Invertible incline matrices and Cramer’s rule over inclines

Song-Chol Han a,b, Hong-Xing Li a,∗

a Department of Mathematics, Beijing Normal University, Beijing 100875, PR China
b Department of Mathematics and Mechanics, Kim Il Sung University, Pyongyang, Democratic People’s Republic of Korea

Received 25 February 2004; accepted 15 March 2004
Submitted by R.A. Brualdi

Abstract

Inclines are the additively idempotent semirings in which the products are less than or equal to factors. Thus inclines generalize Boolean algebra, fuzzy algebra and distributive lattice. And the Boolean matrices, the fuzzy matrices and the lattice matrices are the prototypical examples of the incline matrices (i.e., the matrices over inclines). In this paper, the complete description of the invertible incline matrices is given. Some necessary and sufficient conditions for an incline matrix to be invertible are studied, Cramer’s rule over inclines is presented and the group of invertible incline matrices is investigated. The main results in the present paper generalize and develop the corresponding results in the literatures for the Boolean matrices, the fuzzy matrices and the lattice matrices.

© 2004 Elsevier Inc. All rights reserved.

AMS classification: 15A09; 15A15; 16Y60

Keywords: Incline matrix; Inverse; Permanent; Cramer’s rule; Permutation matrix; Boolean matrix; Fuzzy matrix; Lattice matrix; Group

∗Supported by National Natural Science Foundation of China (60174013), Research Fund for Doctoral Program of Higher Education (20020027013), Science and Technology Key Project Fund of Ministry of Education (03184) and Major State Basic Research Development Program of China (2002CB312200).

Corresponding author.

E-mail address: lhxqx@bnu.edu.cn (H.-X. Li).

0024-3795/ - see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2004.03.025
1. Introduction

The notion of inclines and their applications were described in Cao et al. [1] comprehensively. Inclines are the additively idempotent semirings in which the products are less than or equal to factors. Thus inclines generalize Boolean algebra, fuzzy algebra and distributive lattice. And the Boolean matrices, the fuzzy matrices and the lattice matrices are the prototypical examples of the incline matrices. All applications of Boolean algebra and fuzzy algebra to automata theory, design of switching circuits, logic of binary relations, medical diagnosis, Markov chains, information system and clustering are instances in which inclines can be applied. Besides, inclines are applied to nervous system, probable reasoning, finite state machines, psychological measurement, dynamical programming and decision theory.

Kim [8] described Boolean matrix theory and its applications in detail, and noted that the permutation matrices are the only invertible Boolean matrices. Zhao [21] proved that a fuzzy square matrix is invertible if and only if it is a permutation matrix, and so generalized the result of Kim [8]. Luce [11] showed that a matrix over a Boolean algebra of at least two elements is invertible if and only if it is an orthogonal matrix. Zarko [17] established Cramer's rule over general Boolean algebras. Give’on [4] developed the theory of invertible lattice matrices firstly, thus generalizing the result of Luce [11]. Zhao [19,20] discussed the conditions for invertibility of the matrices over a kind of Brouwerian lattices and arbitrary distributive lattice, respectively. Skornyakov [13] gave the complete and abundant description of the invertible lattice matrices. On the other hand, Kim et al. [7] developed the determinant theory for the fuzzy square matrices. Zhang [18] studied the determinant theory for the square lattice matrices and generalized the results of Kim et al. [7]. Yi and Liu [16] defined the permanent of the rectangular matrices over complete and completely distributive lattices and proved the expansion theorem for it. Tian et al. [15] presented Cramer’s rule over complete and completely distributive lattices. Cao et al. [1] firstly studied the condition for an incline matrix to be invertible, thus showed that the statement of Luce [11] holds for the incline matrices as well.

The aim of this paper is to generalize the results of the literatures [1,4,11,13,15,17,20] and give the complete description of the invertible incline matrices. We study some necessary and sufficient conditions for an incline matrix to be invertible, present Cramer’s rule over inclines and discuss the group of invertible incline matrices. The main results in the present paper generalize and develop the corresponding results in the literatures for the Boolean matrices, the fuzzy matrices and the lattice matrices.

2. Preliminaries and some lemmas

In this section, we give some definitions and preliminary lemmas.
Definition 2.1 [1]. A nonempty set $L$ with two binary operations $+$ and $\cdot$ is called an incline if it satisfies the following conditions:

1. $(L, +)$ is a semilattice,
2. $(L, \cdot)$ is a commutative semigroup,
3. $x(y + z) = xy + xz$ for all $x, y, z \in L$,
4. $x + xy = x$ for all $x, y \in L$.

In an incline $L$, define a relation $\leq$ by $x \leq y \iff x + y = y$. Obviously, $xy \leq x$ for all $x, y \in L$.

The Boolean algebra $(\{0, 1\}, \lor, \land)$ is an incline. The fuzzy algebra $([0, 1], \lor, T)$ is also an incline, where $T$ is a t-norm. And the distributive lattice is a kind of inclines.

For any positive integer $n$, $n$ always stands for the set $\{1, 2, \ldots, n\}$ and $[n]$ denotes the least common multiple of the integers $1, 2, \ldots, n$.

Throughout this paper, unless otherwise stated, $L$ always denotes any given incline with the additive identity 0 and the multiplicative identity 1. It follows that 0 is the least element and 1 is the greatest element in $L$.

Denote by $L^{m \times n}$ and $L^n$ the set of all $m \times n$ matrices over $L$ and the set of all column vectors of order $n$ over $L$, respectively. Especially, we denote $M_n(L) := L^{n \times n}$.

Given $A = (a_{ik}) \in L^{m \times n}$ and $B = (b_{kj}) \in L^{n \times l}$, the product $A \cdot B \in L^{m \times l}$ is defined by

$$A \cdot B := \left( \sum_{k \in n} a_{ik} b_{kj} \right).$$

For $A = (a_{ij}) \in M_n(L)$ and positive integer $l$, denote by $a_{ij}^{(l)}$ the $(i, j)$-entry of $A^l$, i.e., $A^l = (a_{ij}^{(l)})$.

Denote by $O$ the zero matrix over $L$, i.e., all entries of $O$ are 0. Denote by $I_n$ the $n \times n$ identity matrix over $L$, i.e., $I_n = (\delta_{ij}) \in M_n(L)$ and

$$\delta_{ij} := \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \text{ for } i, j \in n.$$ 

For a convenience, $I_n$ is sometimes written as $I$.

The set $M_n(L)$ constitutes a partially ordered monoid with respect to the matrix multiplication.

Definition 2.2 [1]. Let $S$ be a semigroup and $a \in S$. Assume that $a^k = a^{k+d}$ for some positive integers $k$ and $d$. The least such positive integers $k$ and $d$ are called an index and a period of $a$, respectively. In this case, we say that $a$ has index.

Definition 2.3 [1]. An incline $L$ with 0 and 1 is called an integral incline if it has no nonzero elements $x$ and $y$ such that $xy = 0$ and $x + y = 1$. 
Definition 2.4 [1]. A matrix $A \in M_n(L)$ is said to be right invertible (left invertible) if $AB = I$ ($BA = I$) for some $B \in M_n(L)$. The matrix $B$ is called a right inverse (left inverse) of $A$. If $A$ is both right and left invertible, then it is said to be invertible and $B$, denoted by $A^{-1}$, is called an inverse of it.

Obviously, if $A$ is invertible, then its right inverse and left inverse coincide with its inverse.

Definition 2.5 [1]. A matrix $P \in M_n(L)$ is called a permutation matrix if only one entry of its every row and every column is 1 and the other entries are 0.

An element $a$ of $L$ is said to be idempotent if $a^2 = a$. The set of all idempotent elements in $L$ is denoted by $I(L)$, i.e., $I(L) = \{a \in L \mid a^2 = a\}$.

Example 2.1. Let $e \in [0, 1]$ and $T_e$ be a binary operation on $[0, 1]$ defined by

$$T_e(x, y) = \begin{cases} x \land y, & x = 1 \text{ or } y = 1, \\ x \land y \land e, & \text{otherwise} \end{cases}$$

Then $T_e$ is a t-norm and $L = ([0, 1], \lor, T_e)$ is an incline. It is clear that $I(L) = [0, e] \cup \{1\}$.

The following are the fundamental lemmas which will be used in the next sections.

Lemma 2.1 [6, Lemma 2.3]. $I(L)$ is a distributive lattice.

Lemma 2.2 [6, Lemma 3.1]. If $A \in M_n(L)$, then $A^{n^2[n]} \leq A^{n^2[n]} - [n]$.

Lemma 2.3. If $D$ is a distributive lattice and $A \in M_n(D)$, then $A^n \leq A^{n^2[n]}$.

Proof. Let $A = (a_{ij})$. Then for any $i, j \in \mathbb{N}$, we have that

$$a_{ij}^{(n)} = \sum_{i_1, i_2, \ldots, i_{n-1} \in \mathbb{N}} a_{i_1i_2 \cdots i_{n-1}ij}.$$ 

Consider any term $a_{i_1i_2 \cdots i_{n-1}ij}$ of $a_{ij}^{(n)}$. Put $i_0 = i$ and $i_n = j$. Since $i_u \in \mathbb{N}$ for all $u$ ($0 \leq u \leq n$), there exist $p$ and $t$ such that $0 \leq p < t \leq n$ and $i_p = i_t$. Then we obtain that

$$a_{i_0i_1i_2 \cdots i_{n-1}ij} = a_{i_0i_1i_2 \cdots i_{p-1}ij \cdot b \cdot a_{i_pi_{i+1} \cdots i_{n-1}ij \leq a_{i_0i_1i_2 \cdots i_{p-1}ij \cdot b \cdots b \cdot a_{i_{t+_1}i_{t+2} \cdots a_{i_{n-1}ij} \leq a_{i_0i_1i_2 \cdots i_{n-1}ij}.$$ 

The following are the fundamental lemmas which will be used in the next sections.
where \( b = a_{i'p+i+1} \cdots a_{i-1} \) and \( l = [n]/(t - p) + 1 \). Since the inequality above holds for any term of \( a_{ij}^{(n)} \), we see that \( a_{ij}^{(n)} \leq a_{ij}^{(n+l)} \). This completes the proof. □

Proposition 2 of Skornyakov [13] is a corollary of Lemma 2.3.

**Lemma 2.4.** If \( D \) is a distributive lattice and \( A \in M_n(D) \), then \( A^{n^2+n} = A^{n^2+n} \).

**Proof.** By Lemma 2.2, we have that
\[
A^{n^2+n} = A^{n^2}A^n \leq A^{n^2-n}A^n = A^{n^2+n}.
\]
From Lemma 2.3, we obtain that
\[
A^{n^2+n} = A^{n+n}A^{n^2-n} \geq A^nA^{n^2-n} = A^{n^2+n}.
\]
This completes the proof. □

### 3. Some conditions for invertibility of incline matrices

In this section, we give some necessary and sufficient conditions for the incline matrices to be invertible.

**Theorem 3.1.** If \( A \in M_n(L) \) is right invertible, then \( A^n = I \).

**Proof.** Since \( A \) is right invertible, \( AX = I \) for some \( X \in M_n(L) \). Let \( k = n^2+n \) and \( d = n^2 \). Then \( A^{k+d} = A^k \) by Lemma 2.2. Hence we have that
\[
A^d = A^k \cdot I = A^{k} (A \cdots (A(A(X)X) \cdots X)^k \text{ times})
\]
\[
= A^{d}A^{k}X^k = A^{k+d}X^k \leq A^{k}X^k = A \cdots AX \cdots X = I,
\]
i.e., \( A^d \leq I \). Now let \( A = (a_{ij}) \) and \( X = (x_{ij}) \). From \( A^dX^d = I \), it follows that
\[
\sum_{s \in n} a_{is}^{(d)} x_{si}^{(d)} = 1.
\]
Since \( A^d \leq I \), we have that \( a_{ij}^{(d)} = 0 \) for \( i \neq j \). Therefore, for every \( i \in n \), we obtain that
\[
\sum_{s \in n} a_{is}^{(d)} x_{si}^{(d)} = 1,
\]
i.e., \( a_{ii}^{(d)} = 1 \). Consequently, \( A^d = I \). □

Theorem 3.1 generalizes Theorem 7 of Give’on [4] and Proposition 5 of Skornyakov [13].

**Theorem 3.2.** If \( A \in M_n(L) \) is an invertible matrix, then \( A^\circ A = AA^\circ = I \).
Proof. Since $A$ is invertible, $AX = XA = I$ for some $X \in M_n(L)$. Put $A = (a_{ij})$ and $X = (x_{ij})$. By $AX = I$, we have that if $i \neq j$, then $\sum_{s \in \mathbb{N}} a_{is} x_{sj} = 0$, and so $a_{is} x_{sj} = 0$ for all $s \in \mathbb{N}$. And by $XA = I$, we obtain that $\sum_{s \in \mathbb{N}} a_{si} \geq \sum_{s \in \mathbb{N}} x_{is} a_{si} = 1$ and $\sum_{s \in \mathbb{N}} x_{is} \geq \sum_{s \in \mathbb{N}} x_{is} a_{si} = 1$ for all $i \in \mathbb{N}$. Hence we see that
\[
\begin{align*}
   a_{ij} &= a_{ij} \cdot 1 = a_{ij} \left( \sum_{s \in \mathbb{N}} x_{js} \right) = \sum_{s \in \mathbb{N}} a_{ij} x_{js} = a_{ij} x_{ji} + \sum_{s \neq i} a_{ij} x_{js} \\
   &= a_{ij} x_{ji} + 0 = a_{ij} x_{ji} + \sum_{s \neq i} a_{sj} x_{ji} = \sum_{s \in \mathbb{N}} a_{sj} x_{ji} \\
   &= \left( \sum_{s \in \mathbb{N}} a_{sj} \right) x_{ji} = 1 \cdot x_{ji} = x_{ji}
\end{align*}
\]
for all $i, j \in \mathbb{N}$. Therefore, $X = A^T$ and this completes the proof. □

Theorem 3.2 generalizes Theorem 6 of Give’on [4] and Theorem 4.2 of Luce [11]. Let $A \in M_n(L)$. A mapping $f_A : L^n \rightarrow L^n$ is defined by $f_A(x) = Ax$ for $x \in L^n$.

Theorem 3.3. If $A \in M_n(L)$ and $f_A$ is surjective mapping, then $A$ is right invertible.

Proof. Since $f_A$ is surjective mapping, there are some column vectors $(x_{1i}, x_{2i}, \ldots, x_{ni})^T \in L^n$ ($i \in \mathbb{N}$) such that
\[
A \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{ni} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, A \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, A \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.
\]
Put $X = (x_{ij})$. Then $X \in M_n(L)$ and $AX = I$, i.e., $A$ is right invertible. □

Let $x, y \in L^n$. A scalar product of $x$ and $y$ is defined as $x^T y \in L$.

Theorem 3.4. If $A \in M_n(L)$, then the following statements are equivalent to each other:

1. $A$ is right invertible;
2. $A$ is left invertible;
3. $A$ is invertible;
4. $AA^T = I$;
5. $A^T A = I$;
6. $A^2 A = AA^2 = I$;
(7) $A^k = I$ for some positive integer $k$;
(8) $f_A$ is a surjective mapping;
(9) $f_A$ is a bijective mapping;
(10) $f_A$ preserves the scalar products;
(11) $A \in M_n(I(L))$ and $f_A$ is an injective mapping;
(12) $A$ has index and $f_A$ is an injective mapping.

**Proof.** (2) $\Rightarrow$ (7) It is proved by the same method as the proof of Theorem 3.1.
The equivalence of the first seven statements is proved in the procedures of
(1) $\Rightarrow$ (7) $\Rightarrow$ (3) $\Rightarrow$ (6) $\Rightarrow$ (4) $\Rightarrow$ (1),
(2) $\Rightarrow$ (7) $\Rightarrow$ (3) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (2)
by using Theorem 3.1 and Theorem 3.2.

(9) $\Rightarrow$ (8) It is obvious.
(8) $\Rightarrow$ (1) It is Theorem 3.3.
(3) $\Rightarrow$ (9) By Theorem 3.2, $A^T A = AA^T = I$. If $f_A(x) = f_A(y)$, then $Ax = Ay$
and so $x = (A^T A)x = A^T (Ax) = A^T (Ay) = (A^T A)y = y$. Hence $f_A$ is injective mapping.
For any $y \in L^n$, put $x = A^2 y$. Then $x \in L^n$ and $f_A(x) = A(A^2 y) = (AA^T)y = I \cdot y = y$.
Hence $f_A$ is surjective mapping.

(10) $\Rightarrow$ (5) Suppose that (10) holds. Then for any $x, y \in L^n$,
$x^T A^2 Ay = (Ax)^T (Ay) = f_A(x)^T f_A(y) = x^T y$.

Let $A = (a_{ij})$. For any $i \in n$, let $e_i$ be the $i$th unit vector in $L^n$, i.e., the only
$i$-entry of it is 1 and the other entries are 0. For any $i, j \in n$, replace $x$ and $y$ with $e_i$
and $e_j$, respectively, in the expression above. Then we have that
$$\sum_{s \in n} a_{is} a_{js} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$
Hence $A^2 = I$.

(5) $\Rightarrow$ (10) If $A^2 = I$, then for any $x, y \in L^n$, we obtain that
$f_A(x)^T f_A(y) = (Ax)^T (Ay) = x^T A^2 Ay = x^T \cdot y = x^T y$.
Hence (10) holds.

(3) $\Rightarrow$ (11) Let $A = (a_{ij}) \in M_n(L)$ be an invertible matrix. It follows from (4) that
for any $i \in n$, $\sum_{s \in n} a_{is} \geq \sum_{s \in n} a_{is}^2 = 1$, i.e., $\sum_{s \in n} a_{is} = 1$.
Hence for any $i, j \in n$, we have that
$$a_{ij} = a_{ij} \cdot 1 = a_{ij} \left( \sum_{s \in n} a_{is} \right) = \sum_{s \in n} a_{is} a_{js} = \sum_{s \neq j} a_{ij} a_{is} + a_{ij}^2.$$ 
On the other hand, by (5), we see that $\sum_{k \in n} a_{ik} a_{ks} = 0$ for $j \neq s$, i.e., $a_{ik} a_{ks} = 0$ for
all $k \in n$. Therefore, $a_{ij} = a_{ij}^2$, which implies that $a_{ij} \in I(L)$ and $A \in M_n(I(L)).$
By (9), $f_A$ is injective mapping.
(11) \(\Rightarrow\) (12) Since \(A \in M_n(I(L))\) and \(I(L)\) is a distributive lattice by Lemma 2.1, it follows from Lemma 2.4 that \(A\) has index.

(12) \(\Rightarrow\) (7) Let \(A\) has the index \(k\) and the period \(d\). Then \(A^{k+d} = A^k\). Since \(f_A\) is an injective mapping, \(f_{A^k}\) is also an injective mapping. Hence, from \(A^k \cdot A^d = A^{k+d} = A^k = A^k \cdot I\), we have that \(A^d = I\). \(\square\)

Theorem 3.4 generalizes Theorem 1 of Skornyakov [13].

If \(A \in M_n(L)\) is an invertible matrix, then \(f_A\) is an injective mapping. But its converse is not true in general.

**Example 3.1.** Let \(L = ([0, 1], \lor, \times)\), where \(\times\) denotes the ordinary multiplication of numbers. Let \(A = (a_{ij}) \in M_n(L)\) be defined by

\[
a_{ij} := \begin{cases} 0.1, & i = j, \\ 0, & i \neq j. \end{cases}
\]

Then we see that

\[
A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0.1x_1 \\ 0.1x_2 \\ \vdots \\ 0.1x_n \end{pmatrix}.
\]

Hence the mapping \(f_A\) is injective. However, \(AA^T \neq I\), and so \(A\) is not an invertible matrix.

**Corollary 3.1.** If \(A \in M_n(L)\) is invertible, then its inverse is \(A^T\).

**Corollary 3.2** [1]. If \(A, B \in M_n(L)\) and \(AB = I\), then \(B = A^T\) and \(BA = I\).

**Proof.** It follows immediately from Theorem 3.4. \(\square\)

**Corollary 3.3** [13]. If \(D\) is a distributive lattice and \(A = (a_{ij}) \in M_n(D)\), then the following statements are equivalent to each other:

1. \(A\) is right invertible;
2. \(A\) is left invertible;
3. \(A\) is invertible;
4. the rows of \(A\) are orthogonal, i.e., \(\sum_{s \in \mathbb{N}} a_{is} = 1\) for all \(i \in \mathbb{N}\) and \(a_{ik}a_{jk} = 0\) for \(i \neq j\) and \(k \in \mathbb{N}\);
5. the columns of \(A\) are orthogonal;
6. the rows and the columns of \(A\) are orthogonal, respectively;
7. \(A^k = I\) for some positive integer \(k\);
(8) \( f_A \) is a bijective mapping;
(9) \( f_A \) is a surjective mapping;
(10) \( f_A \) is an injective mapping;
(11) \( f_A \) preserves the scalar products.

\[ \text{Proof.} \quad \text{Since } D \text{ is a distributive lattice, } I(D) = D \text{ and } A \text{ has index by Lemma 2.4. Hence, the statement (11) coincides with (12) in Theorem 3.4. This completes the proof.} \]

Zhao [20] proved that the statements (3)–(5) in Corollary 3.3 are equivalent. Give’on [4] proved that the statements (3), (6) and (7) in Corollary 3.3 are equivalent.

4. Permanent and Cramer’s rule over inclines

In this section, we first define the permanent of an \( m \times n \) incline matrix and prove the expansion theorem for it.

**Definition 4.1.** Let \( A = (a_{ij}) \in L^{m \times n} \) and \( m \leq n \). A permanent \( \text{per}(A) \) of \( A \) is defined by

\[
\text{per}(A) := \sum_{\sigma \in \mathcal{S}} \prod_{i \in m} a_{i \sigma(i)},
\]

where \( \mathcal{S} \) is the set of all injective mappings from \( m \) to \( n \).

If \( A \in M_n(L) \), then the permanent \( \text{per}(A) \) is the sum of \( n! \) terms which are all possible products of \( n \) entries of \( A \) taken exactly one in each row and each column.

**Lemma 4.1.** If \( A \in M_n(L) \), then \( \text{per}(A) = \text{per}(A^T) \).

\[ \text{Proof.} \quad \text{Let } A = (a_{ij}). \text{ Consider any term } a_{1 \sigma(1)}a_{2 \sigma(2)} \cdots a_{n \sigma(n)} \text{ of } \text{per}(A). \text{ Since } \sigma(1), \sigma(2), \ldots, \sigma(n) \text{ form an arrangement of the symbols } 1, 2, \ldots, n, \text{ all factors } a_{1 \sigma(1)}, a_{2 \sigma(2)}, \ldots, a_{n \sigma(n)} \text{ stand in the different rows and the different columns of } A^T. \text{ Hence } a_{1 \sigma(1)}a_{2 \sigma(2)} \cdots a_{n \sigma(n)} \text{ serves as a term of } \text{per}(A^T), \text{ too. The converse is also true. Therefore, } \text{per}(A) \text{ and } \text{per}(A^T) \text{ consist of the same terms. This completes the proof.} \]

Lemma 4.1 generalizes (1) of Theorem 1 in Zhang [18] and Property 4 of Yi and Liu [16].

**Lemma 4.2.** Let \( A \in L^{m \times n} \) and \( m \leq n \). If \( P \in M_m(L) \) and \( Q \in M_n(L) \) are the permutation matrices, then \( \text{per}(PAQ) = \text{per}(A) \).
Proof. Let $A = (a_{kj})$ and $P = (p_{ik})$. Suppose that $p_{ik} = 1$ for any $i \in m$, where $k_1, k_2, \ldots, k_m$ are different each other. Then $(i, j)$-entry of $PA$ is $a_{k_j}$. Hence we have that

$$\text{per}(PA) = \sum_{\sigma} \prod_{i \in m} a_{k_{\sigma(k_i)}} = \sum_{\sigma} \prod_{k_i \in m} a_{k_{\sigma(k_i)}} = \text{per}(A).$$

Similarly, it is proved that $\text{per}(AQ) = \text{per}(A)$. Therefore, $\text{per}(PAQ) = \text{per}(AQ) = \text{per}(A)$. □

Lemma 4.2 generalizes Proposition 1.1 of Yi and Liu [16].

If $k \in l$, then $\Omega_{kl}$ denotes the following set of $k$-tuples of integers:

$$\Omega_{kl} := \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \mid 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq l\}.$$

Let $A \in L^{m \times n}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \Omega_{rm}$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_t) \in \Omega_{rn}$. Then

$$A[\alpha|\beta] = A[\alpha_1, \alpha_2, \ldots, \alpha_r|\beta_1, \beta_2, \ldots, \beta_t]$$

denotes the $r \times t$ submatrix of $A$ whose $(i, j)$-entry is equal to $a_{\alpha_i\beta_j}$ ($i \in r$, $j \in t$), and

$$A(\alpha|\beta) = A(\alpha_1, \alpha_2, \ldots, \alpha_r|\beta_1, \beta_2, \ldots, \beta_t)$$

denotes the $(m - r) \times (n - t)$ submatrix of $A$ obtained from $A$ by deleting the rows and the columns where the entries of $A[\alpha|\beta]$ stand. $A(\alpha|\beta)$ is called a complementary submatrix of $A[\alpha|\beta]$.

Definition 4.2. Let $A \in M_n(L)$. An adjoint matrix $\text{adj}(A) \in M_n(L)$ of $A$ is the matrix whose $(i, j)$-entry is $\text{per}(A(j|i))$ for all $i, j \in n$.

The following is the expansion theorem for the permanent of incline matrices.

Lemma 4.3. Let $A = (a_{ij}) \in L^{m \times n}$ and $m \leq n$.

1. If $\alpha \in \Omega_{rm}$ ($r < m$), then

$$\text{per}(A) = \sum_{\beta \in \Omega_{rn}} \text{per}(A[\alpha|\beta]) \cdot \text{per}(A(\alpha|\beta)).$$

2. For any $i \in m$,

$$\text{per}(A) = \sum_{j \in n} a_{ij} \cdot \text{per}(A(i|j)).$$

3. If $m = n$, then for any $j \in n$,

$$\text{per}(A) = \sum_{i \in n} a_{ij} \cdot \text{per}(A(i|j)).$$
Proof. (1) Put \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \). For a fixed \( \beta = (\beta_1, \beta_2, \ldots, \beta_r) \) \( \in \Omega_{m-r} \), we have that

\[
\text{per}(A[\alpha | \beta]) = \sum_{\sigma_1 \in S_1} \prod_{i \in r} a_{\alpha_i \sigma_1(\alpha_i)},
\]

where \( S_1 \) is the set of all injective mappings from \( \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \) to \( \{\beta_1, \beta_2, \ldots, \beta_r\} \). Hence \( \text{per}(A[\alpha | \beta]) \) consists of \( r! \) terms. On the other hand,

\[
\text{per}(A(\alpha | \beta)) = \sum_{\sigma_2 \in S_2} \prod_{j \in m} a_{\alpha_j \sigma_2(\alpha_j)},
\]

where \( \{\alpha_{r+1}, \alpha_{r+2}, \ldots, \alpha_m\} = m \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \) and \( S_2 \) is the set of all injective mappings from \( \{\alpha_{r+1}, \alpha_{r+2}, \ldots, \alpha_m\} \) into \( n \setminus \{\beta_1, \beta_2, \ldots, \beta_r\} \). Hence \( \text{per}(A(\alpha | \beta)) \) has \( (m-r)!\binom{m-r}{n-r} \) terms. The product of a term \( \prod_{i \in r} a_{\alpha_i \sigma_1(\alpha_i)} \) of \( \text{per}(A[\alpha | \beta]) \) by a term \( \prod_{j \in m} a_{\alpha_j \sigma_2(\alpha_j)} \) of \( \text{per}(A(\alpha | \beta)) \) is a term of \( \text{per}(A) \). Thus, \( \text{per}(A[\alpha | \beta]) \cdot \text{per}(A(\alpha | \beta)) \) is a sum of \( r!(m-r)!\binom{m-r}{n-r} \) distinct terms of \( \text{per}(A) \). Furthermore, for the different \( \beta \) and \( \omega \) in \( \Omega_{m-r} \), the terms of \( \text{per}(A[\alpha | \beta]) \cdot \text{per}(A(\alpha | \beta)) \) and the terms of \( \text{per}(A[\alpha | \omega]) \cdot \text{per}(A(\alpha | \omega)) \) are distinguished. However, \( |\Omega_{m-r}| = C_n^r \). Hence \( \sum_{\beta \in \Omega_{m-r}} \text{per}(A[\alpha | \beta]) \cdot \text{per}(A(\alpha | \beta)) \) is the sum of \( C_n^r \cdot r!(m-r)!\binom{m-r}{n-r} = m!C_n^m \) terms of \( \text{per}(A) \), and so it is equal to \( \text{per}(A) \) since the total number of the terms of \( \text{per}(A) \) is just \( m!C_n^m \).

(2) It is a corollary of (1).

(3) If \( m = n \), then \( \text{per}(A) = \text{per}(A^T) \) by Lemma 4.1 and the columns of \( A \) are the rows in \( A^T \). By applying (2) into \( \text{per}(A^T) \), we obtain the expansion for \( \text{per}(A) \) by the column. \( \square \)

(1) of Lemma 4.3 generalizes Theorem 2.1 of Yi and Liu [16]. (2) of Lemma 4.3 generalizes (1) of Theorem 2 in Zhang [18].

Corollary 4.1. Let \( A \in L_{m \times n} \) (\( m \leq n \)) have the following form:

\[
A = \begin{pmatrix} B & C \\ O & D \end{pmatrix},
\]

where \( D \) is the square matrix. Then \( \text{per}(A) = \text{per}(B) \cdot \text{per}(D) \).

Proof. Expand \( \text{per}(A) \) by the rows in which the submatrix \( D \) stands. Then the conclusion follows from (1) of Lemma 4.3. \( \square \)

Corollary 4.1 generalizes (5) of Theorem 1 in Zhang [18]. Corollary 4.1 corrects and generalizes Corollary 2.2 of Yi and Liu [16].

By induction and Corollary 4.1, we obtain the following result.
Corollary 4.2. Let \( A \in L_{m \times n}^n (m \leq n) \) have the following form:

\[
A = \begin{pmatrix}
    A_{11} & A_{12} & \cdots & A_{1r} \\
    O & A_{22} & \cdots & A_{2r} \\
    \vdots & \vdots & \ddots & \vdots \\
    O & O & \cdots & A_{rr}
\end{pmatrix},
\]

where \( A_{ii} (2 \leq i \leq r) \) are the square matrices. Then \( \text{per}(A) = \prod_{i \in I} \text{per}(A_{ii}) \).

Corollary 4.2 corrects and generalizes Corollary 2.3 of Yi and Liu [16].

The next example shows that in Corollary 4.1, the condition that submatrix \( D \) is the square matrix cannot be omitted.

Example 4.1. Let \( L = ([0, 1], \vee, \wedge) \) and let \( A \) be as follows:

\[
A = \begin{pmatrix}
    0.6 & 0.5 & 1 \\
    0 & 1 & 0.7
\end{pmatrix}.
\]

Put \( B = (0.6), \ C = (0.5, 1) \) and \( D = (1, 0.7) \). Then \( \text{per}(A) = 1, \ \text{per}(B) = 0.6 \) and \( \text{per}(D) = 1 \). Hence we have that

\[
\text{per}(B) \wedge \text{per}(D) = 0.6 \wedge 1 \neq 1 = \text{per}(A).
\]

We next show that the permanent of the invertible incline matrix is equal to 1 and establish Cramer’s rule over inclines.

Theorem 4.1. If \( A \in M_n(L) \) is invertible, then \( \text{per}(A) = 1 \).

Proof. By Theorem 3.2, \( AA^T = I \). Let \( A = (a_{ij}) \). Then for any \( i \in N \), we have that

\[
\sum_{s \in N} a_{is} = 1 \quad \text{and} \quad a_{ij} a_{jk} = 0 \quad \text{for} \quad j \neq k.
\]

If \( t = j \), then every term of \( \text{per}(A(i/t)) \) has a factor which is an entry of the \( j \)-th column of \( A \) and is different from \( a_{ij} \). Hence we see that

\[
1 = \prod_{i \in N} 1 = \prod_{i \in N} \left( \sum_{s \in N} a_{is} \right) = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},
\]

where \( S_n \) is the set of all permutations of the symbols 1, 2, \ldots, \( n \). Therefore, \( 1 = \text{per}(A) \). \( \square \)

Theorem 4.1 generalizes (1) of Lemma 2.10 in Tan [14].

Theorem 4.2. If \( A \in M_n(L) \) is invertible, then \( \text{adj}(A) = A^T \).

Proof. By Theorem 3.2, \( AA^T = A^T A = I \). Hence for any \( i \in N \), we have that

\[
\sum_{s \in N} a_{is} = 1 \quad \text{and} \quad a_{ij} a_{ik} = 0 \quad \text{for} \quad j \neq k.
\]

If \( t \neq j \), then every term of \( \text{per}(A(i/t)) \) has a factor which is an entry of the \( j \)-th column of \( A \) and is different from \( a_{ij} \). Hence we see that

\[
1 = \prod_{i \in N} 1 = \prod_{i \in N} \left( \sum_{s \in N} a_{is} \right) = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},
\]

where \( S_n \) is the set of all permutations of the symbols 1, 2, \ldots, \( n \). Therefore, \( 1 = \text{per}(A) \). \( \square \)
from $a_{ij}$. Hence we see that $a_{ij} \cdot \text{per}(A(i|t)) = 0$. By Theorem 4.1 and (2) of Lemma 4.3, we obtain that

$$1 = \text{per}(A) = \sum_{j \in \mathbb{N}} a_{ij} \cdot \text{per}(A(i|j)) \leq \sum_{j \in \mathbb{N}} \text{per}(A(i|j)),$$

i.e., $\sum_{j \in \mathbb{N}} \text{per}(A(i|j)) = 1$. Therefore, we have that

$$a_{ij} = a_{ij} \cdot 1 = a_{ij} \left( \sum_{t \in \mathbb{N}} \text{per}(A(i|t)) \right) = \sum_{t \in \mathbb{N}} a_{ij} \text{per}(A(i|t)) = a_{ij} \text{per}(A(i|j)) + \sum_{t \neq j} a_{it} \text{per}(A(i|j))$$

$$= \sum_{t \in \mathbb{N}} a_{it} \text{per}(A(i|j)) = \left( \sum_{t \in \mathbb{N}} a_{it} \right) \cdot \text{per}(A(i|j)) = 1 \cdot \text{per}(A(i|j)) = \text{per}(A(i|j)),$$

and so $a_{ij} = \text{per}(A(i|j))$ for all $i, j \in \mathbb{N}$. This means that $A^T = \text{adj}(A)$. □

Theorem 4.2 generalizes (2) of Lemma 2.10 in Tan [14].

The following is the Cramer’s rule for a matrix equation over an incline.

**Theorem 4.3.** Let $A = (a_{ij}) \in M_n(L)$ and $b = (b_i) \in L^n$. If $A$ is invertible, then the matrix equation $Ax = b$ has a unique solution $A^Tb = (d_j) \in L^n$, where $d_j$ ($j \in \mathbb{N}$) is the permanent of $n \times n$ matrix formed from $A$ by replacing the $j$th column of $A$ with the vector $b_i$.

**Proof.** By Theorem 3.2, $AA^T = A^TA = I$. Hence $A(A^Tb) = (AA^T)b = I \cdot b = b$, which implies that the equation $Ax = b$ has a solution $A^Tb$. Let $y$ be any solution of it. Then $Ay = b$, and so $y = I \cdot y = (A^TA)y = A^T(Ay) = A^Tb$, which means that the equation $Ax = b$ has a unique solution.

Now let $(d_j) = A^Tb$. By Theorem 4.2, for any $j \in \mathbb{N}$, we see that

$$d_j = \sum_{i \in \mathbb{N}} a_{ij} b_i = \sum_{i \in \mathbb{N}} \text{per}(A(i|j)) \cdot b_i.$$
On the other hand, we have that
\[
\text{per} \left( \begin{array}{cccc}
a_{11} & a_{12} & \cdots & b_1 \\
a_{21} & a_{22} & \cdots & b_2 \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & b_n
\end{array} \right) = \sum_{i \in \mathbb{N}} b_i \cdot \text{per}(A(i|j))
\]
by applying (3) of Lemma 4.3 into the expansion by the \( j \)th column. Hence the conclusion holds. \( \square \)

Theorem 4.3 generalizes a theorem of Zarko [17]. Theorem 4, Corollaries 7 and 8 of Tian et al. [15] are the corollaries of Theorem 4.1.

5. Group of invertible incline matrices

In this section, we study some properties of the group of all \( n \times n \) invertible incline matrices.

Denote by \( G_n(L) \) the set of all \( n \times n \) invertible matrices over an incline \( L \), and denote by \( P_n(L) \) the set of all \( n \times n \) permutation matrices over \( L \). It is clear that \( G_n(L) \) forms a subgroup of the monoid \( M_n(L) \) and \( P_n(L) \) is a subgroup of \( G_n(L) \).

Obviously, \( P_n(L) \cong S_n \), where \( S_n \) is the symmetric group of degree \( n \).

Theorem 5.1. If \( H \) is a subsemigroup of \( G_n(L) \), then \( H \) is a subgroup of \( G_n(L) \).

Proof. Let \( A, B \in H \). By (7) of Theorem 3.4, there is an integer \( l \) such that \( l \geq 2 \) and \( A^l = I \). So \( A^{-1} = A^{l-1} \in H \). Hence \( A^{-1}B \in H \). This means that \( H \) is a subgroup of \( G_n(L) \). \( \square \)

Theorem 5.1 generalizes Lemma 2.5 of Tan [14].

Theorem 5.2. The following statements are equivalent to each other:

1. An incline \( L \) is an integral incline;
2. \( G_n(L) = P_n(L) \) for any \( n (n \geq 2) \);
3. \( G_n(L) = P_n(L) \) for some \( n (n \geq 2) \).

Proof. (1) \( \Rightarrow \) (2) Suppose that \( G_n(L) \neq P_n(L) \) for some \( n (n \geq 2) \). Let \( A = (a_{ij}) \in G_n(L) \setminus P_n(L) \). Then there is an entry \( a_{ij} \) of \( A \) such that \( a_{ij} \neq 0 \) and \( a_{ij} \neq 1 \). Put \( b = \sum_{s \neq j} a_{is} \). Then by (4) of Theorem 3.4, we have that \( a_{ij} + b = \sum_{s \in \mathbb{N}} a_{is} = 1 \). Thus \( b \neq 0 \), and by (5) of Theorem 3.4, we obtain that
\[
a_{ij} \cdot b = a_{ij} \cdot \left( \sum_{s \neq j} a_{is} \right) = \sum_{s \neq j} a_{ij}a_{is} = 0.
\]
This means that \( L \) is not an integral incline.

(2) \( \Rightarrow \) (3) It is trivial.

(3) \( \Rightarrow \) (1) Suppose that \( L \) is not integral. Then there exist nonzero elements \( a \) and \( b \) in \( L \) such that \( a + b = 1 \) and \( ab = 0 \). Then \( a \neq 1 \) and \( b \neq 1 \). In fact, if \( a = 1 \), then \( b = 1 \cdot b = ab = 0 \). If \( b = 1 \), then \( a = a \cdot 1 = ab = 0 \).

And we have that \( a, b \in I(L) \) since
\[
\begin{align*}
a &= a \cdot 1 = a(a + b) = a^2 + ab = a^2 + 0 = a^2, \\
b &= 1 \cdot b = (a + b)b = ab + b^2 = 0 + b^2 = b^2.
\end{align*}
\]

Let \( n \geq 2 \) and define \( A \in M_n(L) \) as follows:
\[
A = \begin{pmatrix} B & O \\ O & I_{n-2} \end{pmatrix},
\]
where
\[
B = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.
\]

Since \( a^2 + b^2 = a + b = 1 \) and \( ab + ba = 0 \), we can see that \( AA^T = I \). By Theorem 3.4, \( A \) is invertible, i.e., \( A \in G_n(L) \). However, \( A \) is not a permutation matrix. Hence \( A \in G_n(L) \setminus P_n(L) \neq \emptyset \). This is a contradiction. 

**Corollary 5.1.** If \( L \) is not an integral incline, then \( G_n(L) \neq P_n(L) \) for all \( n \geq 2 \).

**Proof.** It follows immediately from Theorem 5.2. 

**Corollary 5.2** [1]. In an integral incline \( L \), if \( XY = I \), then \( X \) and \( Y \) are the permutation matrices.

**Proof.** It follows from Theorems 3.4 and 5.2. 

**Corollary 5.3.** If \( D \) is a distributive lattice, then the following statements are equivalent:

(1) \( D \) is anti-diamond-type [12];
(2) \( G_n(D) = P_n(D) \) for all \( n \geq 2 \);
(3) \( G_n(D) = P_n(D) \) for some \( n \geq 2 \).

**Proof.** A distributive lattice is anti-diamond-type if and only if it is integral as an incline. Hence the conclusion holds by Theorem 5.2. 

Theorem 5.2 and Corollary 5.3 generalize and develop Theorem 2 of Skornyakov [13].

**Theorem 5.3.** $P_2(L)$ is a normal subgroup of $G_2(L)$.

**Proof.** It is easy to see that 

$$P_2(L) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$ 

Let 

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_2(L).$$

By (6) of Theorem 3.4, $AA^T = A^T A = I$ and $A^{-1} = A^T$. Hence $a + b = c + d = 1$ and $ac = bd = ab = cd = 0$, so 

$$a = a(c + d) = ac + ad = ad + bd = (a + b)d = d,$$

$$b = b(c + d) = bc + bd = bc = ac + bc = (a + b)c = c.$$ 

Thus $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$. We now show that $A \cdot P_2(L) \cdot A^{-1} \subseteq P_2(L)$. In fact, we have that 

$$A \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A^{-1} = AA^{-1} = AA^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in P_2(L)$$

and similarly, 

$$A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A^{-1} = \begin{pmatrix} ba + ab & b^2 + a^2 \\ a^2 + b^2 & ab + ba \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in P_2(L).$$

This completes the proof. □

The following example shows that there is an incline $L$ such that $P_n(L)$ is not a normal subgroup of $G_n(L)$ for $n \geq 3$.

**Example 5.1.** Let $D = \{a, b, c\}$ and $L = 2^D$. Then $L$ forms an incline (a distributive lattice) with respect to $\cup$ and $\cap$. Put 

$$A = \begin{pmatrix} \{a\} & \{b\} & \{c\} \\ \{b\} & \{c\} & \{a\} \\ \{c\} & \{a\} & \{b\} \end{pmatrix}, \quad P = \begin{pmatrix} \emptyset & D & \emptyset \\ D & \emptyset & \emptyset \\ \emptyset & \emptyset & D \end{pmatrix}.$$ 

By Theorem 3.4, $A \in G_3(L)$ and $P \in P_3(L)$. And we have that 

$$APA^{-1} = APA^T = APA = \begin{pmatrix} \{c\} & \{b\} & \{a\} \\ \{b\} & \{a\} & \{c\} \\ \{a\} & \{c\} & \{b\} \end{pmatrix} \notin P_3(L).$$
Hence $P_3(L)$ is not a normal subgroup of $G_3(L)$. If $n \geq 4$, then consider

$$
B = \begin{pmatrix} A & O \\ O & I_{n-3} \end{pmatrix}, \quad Q = \begin{pmatrix} P & O \\ O & I_{n-3} \end{pmatrix}.
$$

Then $B \in G_n(L)$ and $Q \in P_n(L)$. However,

$$
BQB^{-1} = BQB^T = BQB = \begin{pmatrix} AP & O \\ O & I_{n-3} \end{pmatrix} \notin P_n(L).
$$

Therefore, $P_n(L)$ is not a normal subgroup of $G_n(L)$.

6. Conclusions

In this paper, we gave the complete description of the invertible matrices over commutative inclines. Some necessary and sufficient conditions for an incline matrix to be invertible were studied, Cramer’s rule over inclines was established and the group of invertible incline matrices was investigated. The main results in the present paper are the generalizations of the corresponding results in the literatures for the Boolean matrices, the fuzzy matrices and the lattice matrices.

It is clear that if $T$ is a noncommutative t-norm [3], then the fuzzy algebra $(\mathbb{R}, \vee, T)$ constitutes a noncommutative incline [1,10]. We believe that our techniques can be applied to study the invertible matrices over noncommutative inclines and many results above can be further generalized to the setting of noncommutative inclines.

References