the Newton mapping $N_P(z)$, which is defined by:

$$N_P(z) = z - P(z)/P'(z) \tag{3}$$

would be defined only if the zeros of $P(z)$ are contained in the open unit disk.

In this study, we present a new protocol for determining the exact number of complex zeros of a given real polynomial in the unit disk using a well-known applied mathematics protocol, the Boubaker polynomials. The polynomials were established by Boubaker (2007, 2008) and have been worked upon by many researchers till now for further developments and its utilities are being investigated to deal various types of case-studies in applied engineering, medical sciences, etc.

Several properties and modified versions of these polynomials have been investigated; to mention a few studies: Boubaker et al. (2007), Labiadh (2007), Oyodum et al. (2009), Zhao et al. (2009, 2010) and Barry and Hennessy (2010). A modified

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version of these polynomials, called $4q$-Boubaker polynomials, was the basis for the development of the Boubaker polynomials expansion scheme (BPES). The scheme has been used by Agida and Kumar (2010) and Kumar (2010) to solve particular integral equations. On the other hand few standard boundary value problems of ordinary differential equations (Boubaker, 2008; Zhang and Naing, 2010; Koçak et al., 2011a) and many physical models involving ordinary differential equations systems (Milgram, 2011; Dubey et al., 2010, Yildirim et al., 2010) were solved more efficiently by BPES as compared to other methods. Physical models in terms of partial differential equations in many fields were reliably addressed through BPES. For example: the works carried out by Ghrib et al. (2008), Guezmir et al. (2009) and Zhang and Li (2010) in general to investigate material and alloy properties and more particularly the works by Zhang (2010b) and Slama et al. (2008a,b, 2009, 2010) in the field of resistance spot welding research to obtain analytical temperature distribution.

The contributions by Ghanouchi and Labiadh (2008), Tabatabaei et al. (2009), Belhadj et al. (2009) and Koçak et al. (2011b) further evoked the use of BPES to solve core studies in the field of Heat and Mass Transfer. Awojoyogbe and Boubaker (2009) and in many other studies jointly explained how NMR blood flow equations can be solved in various heart models to find magnetic phase shift, and in Bio-medical engineering to find net magnetization under the MRI exposure in various geometries. The work carried out by Fridjine and Amlouk (2009) discusses the case of optimizing functional materials in hybrid solar energy devices.

The main idea in this paper consists of constructing the Sturm-sequences which are built using the properties of BPES. The idea of this construction is based on the work of Schelin (1983) who first used Chebyshev polynomials to construct Sturm-like sequence to count zeros of real polynomials. A similar construction using Chebyshev polynomials appears in the works of Locher and Skrzipek (1995) and Gleyse (1997). The examination of the number of sign changes and the sign repetitions in the built-off Sturm sequences in this work using $4q$-Boubaker polynomials finally leads to define the complete protocol to achieve the goal of computing the number of complex zeros of real polynomials.

The concept of sign changes and sign repetitions dates back to Seventeenth century when Rene Descarte proposed a rule of signs to find upper bound on the count of positive and negative real zeros of a polynomial. Another concept of examining signs appears in Routh–Hurwitz test and its extensions (Gantmacher, 1960) which is used to determine if all zeros of a real polynomial lie in the open left-half plane and hence to comment on polynomial stability. However, the criterion of counting the number of sign changes and the sign repetitions used to develop method in this paper is based on a similar concept used in Sturm theorem (Collins and Rudiger, 1983) to count real zeros of polynomials defined in interval $[-1, 1]$. We demonstrate through worked out examples in Section 3 that the proposed protocol – which uses little extension of the concepts in the Sturm theorem – yields encouraging results when it comes to count the number of complex zeros of real polynomials in the open unit disk.

The structure of this paper is as follows:

We begin by introducing, in Section 2, some necessary definitions and mathematical preliminaries of the Boubaker Polynomials which are required for establishing our results. This follows the procedure of constructing the Sturm-like shaped sequence of the polynomials. In Section 3, we use the protocol to determine the exact number of complex zeros of some variable degree polynomials in the open unit disk. We end with illustrating conclusion and future work.

2. Materials and methods

2.1. The Boubaker polynomials

The first monomial definition of the Boubaker polynomials (Boubaker, 2007, 2008; Ghanouchi and Labiadh, 2008; Belhadj et al., 2009) appeared in a physical study that yielded an analytical solution to the heat equation inside a physical model.

Definition 1. Boubaker polynomials monomial definition is given by:

$$B_n(X) = \sum_{p=0}^{\xi(n)} \left[\frac{(n-4p)}{(n-p)} C_{n-p}^{p} \right] \cdot (-1)^p \cdot X^{n-2p} \tag{4}$$

where:

$$\xi(n) = \left\lfloor \frac{n}{2} \right\rfloor = \frac{2n + ((-1)^n - 1)}{4}$$

(The symbol: $\lfloor\rfloor$ designates the floor function).

The Boubaker polynomials have also the explicit monic expression:

$$B_n(X) = X^n - (n-4).X^{n-2} + \sum_{p=2}^{\xi(n)} \left[\frac{(n-4p)}{p!} \prod_{j=p+1}^{2p-1} (n-j) \right] \cdot (-1)^p \cdot X^{n-2p} \tag{5}$$

Theorem 1. The characteristic recurrence relation for the Boubaker polynomials is:

$$B_m(X) = X \cdot B_{m-1}(X) - B_{m-2}(X) \quad \text{for :} \quad m > 2$$

Proof. For $m > 2$: $B_{m-1}(X) = \sum_{p=0}^{\xi(m-1)} \left[\frac{(m-1-4p)}{(m-1-p)} C_{m-1-p}^{p} \right] \cdot (-1)^p \cdot X^{m-1-2p}$, and: $B_{m-2}(X) = \sum_{p=0}^{\xi(m-2)} \left[\frac{(m-2-4p)}{(m-2-p)} C_{m-2-p}^{p} \right] \cdot (-1)^p \cdot X^{m-2-2p}$. By calculating the amount: $\Delta = X \cdot B_{m-1}(X) - B_{m-2}(X)$, it gives:

$$\Delta = X^m \left[\sum_{p=0}^{\xi(m-1)} \left[\frac{(m-1-4p)}{(m-1-p)} C_{m-1-p}^{p} \right] \cdot (-1)^p \cdot X^{-2p} - \sum_{p=0}^{\xi(m-2)} \left[\frac{(m-2-4p)}{(m-2-p)} C_{m-2-p}^{p} \right] \cdot (-1)^p \cdot X^{-2-2p} \right]$$

which gives:

$$\Delta = X \cdot B_{m-1}(X) - B_{m-2}(X) = X^m \sum_{p=0}^{\xi(n)} \left[\frac{(n-4p)}{(n-p)} C_{n-p}^{p} \right] \cdot (-1)^p \cdot X^{-2p} = B_m(X) \qquad \square$$

The ordinary generating function of the Boubaker polynomials is:

$$f_{\tilde{B}}(X,t) = \frac{1+3t^2}{1+t(t-X)} \tag{6}$$

Zhao et al. (2010) investigated some special properties of the Boubaker polynomials B_n for the case $n = 4q$ which include involvement of only even powers of x in the polynomials and removal of the $2q$ rank monomial terms from the explicit form. In particular, these properties lead to explicit expressions with only $2q$ effective terms and hence to a class of polynomials which are all even functions. Correspondent $4q$-order Boubaker polynomials (Zhao et al., 2010) are presented in Eq. (7) as a general form and Eq. (8) as first functions:

$$B_{4q}(X) = 4\sum_{p=0}^{2q}\left[\frac{(q-p)}{(4q-p)}C_{4q-p}^{p}\right]\cdot(-1)^p\cdot X^{2(2q-p)} \tag{7}$$

$$\begin{cases} B_0(X) = 1; \\ B_4(X) = X^4 - 2; \\ B_8(X) = X^8 - 4X^6 + 8X^2 - 2; \\ B_{12}(X) = X^{12} - 8X^{10} + 18X^8 - 35X^4 + 24X^2 - 2; \\ B_{16}(X) = X^{16} - 12X^{14} + 52X^{12} - 88X^{10} + 168X^6 - 168X^4 + 48X^2 - 2; \\ B_{20}(X) = X^{20} - 16X^{18} + 102X^{16} - 320X^{14} + 455X^{12} - 858X^8 + 1056X^6 \\ -495X^4 + 80X^2 - 2; \\ \ldots \end{cases} \tag{8}$$

The proposed protocol in this paper is based on $4q$-order Boubaker polynomials instead of original polynomials B_n due to the benefits that all $4q$-order polynomial are even functions and result in less computational cost (to be elaborated in Section 3). We quote the following important results of $4q$-Boubaker polynomials (Zhao et al., 2010) which will be useful in the construction of the Sturm shaped sequences and the final implementation of the protocol to follow. Readers can refer to (Zhao et al., 2010) for detailed proofs.

Theorem 2. The following equality holds:

$$\sum_{k=0}^{n} B_k(x)B_k(y) = 3 + \frac{B_{n+1}(x)B_n(y) - B_n(x)B_{n+1}(y)}{x-y} \quad \text{for all} \quad x \neq y$$

Proof. As a consequence of recurrence relation (Theorem 1) and assuming:

$$B_k(x)B_k(y) = \frac{\Delta_k - \Delta_{k-1}}{x-y} \quad \text{for} \quad k = 2, 3, \ldots \tag{8}$$

where: $\Delta_k = B_{k+1}(x)B_k(y) - B_k(x)B_{k+1}(y)$ (8) summed from 0 to n gives the desired formula. □

If $x \to y$ in (8), we obtain the following Corollary.

Corollary 1. The following equality is satisfied

$$\sum_{k=0}^{n} B_k^2(x) = 3 + B'_{n+1}(x)B_n(x) - B'_n(x)B_{n+1}(x), n > 0 \tag{9}$$

2.2. Built-off Sturm shaped sequence

Definition 2. A Sturm shaped sequence of polynomials is a set:

$$\{P_0(x), P_1(x), P_2(x), \ldots, P_M(x)\} \tag{10}$$

with P_0, P_1 and P_2 three initializing nonzero polynomials, M a given integer and $P_i|_{i=1\ldots M}$ verifying:

$$P_i(x) = \Phi_i(x)P_{i-1}(x) + P_{i-2}(x), \quad i \geqslant 2 \tag{11}$$

where $\Phi_i(x)|_{i=2\ldots N}$ is a given polynomial sequence.

Let us consider a real polynomial $Q(x) = \sum_{i=0}^{N}\xi_i x^i$, along with the sequence $\{P_0(x), P_1(x), P_2(x), \ldots, P_N(x)\}$:

$$\begin{cases} P_0(x) = \sum_{i=0}^{n}\xi_i B_{4i}(x) \\ P_1(x) = \sum_{i=1}^{n}\xi_i B_{4i-4}(x) \\ P_2(x) = B_4(x)P_1(x) - P_0(x) \\ \ldots \\ P_{k+1}(x) = B_4(x)P_k(x) - P_{k-1}(x) \\ P_N(x) = B_4(x)P_{N-1}(x) - P_{N-2}(x) \end{cases} \tag{12}$$

Here N is order of the real polynomial $Q(x)$ and n is number of non-zero terms in $Q(x)$.

Theorem 3. The sequence $\tilde{P}_N(x) = \{P_0(x), P_1(x), P_2(x), \ldots, P_N(x)\}$ with polynomials as in (12) is a Sturm shaped sequence constructed from Boubaker polynomials.

Proof. We have, for all values of $0 \leqslant k < M$:

$$P_{k-1}(x) = B_4(x)P_k(x) - P_{k+1}(x) = \Lambda P_k(x) + r,$$

the remainder r of the Euclidian division of $P_{k-1}(x)$ by $P_k(x)$ is hence: $r = -P_{k+1}(x)$. □

Proposed protocol. For a sequence $\tilde{P}_N(x) = \{P_0(x), P_1(x), P_2(x), \ldots, P_N(x)\}$, associated to a polynomial $Q(x) = \sum_{i=0}^{N}\xi_i x^i$, defined in the domain $[-1, 1]$, the number $Z_{1,Q}$ of complex zeros inside the unit disk is given by:

$$Z_{1,Q} = S^*(-1) + S^*(1) \tag{13}$$

where $S^*(x) = S^C(x) - S^R(x)$ represents the difference between the number of sign changes and sign repetitions in the sequence $\tilde{P}_N(x)$.

This protocol is an extension of the Sturm theorem for real zeros of real-coefficient polynomials. For a proof, refer to Collins and Rudiger (1983). While the usual Sturm theorem and related works on Sturm-like sequence using Chebyshev polynomials in literature (Schelin, 1983; Locher and Skrzipek, 1995; Gleyse, 1997) target only the number of real zeros of real coefficient polynomials in open unit disk or other annulus, we demonstrate through examples in the next section that the proposed protocol – an extension to the theorem – can be used to count number of complex zeros of real polynomials in open unit disk.

Since the built-off Sturm sequence $\tilde{P}_N(x)$ is constructed through $4q$-Boubaker polynomials, which are all even, as a consequence the number of sign changes and also the sign repetitions at -1 and 1 will be the same, i.e. $S^C(-1) = S^C(1)$ and $S^R(-1) = S^R(1) \Rightarrow S^*(-1) = S^*(1)$. Thus, (13) can equivalently be expressed as:

$$Z_{1,Q} = 2S^*(-1) = 2S^*(1) \tag{14}$$

It can be noted that the use of 4q-Boubaker polynomials minimizes the computational cost of (13) by half as one needs to count the sign changes and repetitions either only at 1 or -1 as in (14).

3. Results and discussion

The described protocol has been applied on following polynomials (all zeros are shown opposite):

Example 1:

$$Q_1(x) = x^8 + \frac{9}{8}x^6 + \frac{281}{256}x^4 - \frac{225}{512}x^2 + \frac{625}{4096}\left(\frac{1}{2} \pm i, -\frac{1}{2} \pm i, \frac{1}{2} \pm \frac{1}{4}i, -\frac{1}{2} \pm \frac{1}{4}i\right)$$

Example 2:

$$Q_2(x) = x^5 + 4x^4 + \frac{15}{2}x^3 + \frac{35}{2}x^2 + 14x + 6\left(-3, \pm 2i, -\frac{1}{2} \pm \frac{1}{2}i\right)$$

Example 3:

$$Q_3(x) = x^7 + \frac{1}{2}x^5 - x^3 - \frac{1}{2}x\left(0, \pm 1, \pm i, \pm \frac{1}{\sqrt{2}}i\right)$$

Example 4:

$$Q_4(x) = x^3 - x^2 + \frac{1}{4}x - \frac{1}{4}\left(1, \pm \frac{1}{2}i\right)$$

Implementation details of the proposed protocol on polynomials in Examples 1–4 are given in Table 1 with specific values, sign sequence and sign patterns in corresponding Sturm-shaped sequences. We explicitly describe implementation on, say, $Q_1(x)$. The application of the protocol on $Q_1(x)$ gives the following Boubaker polynomial built Sturm-shaped sequence:

$$\begin{cases} P_0(x) = \sum_{i=0}^{n} \xi_i B_{4i}(x) = B_{16}(x) + \frac{9}{8}B_{12}(x) + \frac{281}{256}B_8(x) \\ -\frac{225}{512}B_4(x) + \frac{625}{4096}B_0(x) \\ P_1(x) = \sum_{i=1}^{n} \xi_i B_{4i-4}(x) = B_{12}(x) + \frac{9}{8}B_8(x) + \frac{281}{256}B_4(x) - \frac{225}{512}B_0(x) \\ P_{k+1}(x) = B_4(x)P_k(x) - P_{k-1}(x), \ \ k = 2, 3, \ldots, 8 \end{cases}$$

and corresponding sign sequence $\{+,-,-,+,-,-,+,-,-\}$ at $x = 1$ or $x = -1$. Consequently:

$$Z_{1,Q1} = 2S^*(-1) = 2S^*(1) = 2(5-3) = 4$$

which is true as only four complex zeros of $Q_1(x)$: $\frac{1}{2} + i, \frac{1}{2} - i, -\frac{1}{2} + i$ and $-\frac{1}{2} - i$ lie in the open unit disk. Zeros loci for $Q_1(x)$ are shown in Fig. 1.

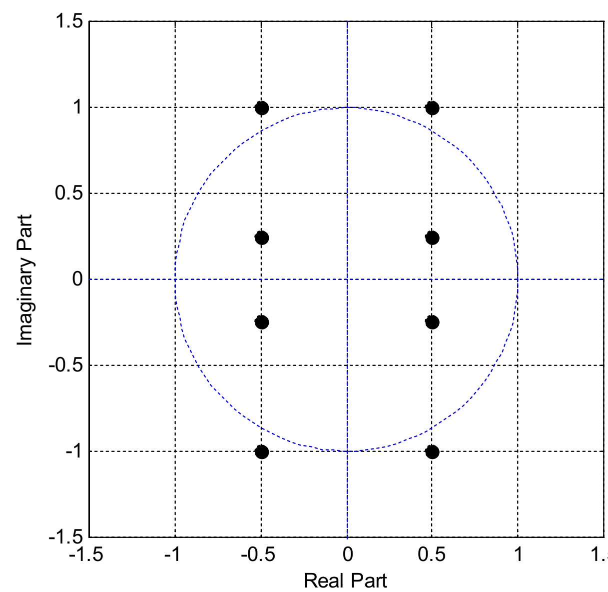

Fig. 1 Zeros loci for $Q_1(x)$.

Table 1 Implementation details of the proposed protocol on example polynomials 1–4.

	$Q_1(x)$	$Q_2(x)$	$Q_3(x)$	$Q_4(x)$
P_0	341/537	57/2	0	$-11/2$
P_1	$-83/512$	10	3/2	17/4
P_2	$-1143/2414$	$-77/2$	$-3/2$	5/4
P_3	341/537	57/2	0	$-11/2$
P_4	$-83/512$	10	3/2	–
P_5	$-1143/2414$	$-77/2$	$-3/2$	–
P_6	341/537	–	0	–
P_7	$-83/512$	–	3/2	–
P_8	$-1143/2414$	–	–	–
Sign sequence at $x = 1$ or -1	$\{+,-,-,+,-,-,+,-,-\}$	$\{+,+,-,+,+,-\}$	$\{+,+,-,+,+,-,+,+\}$	$\{-,+,+,-\}$
$S^C(-1)$ or $S^C(1)$	5	3	4	2
$S^R(-1)$ or $S^R(1)$	3	2	3	1
$S^*(-1)$ or $S^*(1)$	2	1	1	1
$Z_{1,Q}$	**4**	**2**	**2**	**2**

Bold values represent main output of the proposed protocol on examples 1–4. These bold values represent the number of complex zeros inside the unit disk, found by the protocol for polynomials in examples 1–4.

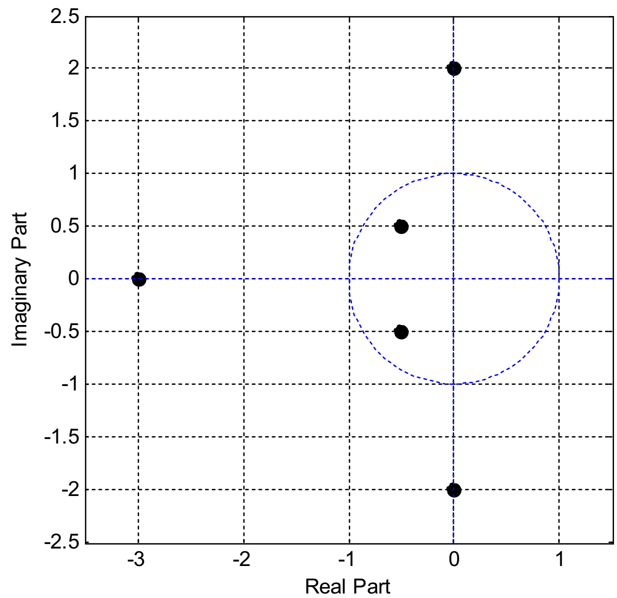

Fig. 2 Zeros loci for $Q_2(x)$.

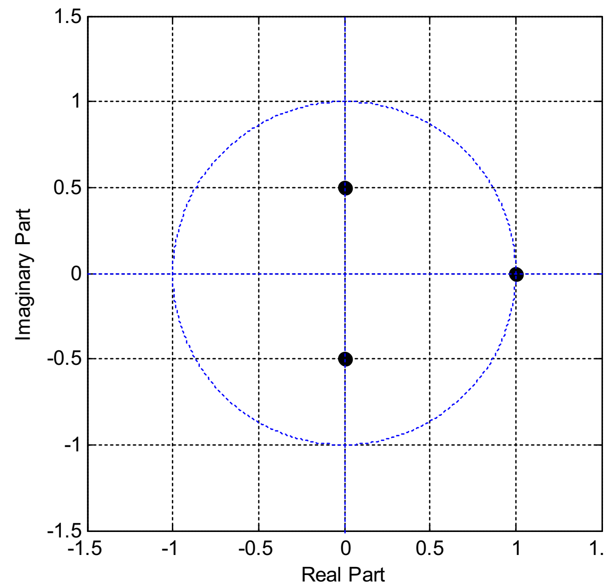

Fig. 4 Zeros loci for $Q_4(x)$.

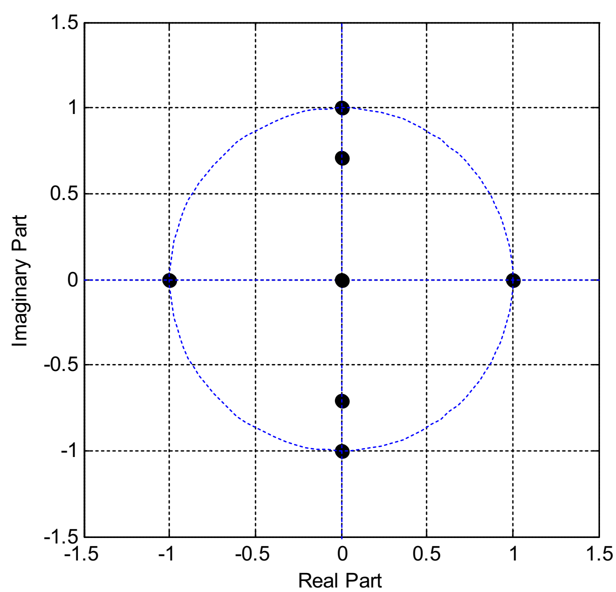

Fig. 3 Zeros loci for $Q_3(x)$.

The results in Table 1 speak for themselves. The proposed protocol shows that polynomials in Examples 1–4 have 4, 2, 2, and 2 complex zeros inside the open unit disk, respectively, which is in good agreement with the loci plots of these polynomials in z-plane (Figs. 1–4). It can be noted that the proposed method counts only those complex zeros inside open unit disk that are purely non-real, i.e. involve some imaginary term. This is evident through Example-3 and meanwhile from Fig. 3 that $x = 0$ (a purely real zero of $Q_3(x)$) – beside located in the open unit disk – is not counted by the proposed algorithm.

4. Conclusion and future work

The exact number of complex zeros in the open unit disk of some polynomials of different orders has been determined using the Boubaker polynomial generated Sturm sequence. The protocol is general and efficient since no restriction is applied to the targeted polynomial. According to the example investigations, this method is a simple and efficient numerical method for computing the number of polynomials complex zeros lying inside the unit disk.

The construction of Boubaker polynomials built-off Sturm shaped sequences in this work to exactly compute the number of complex zeros for real polynomials, as a start, through this work further encourages researchers to revisit the approximations, work to refine it for other similar problems and devise extended methods. Investigation of further accuracy and suitability of this protocol along with proposition of its utility to address case studies in complex analysis and control theory are the topics of future research. It can be observed that no conditions are presumed for the polynomial $Q(x)$, as opposed to the methods (Gleyse, 1997; Gleyse and Larabi, 2011). Comparison of our method will be made with those of the Cauchy-indices-related method, used by Gleyse (1997), or those of methods using Schur-Cohn, Brown and Cohn transforms (Gleyse and Larabi, 2011) from the view-point of analysis of order of complexity in future.

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