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Bull. Sci. math. 135 (2011) 442–466

BULLETIN DES SCIENCES MATHÉMATIQUES

www.elsevier.com/locate/bulsci

General existence results for reflected BSDE and BSDE ☆

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Received 6 February 2011

Available online 22 April 2011

Abstract

In this paper, we are concerned with the problem of existence of solutions for generalized reflected backward stochastic differential equations (GRBSDEs for short) and generalized backward stochastic differential equations (GBSDEs for short) when the generator $f\,ds + g\,dA_s$ is continuous with general growth with respect to the variable y and stochastic quadratic growth with respect to the variable z. We deal with the case of a bounded terminal condition ξ and a bounded barrier L as well as the case of unbounded ones. This is done by using the notion of generalized BSDEs with two reflecting barriers studied in Essaky and Hassani (submitted for publication) [14]. The work is suggested by the interest the results might have in finance, control and game theory.

MSC: 60H10; 60H20

Keywords: Generalized reflected BSDE; Generalized BSDE; Stochastic quadratic growth; Itô–Tanaka formula

1. Introduction

Originally motivated by questions arising in stochastic control theory, backward stochastic differential equations have found important applications in fields as stochastic control, mathematical finance, Dynkin games and the second order PDE theory (see, for example, [11,16,25,24,8,9] and the references therein).

☆ This work is supported by Hassan II Academy of Science and Technology and Action Intégrée MA/10/224.

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0007-4497/$ – see front matter
doi:10.1016/j.bulsci.2011.04.003

The particular case of linear BSDEs has appeared long time ago both as the equations for the adjoint process in stochastic control, as well as the model behind the Black and Scholes formula for the pricing and hedging of options in mathematical finance. However the notion of nonlinear BSDEs has been introduced in 1990 by Pardoux and Peng [24]. A solution for such an equation is a couple of adapted processes (Y, Z) with values in $\mathbb{R} \times \mathbb{R}^d$ satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dB_s, \quad 0 \leqslant t \leqslant T. \tag{1}$$

In [24], the authors have proved the existence and uniqueness of the solution under conditions including basically the Lipschitz continuity of the generator f.

From the beginning, many authors attempted to improve the result of [24] by weakening the Lipschitz continuity of the coefficient f, see e.g. [1–5,10,15,17,18,22,7], or the L^2-integrability of the initial data ξ, see [11,5].

When the generator f is only continuous there exists a solution to Eq. (1) under one of the following group of conditions:

- ξ is square integrable and f has a uniform linear growth in y, z (see Lepeltier and San Martin [21]).
- ξ is bounded and f has a superlinear growth in y and quadratic growth in z, i.e. there exist a positive constant C and a positive function ϕ, such that
$$\left|f(t, \omega, y, z)\right| \leqslant \phi(y) + C|z|^2,$$
where $\int_0^{+\infty} \frac{ds}{\phi(s)} = \int_{-\infty}^0 \frac{ds}{\phi(s)} = \infty$ (see Lepeltier and San Martin [20]; Kobylanski [18]).
- ξ is bounded and f satisfies the following condition
$$\left|f(t, \omega, y, z)\right| \leqslant C + R_t|z| + \frac{1}{2}|z|^2,$$
where C is a positive constant and R is a square integrable process with respect to the measure $dt\, dP$ (see Hamadène and El Karoui [13]).
- There exist two constants $\beta \geqslant 0$ and $\gamma > 0$ together with a progressively measurable non-negative stochastic process $\{\alpha(t)\}_{t \leqslant T}$ and a deterministic continuous nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that, P-a.s.,
  (i) for each (t, y, z), $y(f(t, y, z) - f(t, 0, z)) \leqslant \beta|y|^2$,
  (ii) for each (t, y, z), $|f(t, \omega, y, z)| \leqslant \alpha(t) + \varphi(|y|) + \frac{\gamma}{2}|z|^2$,
  (iii) $\mathbb{E}e^{\gamma e^{\beta T}(|\xi| + \int_0^T \alpha(s)\, ds)} < +\infty$
  (see Briand and Hu [6]).

The notion of reflected BSDE has been introduced by El Karoui et al. [12]. A solution of such an equation, associated with a coefficient f; a terminal value ξ and a barrier L, is a triple of processes (Y, Z, K) with values in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+$ satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + K_T - K_t - \int_t^T Z_s\, dB_s, \qquad Y_t \geqslant L_t, \quad \forall t \leqslant T. \tag{2}$$

Here the additional process K is continuous nondecreasing and its role is to push upwards the process Y in order to keep it above the barrier L and moreover it satisfies $\int_0^T (Y_s - L_s)\, dK_s = 0$, this means that the process K acts only when the process Y reaches the barrier L. Once more under square integrability of the terminal condition ξ and the barrier L and Lipschitz property of the coefficient f, the authors have proved that Eq. (2) has a unique solution.

When the generator f is only continuous there exists a solution to Eq. (2) under one of the following group of conditions:

- ξ and L are square integrable and f has a uniform linear growth in y, z (see Matoussi [23]).
- ξ and L are bounded and f has a superlinear growth in y and quadratic growth in z, i.e. there exist a positive constant C and a positive function ϕ, such that

$$\left|f(t,\omega,y,z)\right| \leqslant \phi(y) + C|z|^2,$$

where $\int_0^{+\infty} \frac{ds}{\phi(s)} = \int_{-\infty}^0 \frac{ds}{\phi(s)} = \infty$ (see Kobylanski, Lepeltier, Quenez and Torres [19]).

We should point out here that, in the previous works, the existence of a solution for RBSDE or BSDE has been proved in the case when the quadratic condition imposed on the coefficient f is uniform in ω and hence those works cannot cover, for example, a generator with stochastic quadratic growth of the form $C_s(\omega)\psi(|y|)|z|^2$. Moreover, most of the previous works require that the terminal condition ξ and the barrier L are bounded random variables in the case of GRBSDEs or ξ is bounded in the case of GBSDEs. These conditions on f, ξ and L seem to be restrictive and are not necessary to have a solution.

One of the main purposes of this work is to study the GRBSDE with one barrier L which is a reflected BSDE which involves an integral with respect to a continuous and increasing process A of the form:

$$\begin{cases} \text{(i) } Y_t = \xi + \int_t^T f(s,Y_s,Z_s)\,ds + \int_t^T g(s,Y_s)\,dA_s + \int_t^T dK_s - \int_t^T Z_s\,dB_s, \quad t \leqslant T, \\ \text{(ii) } \forall t \leqslant T, \quad L_t \leqslant Y_t, \\ \text{(iii) } \int_0^T (Y_t - L_t)\,dK_t = 0, \quad \text{a.s.}, \\ \text{(iv) } Y \in \mathcal{C}, \quad K \in \mathcal{K}, \quad Z \in \mathcal{L}^{2,d}. \end{cases} \tag{3}$$

We prove existence of solutions for GRBSDE (3) when the generator $f\,ds + g\,dA_s$ is continuous with general growth with respect to the variable y and stochastic quadratic growth with respect to the variable z. This allows us to cover some BSDEs having a generator satisfying, for example, the following condition: for each (s,ω,y,z)

$$\left|f(s,\omega,y,z)\right| \leqslant \alpha_s\phi\big(|y|\big) + \frac{C_s\psi(|y|)}{2}|z|^2 + R_s|z|,$$

$$\left|g(s,\omega,y)\right| \leqslant \beta_s\phi\big(|y|\big),$$

where α, ϕ, C, ψ, R and β are given later. We deal with the case of a bounded terminal condition ξ and a bounded barrier L as well as the case of unbounded ones. We give some examples which are covered by our result and, to our knowledge, not covered by the previous works. Moreover, as we will see later, the existence of a solution for our GRBSDE (3) is related to the existence of a solution (x, z, k) for the following BSDE:

$$\begin{cases} x_t = \xi \vee \sup\limits_{s \leqslant T} L_s + \int_t^T \phi(x_s)\, d\eta_s + \int_t^T \frac{C_s \psi(x_s)}{2} |z_s|^2\, ds \\ \qquad + \int_t^T R_s |z_s|\, ds + \int_t^T dk_s - \int_t^T z_s\, dB_s, \\ x_s \geqslant 0, \quad \forall s \leqslant T, \quad k \in \mathcal{K}, \quad z \in \mathcal{L}^{2,d}. \end{cases} \tag{4}$$

Roughly speaking, we prove that if the BSDE (4) has a solution and the coefficient $f\, ds + g\, dA_s$ is continuous with general growth with respect to the variable y and stochastic quadratic growth with respect to the variable z (see condition (H.2) below), then the GRBSDE (3) has a solution. Therefore a natural question arises: under which condition on $(\xi, L, \phi, \psi, C, \eta)$, the BSDE (4) has a solution? This is the second purpose of this work.

The third purpose of this work is to prove the existence of solutions for the GRBSDE (3) when the barrier $L \equiv -\infty$ which is nothing else than a GBSDE of the form:

$$\begin{cases} \text{(i) } Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s)\, dA_s - \int_t^T Z_s\, dB_s, \quad t \leqslant T, \\ \text{(ii) } Y \in \mathcal{C}, \quad Z \in \mathcal{L}^{2,d}. \end{cases} \tag{5}$$

As a very particular case of our result, when ξ is not bounded, we obtain that the following BSDE

$$Y_t = \xi + \int_t^T \frac{\gamma_s}{2} |Z_s|^2\, ds - \int_t^T Z_s\, dB_s \tag{6}$$

has a solution if

$$\mathbb{E}\left[\frac{e^{C_T|\xi|} - 1}{C_T} 1_{\{C_T > 0\}} + |\xi| 1_{\{C_T = 0\}}\right] < +\infty,$$

where γ be a nonnegative process which is $\mathcal{F}_t$-adapted and $C_t = \sup_{0 \leqslant s \leqslant t} |\gamma_s|$, $\forall t \in [0, T]$. Moreover

$$|Y_t| \leqslant \frac{\ln(1 + C_t \mathbb{E}(\overline{\Lambda}|\mathcal{F}_t))}{C_t} 1_{\{C_t > 0\}} + \mathbb{E}(\overline{\Lambda}|\mathcal{F}_t) 1_{\{C_t = 0\}},$$

where $\overline{\Lambda} = \frac{e^{C_T|\xi|} - 1}{C_T} 1_{\{C_T > 0\}} + |\xi| 1_{\{C_T = 0\}}$.

To prove our results, we will use an approach based upon the recent result obtained in the preprint of Essaky and Hassani [14] where the authors have proved the existence of a solution for a generalized BSDE with two reflecting barriers when the generator $f\, ds + g\, dA_s$ is continuous with general growth with respect to the variable y and stochastic quadratic growth with respect to the variable z and without assuming any P-integrability conditions on the data. This result allows a simple treatment of the problem of existence of solutions for one barrier reflected BSDEs and also for BSDEs without reflection. This approach seems to be new.

Let us describe our plan. First, some notation is fixed in Section 2. In Section 3, we recall the existence of solutions for GBSDE with two reflecting barriers studied in [14]. Section 4 is devoted to the proof of a general existence result for GRBSDE and GBSDE when the coefficient $f\,ds + g\,dA_s$ is continuous with general growth with respect to the variable y and stochastic quadratic growth with respect to the variable z. In Section 5, we give sufficient conditions under which the BSDE (4) has a solution. In Section 6, we give some important consequences and examples of our results.

2. Notations

The purpose of this section is to introduce some basic notations, which will be needed throughout this paper.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\leqslant T}, P)$ be a stochastic basis on which is defined a Brownian motion $(B_t)_{t\leqslant T}$ such that $(\mathcal{F}_t)_{t\leqslant T}$ is the natural filtration of $(B_t)_{t\leqslant T}$ and $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. Note that $(\mathcal{F}_t)_{t\leqslant T}$ satisfies the usual conditions, i.e. it is right continuous and complete.

Let us now introduce the following notation. We denote:

- $\mathcal{P}$ to be the sigma algebra of $\mathcal{F}_t$-progressively measurable sets on $\Omega \times [0, T]$.
- $\mathcal{C}$ to be the set of $\mathbb{R}$-valued $\mathcal{P}$-measurable continuous processes $(Y_t)_{t\leqslant T}$.
- $\mathcal{L}^{2,d}$ to be the set of $\mathbb{R}^d$-valued and $\mathcal{P}$-measurable processes $(Z_t)_{t\leqslant T}$ such that

$$\int_0^T |Z_s|^2\,ds < \infty, \quad P\text{-a.s.}$$

- $\mathcal{K}$ to be the set of $\mathcal{P}$-measurable continuous nondecreasing processes $(K_t)_{t\leqslant T}$ such that $K_0 = 0$ and $K_T < +\infty$, P-a.s.

  The following assumptions will be needed throughout the paper:
- ξ is an $\mathcal{F}_T$-measurable one-dimensional random variable.
- $f : [0, T] \times \Omega \times \mathbb{R}^{1+d} \to \mathbb{R}$ is a function which to (t, ω, y, z) associates $f(t, \omega, y, z)$ which is continuous with respect to (y, z) and $\mathcal{P}$-measurable.
- $g : [0, T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a function which to (t, ω, y) associates $g(t, \omega, y)$ which is continuous with respect to y and $\mathcal{P}$-measurable.
- A is a process in $\mathcal{K}$.
- $L := \{L_t,\ 0 \leqslant t \leqslant T\}$ is a real valued barrier which is $\mathcal{P}$-measurable and continuous process such that $\xi \geqslant L_T$.

3. Generalized BSDE with two reflecting barriers

In view of clarifying this issue, we recall some results concerning GRBSDEs with two barriers which shall play a central role in our proofs. Let us start by recalling the following definition of two singular measures.

Definition 3.1. Let μ_1 and μ_2 be two positive measures defined on a measurable space (Λ, Σ), we say that μ_1 and μ_2 are singular if there exist two disjoint sets A and B in Σ whose union is Λ such that μ_1 is zero on all measurable subsets of B while μ_2 is zero on all measurable subsets of A. This is denoted by $\mu_1 \perp \mu_2$.

Let us now define the notion of solution of the GRBSDE with two obstacles L and U.

Definition 3.2. We call $(Y, Z, K^+, K^-) := (Y_t, Z_t, K_t^+, K_t^-)_{t\leqslant T}$ a solution of the generalized reflected BSDE, associated with coefficient $f\,ds + g\,dA_s$; terminal value ξ and barriers L and U, if the following hold:

$$\begin{cases} \text{(i) } Y_t = \xi + \displaystyle\int_t^T f(s, Y_s, Z_s)\,ds + \int_t^T g(s, Y_s)\,dA_s \\ \qquad\qquad + \displaystyle\int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s\,dB_s, \quad t \leqslant T, \\ \text{(ii) } Y \text{ is between } L \text{ and } U, \text{ i.e. } \forall t \leqslant T,\ L_t \leqslant Y_t \leqslant U_t, \\ \text{(iii) the Skorohod conditions hold:} \\ \quad \displaystyle\int_0^T (Y_t - L_t)\,dK_t^+ = \int_0^T (U_t - Y_t)\,dK_t^- = 0, \quad \text{a.s.}, \\ \text{(iv) } Y \in \mathcal{C}, \quad K^+, K^- \in \mathcal{K}, \quad Z \in \mathcal{L}^{2,d}, \\ \text{(v) } dK^+ \perp dK^-. \end{cases} \tag{7}$$

We introduce also the following assumptions:

(A.0) $U_t := U_0 - V_t - \int_0^t \rho_s\,ds - \int_0^t \theta_s\,dA_s + \int_0^t \chi_s\,dB_s$, with $U_0 \in \mathbb{R}$, $V \in \mathcal{K}$, $\chi \in \mathcal{L}^{2,d}$, ρ and θ are nonnegative predictable processes satisfying $\int_0^T \rho_s\,ds + \int_0^T \theta_s\,dA_s < +\infty$ P-a.s., such that $L_t \leqslant U_t$, $\forall t \in [0, T]$ and $\xi \leqslant U_T$.

(A.1) There exist two processes $\eta' \in L^0(\Omega, L^1([0, T], ds, \mathbb{R}_+))$ and $C' \in \mathcal{C}$ such that

$$\forall(s, \omega), \quad \left|f(s, \omega, y, z)\right| \leqslant \eta'_s(\omega) + \frac{C'_s(\omega)}{2}|z|^2, \quad \forall y \in \left[L_s(\omega), U_s(\omega)\right],\ \forall z \in \mathbb{R}^d.$$

(A.2) There exists a process $\eta'' \in L^0(\Omega, L^1([0, T], dA_s, \mathbb{R}_+))$ such that

$$\forall(s, \omega), \quad \left|g(s, \omega, y)\right| \leqslant \eta''_s, \quad \forall y \in \left[L_s(\omega), U_s(\omega)\right].$$

The following result is obtained by Essaky and Hassani [14] and it is related to the existence of maximal (resp. minimal) solution of (7), that is, there exists a quadruple $(Y_t, Z_t, K_t^+, K_t^-)_{t\leqslant T}$ which satisfies (7) and if in addition $(Y'_t, Z'_t, K'^+_t, K'^-_t)_{t\leqslant T}$ is another solution of (7), then P-a.s. holds, for all $t \leqslant T$, $Y'_t \leqslant Y_t$ (resp. $Y'_t \geqslant Y_t$).

Theorem 3.1. *Let assumptions* (A.0)–(A.2) *hold true. Then there exists a maximal* (*resp. minimal*) *solution for GRBSDE with two barriers* (7). *Moreover for all solutions* (Y, Z, K^+, K^-) *of Eq.* (7) *we have*

$$dK_s^- \leqslant \left(f(s, U_s, \chi_s) - \rho_s\right)^+ ds + \left(g(s, U_s) - \theta_s\right)^+ dA_s. \tag{8}$$

Furthermore, if the following condition holds

$$L_t := L_0 + \overline{V}_t + \int_0^t \overline{\rho}_s\,ds + \int_0^t \overline{\theta}_s\,dA_s + \int_0^t \overline{\chi}_s\,dB_s,$$

with $L_0 \in \mathbb{R}$, $\overline{V} \in \mathcal{K}$, $\overline{\chi} \in \mathcal{L}^{2,d}$, $\bar{\rho}$ *and* $\bar{\theta}$ *are nonnegative predictable processes satisfying* $\int_0^T \bar{\rho}_s\,ds + \int_0^T \bar{\theta}_s\,dA_s < +\infty$ P*-a.s., then*

$$dK_s^+ \leqslant \left(-f(s, L_s, \overline{\chi}_s) - \bar{\rho}_s\right)^+ ds + \left(-g(s, L_s) - \bar{\theta}_s\right)^+ dA_s. \tag{9}$$

Proof. The existence result follows from Essaky and Hassani [14]. By applying the Itô–Tanaka formula to $(U_t - Y_t)^+ = U_t - Y_t$, we find

$$(\chi_s - Z_s)1_{\{U_s = Y_s\}}\,ds = 0,$$

and

$$dK_s^- \leqslant 1_{\{U_s = Y_s\}}\left(dK_s^+ + \left(f(s, U_s, \chi_s) - \rho_s\right)ds + \left(g(s, U_s) - \theta_s\right)dA_s\right).$$

Using now the fact that $dK^+ \perp dK^-$, we obtain inequality (8).

Inequality (9) follows by the same way by applying Itô–Tanaka's formula to $(Y_t - L_t)^+ = Y_t - L_t$ and using the fact that $dK^+ \perp dK^-$. $\square$

Remark 3.1. We should point out here that Theorem 3.1 does not involve any P-integrability conditions about the data.

4. General existence result for GRBSDE and GBSDE

The main objective of this section is to show existence results of solutions for GRBSDEs and GBSDEs assuming general conditions on the data. As we will see later, we prove that the existence of solutions for GRBSDE and BSDE is related to the existence of solutions for another BSDE.

4.1. One barrier generalized reflected BSDE

Let us introduce the definition of our GRBSDE with lower obstacle L.

Definition 4.1. We call $(Y, Z, K) := (Y_t, Z_t, K_t)_{t \leqslant T}$ a solution of the generalized reflected BSDE, associated with coefficient $f\,ds + g\,dA_s$; terminal value ξ and a lower barrier L, if the following hold:

$$\begin{cases} \text{(i) } Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,ds + \int_t^T g(s, Y_s)\,dA_s + \int_t^T dK_s - \int_t^T Z_s\,dB_s, \quad t \leqslant T, \\ \text{(ii) } \forall t \leqslant T, \quad L_t \leqslant Y_t, \\ \text{(iii) } \int_0^T (Y_t - L_t)\,dK_t = 0, \quad \text{a.s.}, \\ \text{(iv) } Y \in \mathcal{C}, \quad K \in \mathcal{K}, \quad Z \in \mathcal{L}^{2,d}. \end{cases} \tag{10}$$

We are now given the following objects:

- an $\mathcal{F}_T$-measurable random variable $\Lambda : \Omega \to \mathbb{R}_+$,
- two positive predictable processes α and β such that $\eta_T < +\infty$ P-a.s., where $\eta_t = \int_0^t \alpha_s\, ds + \int_0^t \beta_s\, dA_s$,
- two continuous functions $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$,
- a nonnegative process $C \in \mathcal{C}$,
- a nonnegative process R in $\mathcal{L}^{2,1}$.

We will make the following assumptions:

(H.1) $\xi \leqslant \Lambda$ and $L_s \leqslant \Lambda, \forall s \in [0, T]$.
(H.2) There exists $(x, z, k) \in \mathcal{C} \times \mathcal{L}^{2,d} \times \mathcal{K}$ such that

$$\text{(i)} \qquad \begin{cases} \text{(j) } x_t = \Lambda + \displaystyle\int_t^T \phi(x_s)\, d\eta_s + \int_t^T \frac{C_s \psi(x_s)}{2} |z_s|^2\, ds \\ \qquad + \displaystyle\int_t^T R_s |z_s|\, ds + \int_t^T dk_s - \int_t^T z_s\, dB_s, \\ \text{(jj) } x_s \geqslant 0, \quad \forall s \leqslant T. \end{cases}$$

From now on, the above equation will be denoted by $E^+(\Lambda, \phi(x)\, d\eta_s + \frac{C_s \psi(x)}{2} |z|^2\, ds + R_s |z|\, ds)$.

(ii) For all $(s, \omega) \in [0, T] \times \Omega$

$$f(s, \omega, x_s, z_s) \leqslant \alpha_s \phi(x_s) + \frac{C_s \psi(x_s)}{2} |z_s|^2 + R_s |z_s|,$$
$$g(s, \omega, x_s) \leqslant \beta_s \phi(x_s).$$

(iii) There exist two positive predictable processes $\bar{\alpha}$ and $\bar{\beta}$ satisfying $\int_0^T \bar{\alpha}_s\, ds + \int_0^T \bar{\beta}_s\, dA_s < +\infty$ P-a.s., and $\overline{\psi} \in \mathcal{C}$ such that $\forall (s, \omega)$ and $\forall (y, z)$ satisfying $L_s \leqslant y \leqslant L_s \vee x_s$

$$\big| f(s, \omega, y, z) \big| \leqslant \bar{\alpha}_s + \frac{\overline{\psi}_s}{2} |z|^2,$$
$$\big| g(s, \omega, y) \big| \leqslant \overline{\beta}_s.$$

Remark 4.1. 1. By using a localization procedure and Fatou's lemma one can prove easily that:

$$x_t \geqslant \mathbb{E}(\Lambda | \mathcal{F}_t) \geqslant L_t, \quad \forall t \in [0, T].$$

2. It is worth noting that condition (H.2)(iii) holds true if the functions f and g satisfy the following:

$$\forall (s, \omega), \quad \big| f(s, \omega, y, z) \big| \leqslant \sigma_s \Phi(s, \omega, y) + \gamma_s \Psi(s, \omega, y) |z|^2,$$
$$\forall y \in \big[L_s(\omega), x_s(\omega) \big],\ \forall z \in \mathbb{R}^d,$$

and

$$\forall (s, \omega), \quad \big| g(s, \omega, y) \big| \leqslant \delta_s \varphi(s, \omega, y), \quad \forall y \in \big[L_s(\omega), x_s(\omega) \big], \tag{11}$$

where Φ, Ψ and φ are continuous functions on $[0,T]\times\mathbb{R}$ and progressively measurable, $\sigma\in L^0(\Omega, L^1([0,T], ds, \mathbb{R}_+))$, $\gamma\in\mathcal{C}$ and $\delta\in L^0(\Omega, L^1([0,T], dA_s, \mathbb{R}_+))$. To do this, we just take $\bar\alpha$, $\overline{\psi}$ and $\overline{\beta}$ as follows:

$$\bar\alpha_t(\omega)=\sigma_t(\omega)\sup_{s\leqslant t}\sup_{\alpha\in[0,1]}\left|\Phi\big(s,\omega,\alpha L_s+(1-\alpha)x_s\big)\right|,$$
$$\overline{\psi}_t(\omega)=2\gamma_t\sup_{s\leqslant t}\sup_{\alpha\in[0,1]}\left|\Psi\big(s,\omega,\alpha L_s+(1-\alpha)x_s\big)\right|,$$
$$\overline{\beta}_t(\omega)=\delta_t(\omega)\sup_{s\leqslant t}\sup_{\alpha\in[0,1]}\left|\varphi\big(s,\omega,\alpha L_s+(1-\alpha)x_s\big)\right|.$$

The following theorem is a consequence of Theorem 3.1.

Theorem 4.1. *Let assumptions* (H.1)–(H.2) *hold. Then the following GRBSDE*

$$\begin{cases} \text{(i) } Y_t=\xi+\int_t^T f(s,Y_s,Z_s)\,ds+\int_t^T g(s,Y_s)\,dA_s+\int_t^T dK_s-\int_t^T Z_s\,dB_s, \quad t\leqslant T,\\ \text{(ii) } \forall t\leqslant T,\quad L_t\leqslant Y_t\leqslant x_t,\\ \text{(iii) } \int_0^T (Y_t-L_t)\,dK_t=0,\quad a.s.,\\ \text{(iv) } Y\in\mathcal{C},\quad K\in\mathcal{K},\quad Z\in\mathcal{L}^{2,d} \end{cases}\tag{12}$$

has a maximal (*resp. minimal*) *solution. Moreover, if the following condition holds*

$$L_t:=L_0+\overline{V}_t+\int_0^t\bar\rho_s\,ds+\int_0^t\bar\theta_s\,dA_s+\int_0^t\overline{\chi}_s\,dB_s,$$

with $L_0\in\mathbb{R}$, $\overline{V}\in\mathcal{K}$, $\overline{\chi}\in\mathcal{L}^{2,d}$, $\bar\rho$ and $\bar\theta$ are nonnegative predictable processes satisfying $\int_0^T\bar\rho_s\,ds+\int_0^T\bar\theta_s\,dA_s<+\infty$ P-a.s., then for all solutions (Y,Z,K) of Eq. (12) *we have*

$$dK_s\leqslant\big(-f(s,L_s,\overline{\chi}_s)-\bar\rho_s\big)^+ds+\big(-g(s,L_s)-\bar\theta_s\big)^+dA_s.\tag{13}$$

Proof. Let (Y,Z,K^+,K^-) be the maximal (resp. minimal) solution of Eq. (7) with $U_t=x_t$. By using inequality (8) of Theorem 3.1 we conclude that

$$\begin{aligned} dK^-&\leqslant\left(f(s,\omega,x_s,z_s)-\alpha_s\phi(x_s)-\frac{C_s\psi(x_s)}{2}|z_s|^2-R_s|z_s|\right)^+ds\\ &\quad+\big(g(s,\omega,x_s)-\beta_s\phi(x_s)\big)^+dA_s\\ &=0. \end{aligned}$$

Therefore $dK^-=0$ and then Eq. (12) has a maximal (resp. minimal) solution.

Inequality (13) follows easily from inequality (9). □

Remark 4.2. It is worth pointing out that the minimal solution of GRBSDE (12) is also the minimal solution of GRBSDE (10). This statement does not hold for maximal solution.

Once established the existence of solutions for GRBSDEs, we are now interested in proving the same result for GBSDEs.

4.2. Generalized BSDE without reflection

To begin with, let us introduce the definition of our GBSDE.

Definition 4.2. We call $(Y, Z) := (Y_t, Z_t)_{t\leqslant T}$ a solution of the generalized reflected BSDE, associated with coefficient $f\,ds + g\,dA_s$; terminal value ξ, if the following hold:

$$\begin{cases} \text{(i) } Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,ds + \int_t^T g(s, Y_s)\,dA_s - \int_t^T Z_s\,dB_s, \quad t \leqslant T, \\ \text{(ii) } Y \in \mathcal{C}, \quad Z \in \mathcal{L}^{2,d}. \end{cases} \tag{14}$$

For $i = 1, 2$, we are given the following objects:

- an $\mathcal{F}_T$-measurable random variable $\Lambda^i : \Omega \to \mathbb{R}_+$,
- two nonnegative predictable processes α^i and β^i such that $\eta^i_T < +\infty$ P-a.s., where $\eta^i_t = \int_0^t \alpha^i_s\,ds + \int_0^t \beta^i_s\,dA_s$,
- two continuous functions $\phi^i, \psi^i : \mathbb{R}_+ \to \mathbb{R}_+$,
- a nonnegative process $C^i \in \mathcal{C}$,
- a nonnegative process R^i in $\mathcal{L}^{2,1}$.

We will need the following assumptions:

(C.1) $-\Lambda^1 \leqslant \xi \leqslant \Lambda^2$.
(C.2) There exists $(x^i, z^i, k^i) \in \mathcal{C} \times \mathcal{L}^{2,d} \times \mathcal{K}$ such that

$$\text{(i)} \quad \begin{cases} \text{(j) } x^i_t = \Lambda^i + \int_t^T \phi^i(x^i_s)\,d\eta^i_s + \int_t^T \frac{C^i_s \psi^i(x^i_s)}{2} |z^i_s|^2\,ds + \int_t^T R^i_s |z^i_s|\,ds \\ \qquad + \int_t^T dk^i_s - \int_t^T z^i_s\,dB_s, \quad s \leqslant T, \\ \text{(jj) } x^i_s \geqslant 0, \quad \forall s \leqslant T. \end{cases}$$

(ii) For all $(s, \omega) \in [0, T] \times \Omega$

$$f\big(s, \omega, x^2_s, z^2_s\big) \leqslant \alpha^2_s \phi^2\big(x^2_s\big) + \frac{C^2_s \psi^2(x^2_s)}{2} |z^2_s|^2 + R^2_s |z^2_s|,$$
$$f\big(s, \omega, -x^1_s, -z^1_s\big) \geqslant -\alpha^1_s \phi^1\big(x^1_s\big) - \frac{C^1_s \psi^1(x^1_s)}{2} |z^1_s|^2 - R^1_s |z^1_s|.$$

(iii) For all $(s, \omega) \in [0, T] \times \Omega$

$$g\big(s, \omega, x^2_s\big) \leqslant \beta^2_s \phi^2\big(x^2_s\big),$$
$$g\big(s, \omega, -x^1_s\big) \geqslant -\beta^1_s \phi^1\big(x^1_s\big).$$

(iv) There exist two nonnegative predictable processes $\bar{\alpha}$ and $\bar{\beta}$ such that $\int_0^T \bar{\alpha}_s\,ds + \int_0^T \bar{\beta}_s\,dA_s < +\infty$ P-a.s., and $\overline{\psi} \in \mathcal{C}$ such that $\forall(s,\omega)$ and $\forall(y,z)$ satisfying $-x_s^1 \leqslant y \leqslant x_s^2$

$$\begin{aligned}&\bigl|f(s,\omega,y,z)\bigr| \leqslant \bar{\alpha}_s + \frac{\overline{\psi}_s}{2}|z|^2,\\&\bigl|g(s,\omega,y)\bigr| \leqslant \overline{\beta}_s.\end{aligned}$$

The proof of the following theorem follows easily from Theorem 4.1.

Theorem 4.2. *Let assumptions* (C.1)–(C.2) *hold. Then the following GRBSDE*

$$\begin{cases}\text{(i) } Y_t = \xi + \displaystyle\int_t^T f(s,Y_s,Z_s)\,ds + \int_t^T g(s,Y_s)\,dA_s - \int_t^T Z_s\,dB_s, \quad t \leqslant T,\\ \text{(ii) } -x_s^1 \leqslant Y_s \leqslant x_s^2, \quad \forall s \leqslant T,\\ \text{(iii) } Y \in \mathcal{C}, \quad Z \in \mathcal{L}^{2,d}\end{cases} \tag{15}$$

has a maximal (*resp. minimal*) *solution.*

The next section is devoted to give immediate consequences of Theorems 4.1 and 4.2 in the case where the terminal condition ξ and/or the barrier L are bounded.

5. First consequences of Theorems 4.1 and 4.2: the bounded case

5.1. One barrier GBSDE

In this subsection, we consider the same notations as in Section 4.1 and we study only the existence of solutions for GRBSDE (10) in the case of bounded terminal value ξ and barrier L. The unbounded case is treated in the next sections. The following result is a consequence of Theorem 4.1.

Corollary 5.1. *Suppose that there exist two nonnegative real numbers D and a such that*

1. $\xi \leqslant D$ *and* $L_t \leqslant D$, $\forall t \in [0,T]$.
2. $\phi(y) > 0$ *for* $y \geqslant D$.
3. $\eta_T = \int_0^T \alpha_s\,ds + \int_0^T \beta_s\,dA_s \leqslant a < \int_D^{+\infty} \frac{dr}{\phi(r)}$.
4. *For all* $(s,\omega) \in [0,T] \times \Omega$

$$\begin{aligned}&f\bigl(s,\omega,H^{-1}(a-\eta_s),0\bigr) \leqslant \alpha_s \phi\bigl(H^{-1}(a-\eta_s)\bigr),\\&g\bigl(s,\omega,H^{-1}(a-\eta_s)\bigr) \leqslant \beta_s \phi\bigl(H^{-1}(a-\eta_s)\bigr),\end{aligned}$$

where H^{-1} denotes the inverse of the function H defined by:

$$H : [D,+\infty[\to \Biggl[0, \int_D^{+\infty} \frac{dr}{\phi(r)}\Biggl[, \qquad H(x) = \int_D^x \frac{dr}{\phi(r)}.$$

5. *There exist two nonnegative predictable processes $\bar{\alpha}$ and $\bar{\beta}$ satisfying $\int_0^T \bar{\alpha}_s\,ds + \int_0^T \bar{\beta}_s\,dA_s < +\infty$ P-a.s., and $\overline{\psi} \in \mathcal{C}$ such that $\forall(s,\omega)$ and $\forall(y,z)$ satisfying $L_s \leqslant y \leqslant H^{-1}(a-\eta_s)$*

$$\left|f(s,\omega,y,z)\right| \leqslant \bar{\alpha}_s + \frac{\overline{\psi}_s}{2}|z|^2 \quad \textit{and} \quad \left|g(s,\omega,y)\right| \leqslant \overline{\beta}_s.$$

Then the GRBSDE (10) *has a solution such that $L_t \leqslant Y_t \leqslant H^{-1}(a-\eta_t)$.*

Proof. Set $x_t = H^{-1}(a-\eta_t)$, for every $t \in [0,T]$. By Itô's formula we have

$$x_t = H^{-1}(a-\eta_T) + \int_t^T \phi(x_s)\,d\eta_s.$$

Set $\Lambda := H^{-1}(a-\eta_T)$. Since $H(D) = 0 \leqslant a - \eta_T$ and H is increasing, it follows then from assumption 1 that $\xi \leqslant \Lambda$ and $L_t \leqslant \Lambda$, $\forall t \in [0,T]$. Hence assumption (H.1) is satisfied. Assumption (H.2)(i) is satisfied also with $(x,0,0)$. The result follows then form Theorem 4.1. □

The following corollaries, with $\phi(x) = x\ln(x)$ and $\phi(x) = e^x$, assure the existence of a solution for the GRBSDE (10). Their proofs follow easily from Corollary 5.1.

Corollary 5.2. *Suppose that there exist two real numbers $D > 1$ and $a \geqslant 0$ such that*

1. *$\xi \leqslant D$ and $L_t \leqslant D$, $\forall t \in [0,T]$.*
2. $\operatorname{ess\,sup}_w(\int_0^T \alpha_s\,ds + \int_0^T \beta_s\,dA_s) \leqslant a$.
3. *For all $(s,\omega) \in [0,T]\times\Omega$*

$$f\big(s,\omega,e^{\ln(D)e^{a-\eta_s}},0\big) \leqslant \alpha_s \ln(D)e^{\ln(D)e^{a-\eta_s}}e^{a-\eta_s},$$
$$g\big(s,\omega,e^{\ln(D)e^{a-\eta_s}}\big) \leqslant \beta_s \ln(D)e^{\ln(D)e^{a-\eta_s}}e^{a-\eta_s}.$$

4. *There exist two nonnegative predictable processes $\bar{\alpha}$ and $\bar{\beta}$ satisfying $\int_0^T \bar{\alpha}_s\,ds + \int_0^T \bar{\beta}_s\,dA_s < +\infty$ P-a.s., and $\overline{\psi} \in \mathcal{C}$ such that $\forall(s,\omega)$ and $\forall(y,z)$ satisfying $L_s \leqslant y \leqslant e^{\ln(D)e^{a-\eta_s}}$*

$$\left|f(s,\omega,y,z)\right| \leqslant \bar{\alpha}_s + \frac{\overline{\psi}_s}{2}|z|^2 \quad \textit{and} \quad \left|g(s,\omega,y)\right| \leqslant \overline{\beta}_s.$$

Then the GRBSDE (10) *has a solution such that $L_t \leqslant Y_t \leqslant e^{\ln(D)e^{a-\eta_t}}$.*

Corollary 5.3. *Suppose that there exist two real nonnegative numbers D and a such that*

1. *$\xi \leqslant D$ and $L_t \leqslant D$, $\forall t \in [0,T]$.*
2. $\operatorname{ess\,sup}_w(\int_0^T \alpha_s\,ds + \int_0^T \beta_s\,dA_s) \leqslant a < e^{-D}$.
3. *For all $(s,\omega) \in [0,T]\times\Omega$*

$$f\big(s,\omega,-\ln\big(e^{-D}-a+\eta_s\big),0\big)\leqslant\frac{\alpha_s}{e^{-D}-a+\eta_s},$$
$$g\big(s,\omega,-\ln\big(e^{-D}-a+\eta_s\big)\big)\leqslant\frac{\beta_s}{e^{-D}-a+\eta_s}.$$

4. *There exist two nonnegative predictable processes $\overline{\alpha}$ and $\overline{\beta}$ satisfying $\int_0^T\overline{\alpha}_s\,ds+\int_0^T\overline{\beta}_s\,dA_s<+\infty$ P-a.s., and $\overline{\psi}\in\mathcal{C}$ such that $\forall(s,\omega)$ and $\forall(y,z)$ satisfying $L_s\leqslant y\leqslant-\ln(e^{-D}-a+\eta_s)$*

$$\big|f(s,\omega,y,z)\big|\leqslant\overline{\alpha}_s+\frac{\overline{\psi}_s}{2}|z|^2\quad\textit{and}\quad\big|g(s,\omega,y)\big|\leqslant\overline{\beta}_s.$$

Then the GRBSDE (10) *has a solution such that $L_t\leqslant Y_t\leqslant-\ln(e^{-D}-a+\eta_t)$.*

5.2. GBSDE without reflection

In this subsection, we consider the same notations as in Section 4.2 and we treat only the existence of solution in the case of bounded terminal value ξ. The unbounded case is treated in the next sections. The following result is a consequence of Theorem 4.2.

Corollary 5.4. *Suppose that there exist four real numbers $D^1\geqslant0$, $D^2\geqslant0$, a^1 and a^2 such that*

(i) $-D^1\leqslant\xi\leqslant D^2$.
(ii) *For $i=1,2$, $\phi^i(y)>0$ for $y\geqslant D^i$.*
(iii) *For $i=1,2$, $\operatorname{ess\,sup}_w\int_0^T\alpha_s^i\,ds+\int_0^T\beta_s^i\,dA_s\leqslant a^i<\int_{D^i}^{+\infty}\frac{dr}{\phi^i(r)}$.*
(iv) *For all $(s,\omega)\in[0,T]\times\Omega$*

$$f\big(s,\omega,\big(H^2\big)^{-1}\big(a^2-\eta_s^2\big),0\big)\leqslant\alpha_s^2\phi^2\big(\big(H^2\big)^{-1}\big(a^2-\eta_s^2\big)\big),$$
$$f\big(s,\omega,-\big(H^1\big)^{-1}\big(a^1-\eta_s^1\big),0\big)\geqslant-\alpha_s^1\phi^1\big(\big(H^1\big)^{-1}\big(a^1-\eta_s^1\big)\big),$$
$$g\big(s,\omega,\big(H^2\big)^{-1}\big(a^2-\eta_s^2\big)\big)\leqslant\beta_s^2\phi\big(\big(H^2\big)^{-1}\big(a^2-\eta_s^2\big)\big),$$
$$g\big(s,\omega,-\big(H^1\big)^{-1}\big(a^1-\eta_s^1\big)\big)\geqslant-\beta_s^1\phi\big(\big(H^1\big)^{-1}\big(a^1-\eta_s^1\big)\big),$$

where, for $i=1,2$, $H^i(x)=\int_{D^i}^x\frac{dr}{\phi^i(r)}$, $x\geqslant D^i$ and $\eta_t^i=\int_0^t\alpha_s^i\,ds+\int_0^t\beta_s^i\,dA_s$.
(v) *There exist two nonnegative predictable processes $\overline{\alpha}$ and $\overline{\beta}$ satisfying $\int_0^T\overline{\alpha}_s\,ds+\int_0^T\overline{\beta}_s\,dA_s<+\infty$ P-a.s., and $\overline{\psi}\in\mathcal{C}$ such that $\forall(s,\omega)$ and $\forall(y,z)$ satisfying $-(H^1)^{-1}(a^1-\eta_s^1)\leqslant y\leqslant(H^2)^{-1}(a^2-\eta_s^2)$*

$$\big|f(s,\omega,y,z)\big|\leqslant\overline{\alpha}_s+\frac{\overline{\psi}_s}{2}|z|^2,$$
$$\big|g(s,\omega,y)\big|\leqslant\overline{\beta}_s.$$

Then the GBSDE (14) *has a solution such that $-(H^1)^{-1}(a^1-\eta_s^1)\leqslant Y_s\leqslant(H^2)^{-1}(a^2-\eta_s^2)$.*

The following corollaries, with $\phi^1(x)=\phi^2(x)=x\ln(x)$ and $\phi^1(x)=\phi^2(x)=e^x$, assure the existence of a solution for the GRBSDE (10). Their proofs follow from Corollary 5.4.

Corollary 5.5. *Suppose that there exist two real numbers $D > 1$ and a such that*

(i) $|\xi| \leqslant D$.
(ii) $\operatorname{ess\,sup}_w\big(\int_0^T \alpha_s\,ds + \int_0^T \beta_s\,dA_s\big) \leqslant a < +\infty$.
(iii) *For all* $(s,\omega) \in [0,T] \times \Omega$

$$f\big(s,\omega,e^{\ln(D)e^{a-\eta_s}},0\big) \leqslant \alpha_s \ln(D)e^{\ln(D)e^{a-\eta_s}}e^{a-\eta_s},$$
$$f\big(s,\omega,-e^{\ln(D)e^{a-\eta_s}},0\big) \geqslant -\alpha_s \ln(D)e^{\ln(D)e^{a-\eta_s}}e^{a-\eta_s},$$
$$g\big(s,\omega,e^{\ln(D)e^{a-\eta_s}}\big) \leqslant \beta_s \ln(D)e^{\ln(D)e^{a-\eta_s}}e^{a-\eta_s},$$
$$g\big(s,\omega,-e^{\ln(D)e^{a-\eta_s}}\big) \geqslant -\beta_s \ln(D)e^{\ln(D)e^{a-\eta_s}}e^{a-\eta_s},$$

where $\eta_t = \int_0^t \alpha_s\,ds + \int_0^t \beta_s\,dA_s$.
(iv) *There exist two nonnegative predictable processes* $\bar\alpha$ *and* $\bar\beta$ *satisfying* $\int_0^T \bar\alpha_s\,ds + \int_0^T \bar\beta_s\,dA_s < +\infty$ *P-a.s., and* $\bar\psi \in \mathcal{C}$ *such that* $\forall(s,\omega)$ *and* $\forall(y,z)$ *satisfying* $|y| \leqslant e^{\ln(D)e^{a-\eta_s}}$

$$\big|f(s,\omega,y,z)\big| \leqslant \bar\alpha_s + \frac{\bar\psi_s}{2}|z|^2,$$
$$\big|g(s,\omega,y)\big| \leqslant \bar\beta_s.$$

Then the GBSDE (14) *has a solution such that* $|Y_t| \leqslant e^{\ln(D)e^{a-\eta_t}}$.

Corollary 5.6. *Suppose that there exist two real numbers $D \geqslant 0$ and a such that*

(i) $\operatorname{ess\,sup}_w |\xi| \leqslant D$.
(ii) $\operatorname{ess\,sup}_w\big(\int_0^T \alpha_s\,ds + \int_0^T \beta_s\,dA_s\big) \leqslant a < e^{-D}$.
(iii) *For all* $(s,\omega) \in [0,T] \times \Omega$

$$f\big(s,\omega,-\ln\big(e^{-D}-a+\eta_s\big),0\big) \leqslant \frac{\alpha_s}{e^{-D}-a+\eta_s},$$
$$f\big(s,\omega,\ln\big(e^{-D}-a+\eta_s\big),0\big) \geqslant \frac{-\alpha_s}{e^{-D}-a+\eta_s},$$
$$g\big(s,\omega,-\ln\big(e^{-D}-a+\eta_s\big)\big) \leqslant \frac{\beta_s}{e^{-D}-a+\eta_s},$$
$$g\big(s,\omega,\ln\big(e^{-D}-a+\eta_s\big)\big) \geqslant \frac{-\beta_s}{e^{-D}-a+\eta_s},$$

where $\eta_t = \int_0^t \alpha_s\,ds + \int_0^t \beta_s\,dA_s$.
(iv) $\forall(s,\omega)$ *and* $\forall(y,z)$ *satisfying* $|y| \leqslant -\ln(e^{-D}-a+\eta_s)$

$$\big|f(s,\omega,y,z)\big| \leqslant \bar\alpha_s + \frac{\bar\psi_s}{2}|z|^2,$$
$$\big|g(s,\omega,y)\big| \leqslant \bar\beta_s.$$

Then the GBSDE (14) *has a solution such that* $|Y_t| \leqslant -\ln(e^{-D}-a+\eta_t)$.

Corollary 5.7. *Suppose that there exists a nonnegative real number D such that*

(i) $|\xi| \leqslant D$.
(ii) $\phi(x) = e^x$ *for* $x \geqslant D$.
(iii) $\operatorname{ess\,sup}_w \big(\int_0^T \alpha_s\, ds + \int_0^T \beta_s\, dA_s\big) := a < e^{-D}$.
(iv) $\forall(s,\omega)$ *and* $\forall(y,z)$ *satisfying* $|y| \leqslant -\ln(e^{-D} - a)$ *we have*

$$\big|f(s,\omega,y,z)\big| \leqslant \alpha_s \phi\big(|y|\big) + \frac{C_s}{2}|z|^2 + R_s|z|,$$
$$\big|g(s,\omega,y)\big| \leqslant \beta_s \phi\big(|y|\big).$$

Then the GBSDE (14) *has a solution such that* $|Y_t| \leqslant -\ln(e^{-D} - a)$.

6. Existence of solutions for $E^+(\Lambda, \phi(x)\, d\eta_s + \frac{C_s\psi(x)}{2}|z|^2\, ds + R_s|z|\, ds)$

As we have seen, by using an approach based upon the recent result obtained in the preprint of Essaky and Hassani [14], Theorems 4.1 and 4.2 follow easily from Theorem 3.1 but there is still an interesting and important question: under which conditions on $(\Lambda, \phi, \psi, C, \eta)$, equation $E^+(\Lambda, \phi(x)\, d\eta_s + \frac{C_s\psi(x)}{2}|z|^2\, ds + R_s|z|\, ds)$

$$\begin{cases} \text{(j) } x_t = \Lambda + \displaystyle\int_t^T \phi(x_s)\, d\eta_s + \int_t^T \frac{C_s\psi(x_s)}{2}|z_s|^2\, ds + \int_t^T R_s|z_s|\, ds \\ \qquad\qquad + \displaystyle\int_t^T dk_s - \int_t^T z_s\, dB_s, \\ \text{(jj) } x_s \geqslant 0, \quad \forall s \leqslant T, \end{cases}$$

has a solution $(x,z,k) \in \mathcal{C} \times \mathcal{L}^{2,d} \times \mathcal{K}$? For that sake, we list all the notations that will be used throughout this section. We denote:

- D to be a nonnegative constant.
- $\Lambda : \Omega \to [D, +\infty[$ to be an $\mathcal{F}_T$-measurable random variable.
- $\phi, \psi : [D, +\infty[\to \mathbb{R}_+$ to be two continuous functions such that ϕ is of class C^1.
- $\eta \in \mathcal{K}$ to be a process such that $\eta_T < \int_\Lambda^{+\infty} \frac{dr}{\phi(r)}$.
- C to be a process in $\mathbb{R}_+ + \mathcal{K}$.
- R to be a nonnegative process in $\mathcal{L}^{2,1}$.

Further we define also the following functions:

- $H : [D, +\infty[\to [0, \int_D^{+\infty} \frac{dr}{\phi(r)}[$, $H(x) = \int_D^x \frac{dr}{\phi(r)}$,
- $F : [D, +\infty[\times [0, +\infty[\to \mathbb{R}_+$, $F(x,c) = \int_D^x e^{c\int_D^t \psi(r)\, dr}\, dt$,
- $H^{-1} : [0, \int_D^{+\infty} \frac{dr}{\phi(r)}[\to [D, +\infty[$ is such that $H^{-1}(y) = x$ if and only if $H(x) = y$,
- $F^{-1} : \mathbb{R}_+ \times [0, +\infty[\to [D, +\infty[$ is such that $F^{-1}(y,c) = x$ if and only if $F(x,c) = y$,
- $G : \mathcal{G} \to [D, +\infty[$, $G(x,c,\eta) = H^{-1}(H(F^{-1}(x,c)) - \eta)$, where $\mathcal{G}$ is the set defined by:

$$\mathcal{G} = \big\{(x,c,\eta) \in (\mathbb{R}_+)^3 :\ H\big(F^{-1}(x,c)\big) \geqslant \eta\big\}. \tag{16}$$

We use also the following notations:

- $\overline{\Lambda} = F(H^{-1}(H(\Lambda) + \eta_T), C_T)$,
- $\widetilde{\Pi} := \{\pi \in \mathcal{L}^{2,d}\colon |\pi_s| \leqslant 1, \text{ a.e.}\}$,
- $\Pi := \{\pi \in \widetilde{\Pi}\colon |\pi_s| \in \{0,1\} \text{ a.e. and } \operatorname{ess\,sup}_\omega \int_0^T R_s^2 |\pi_s|^2\, ds < +\infty\}$,
- $\Gamma^\pi_{t,s} := e^{\int_t^s R_u \pi_u\, dB_u - \frac{1}{2}\int_t^s R_u^2 |\pi_u|^2\, du}$, for $\pi \in \widetilde{\Pi}$ and $s, t \in [0,T]$.

We are now ready to give necessary and sufficient conditions for the existence of a solution for a particular case of $E^+(\Lambda, \phi(x)\, d\eta_s + \frac{C_s \psi(x)}{2}|z|^2\, ds + R_s|z|\, ds)$.

Proposition 6.1. $\sup_{\pi\in\Pi} \mathbb{E}\Gamma^\pi_{0,T}\overline{\Lambda} < +\infty$ *if and only if there exists* $(x^1, z^1) \in \mathcal{C} \times \mathcal{L}^{2,d}$ *solution of the following BSDE*

$$\begin{cases} x_t^1 = \overline{\Lambda} + \int_t^T R_s |z_s^1|\, ds - \int_t^T z_s^1\, dB_s, \quad t \leqslant T, \\ x_t^1 \geqslant 0, \quad \forall t \leqslant T. \end{cases} \tag{17}$$

In this case, there exist $\bar{z} \in \mathcal{L}^{2,d}$ *and* $\bar{x}_t := \operatorname{ess\,sup}_{\pi\in\Pi} \mathbb{E}(\Gamma^\pi_{t,T}\overline{\Lambda}|\mathcal{F}_t) = \operatorname{ess\,sup}_{\pi\in\widetilde{\Pi}} \mathbb{E}(\Gamma^\pi_{t,T}\overline{\Lambda}|\mathcal{F}_t)$ *such that* $(\bar{x}, \bar{z})$ *is the minimal solution of Eq.* (17)*, that is for all solutions* (x^1, z^1) *of Eq.* (17) *we have* $\bar{x}_t \leqslant x_t^1$.

Proof. Let $(\tau_n)_{n\geqslant 2}$ be the sequence of stopping times defined by $\tau_n := \inf\{t \geqslant 0\colon \int_0^t R_s^2\, ds \geqslant n\} \wedge T$. According to Theorem 4.2, there exists $(x^n, z^n) \in \mathcal{C} \times \mathcal{L}^{2,d}$ such that

$$\begin{cases} x_t^n = \overline{\Lambda} 1_{\{\overline{\Lambda} \leqslant n\}} + \int_t^T R_s 1_{\{s \leqslant \tau_n\}} |z_s^n|\, ds - \int_t^T z_s^n\, dB_s, \quad t \leqslant T, \\ 0 \leqslant x_t^n \leqslant n, \quad \forall t \in [0,T]. \end{cases} \tag{18}$$

By using a localization procedure and Lebesgue's convergence theorem we have that, for all stopping times ν and $n \geqslant 2$,

$$x_0^n = \mathbb{E}\left(x_\nu^n + \int_0^\nu R_s 1_{\{s \leqslant \tau_n\}} |z_s^n|\, ds \right). \tag{19}$$

On the other hand, it follows from Itô's formula that, for all stopping times $\nu \leqslant \sigma \leqslant T$,

$$\begin{cases} x_\nu^n = \Gamma^{\pi^n}_{\nu,\sigma} x_\sigma^n - \int_\nu^\sigma \Gamma^{\pi^n}_{\nu,s}\left(z_s^n + R_s x_s^n \pi_s^n\right) dB_s, \quad t \leqslant T, \\ 0 \leqslant x_\nu^n \leqslant n, \end{cases}$$

where

$$\pi_s^n := \begin{cases} \frac{z_s^n}{|z_s^n|} 1_{\{s \leqslant \tau_n\}} & \text{if } z_s^n \neq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Using standard localization procedure and Lebesgue's convergence theorem we obtain that, for all stopping times $\nu \leqslant \sigma \leqslant T$ and for all $n \geqslant 2$,

$$x_\nu^n = \mathbb{E}\big(\Gamma_{\nu,\sigma}^{\pi^n} x_\sigma^n \big| \mathcal{F}_\nu\big) = \mathbb{E}\big(\Gamma_{\nu,T}^{\pi^n} x_\sigma^n \big| \mathcal{F}_\nu\big) = \mathbb{E}\big(\Gamma_{\nu,T}^{\pi^n} \overline{\Lambda} 1_{\{\overline{\Lambda} \leqslant n\}} \big| \mathcal{F}_\nu\big) \leqslant \operatorname*{ess\,sup}_{\pi \in \Pi} \mathbb{E}\big(\Gamma_{\nu,T}^{\pi} \overline{\Lambda} \big| \mathcal{F}_\nu\big), \tag{20}$$

where we have used the fact that $\Gamma_{\nu,.}^{\pi^n}$ is a martingale on $[\nu, T]$.

It follows from comparison theorem that $x^n \leqslant x^{n+1}$. Set then $\bar{x}_t := \lim_{n\to+\infty} \uparrow x_t^n$. Therefore, in view of (19) and (20) we get for all stopping times $0 \leqslant \nu \leqslant T$,

$$\mathbb{E}\bar{x}_\nu \leqslant \bar{x}_0 \leqslant \sup_{\pi\in\Pi} \mathbb{E}\big(\Gamma_{0,T}^{\pi} \overline{\Lambda}\big) \quad \text{and} \quad \bar{x}_\nu \leqslant \operatorname*{ess\,sup}_{\pi\in\Pi} \mathbb{E}\big(\Gamma_{\nu,T}^{\pi} \overline{\Lambda} \big| \mathcal{F}_\nu\big). \tag{21}$$

Let us now define the sequences of stopping times $(\delta_i^n)_{i\geqslant 2}$ by $\delta_i^n := \inf\{s \geqslant 0\colon x_s^n \geqslant i\} \wedge T$ and $\delta_i := \inf_n \delta_i^n = \lim_n \delta_i^n$. Note that $0 \leqslant \bar{x}_t = \lim_n x_t^n \leqslant i$, for all $t \leqslant \delta_i$.

Define also $\lambda_i := \delta_i \wedge \tau_i$ and let

$$\begin{cases} \bar{x}_t^i = \bar{x}_{\lambda_i} + \displaystyle\int_t^{\lambda_i} R_s \big|\bar{z}_s^i\big|\,ds - \int_t^{\lambda_i} \bar{z}^i{}_s\,dB_s, \\ 0 \leqslant \bar{x}_t^i \leqslant i, \quad \forall t \in [0, \lambda_i]. \end{cases} \tag{22}$$

Applying Itô's formula to $(\bar{x}_t^i - x_t^n)^2 e^{\int_0^t R_s^2\,ds}$ and using a localization procedure, we conclude that

$$\mathbb{E}\big(\bar{x}_{t\wedge\lambda_i}^i - x_{t\wedge\lambda_i}^n\big)^2 \leqslant e^i\,\mathbb{E}\big(\bar{x}_{\lambda_i} - x_{\lambda_i}^n\big)^2, \quad \forall n \geqslant i.$$

By letting n go to infinity we get $\bar{x}_{t\wedge\lambda_i}^i = \bar{x}_{t\wedge\lambda_i}$ and then $\bar{z}^i = \bar{z}^{i+1}$ on $[0, \lambda_i]$. Set $\bar{z}_s := \lim_i \bar{z}_s^i 1_{\{s\leqslant\lambda_i\}} = \bar{z}_s^j$ on $[0, \lambda_j]$. Hence, for all $i \geqslant 2$

$$\begin{cases} \bar{x}_t = \bar{x}_{\lambda_i} + \displaystyle\int_t^{\lambda_i} R_s |\bar{z}_s|\,ds - \int_t^{\lambda_i} \bar{z}_s\,dB_s, \\ 0 \leqslant \bar{x}_t \leqslant i, \quad \forall t \in [0, \lambda_i]. \end{cases}$$

Suppose now that $\sup_{\pi\in\Pi} \mathbb{E}(\Gamma_{0,T}^{\pi} \overline{\Lambda}) < +\infty$. Since $\liminf_n x_{\delta_i^n}^n 1_{\{\delta_i<T\}} = i 1_{\{\delta_i<T\}}$ we have $i P(\delta_i < T) \leqslant \mathbb{E}(\liminf_n x_{\delta_i^n}^n) \leqslant \liminf_n \mathbb{E}(x_{\delta_i^n}^n) \leqslant \sup_{\pi\in\Pi} \mathbb{E}(\Gamma_{0,T}^{\pi} \overline{\Lambda}) < +\infty$. Therefore $P(\bigcup_{i\geqslant 2}(\delta_i = T)) = 1$, and then $P(\bigcup_{i\geqslant 2} (\lambda_i = T)) = 1$. Moreover, it is easy seen that $\bar{z} \in \mathcal{L}^{2,d}$. Now passing to the limit as i goes to infinity in Eq. (22) we obtain

$$\begin{cases} \bar{x}_t = \overline{\Lambda} + \displaystyle\int_t^{T} R_s |\bar{z}_s|\,ds - \int_t^{T} \bar{z}_s\,dB_s, \quad t \leqslant T, \\ 0 \leqslant \bar{x}_t, \quad \forall t \in [0, T]. \end{cases}$$

Henceforth $(\bar{x}, \bar{z})$ is a solution of Eq. (17) which satisfies $\bar{x}_\nu \leqslant \operatorname{ess\,sup}_{\pi\in\Pi} \mathbb{E}(\Gamma_{\nu,T}^{\pi} \overline{\Lambda}|\mathcal{F}_\nu)$, for all stopping times $0 \leqslant \nu \leqslant T$.

On the other hand, let $(x^1, z^1) \in \mathcal{C} \times \mathcal{L}^{2,d}$ be a solution of Eq. (17) and consider for all $\pi \in \widetilde{\Pi}$, $(x^\pi, z^\pi) \in \mathcal{C} \times \mathcal{L}^{2,d}$ a solution of the following BSDE

$$\begin{cases} x_t^\pi = \overline{\Lambda} + \displaystyle\int_t^{T} R_s \big\langle \pi_s, z_s^\pi \big\rangle\,ds - \int_t^{T} z_s^\pi\,dB_s, \quad t \leqslant T, \\ 0 \leqslant x_t^\pi \leqslant x_t^1, \quad \forall t \in [0, T], \end{cases}$$

which exists according to Theorem 4.2. It follows then from Itô's formula that, for all stopping times $\nu \leqslant \sigma \leqslant T$,

$$\begin{cases} x_\nu^\pi = \Gamma_{\nu,\sigma}^\pi x_\sigma^\pi - \int_\nu^\sigma \Gamma_{\nu,s}^\pi \big(z_s^\pi + R_s x_s^\pi \pi_s\big)\, dB_s, \quad t \leqslant T, \\ 0 \leqslant x_\nu^\pi \leqslant x_\nu^1. \end{cases}$$

Consequently, for all stopping times $\nu \leqslant T$, we have by Fatou's lemma and standard localization procedure

$$x_\nu^\pi \geqslant \mathbb{E}\big(\Gamma_{\nu,T}^\pi \overline{\Lambda} \big| \mathcal{F}_\nu\big).$$

Hence, for all stopping times $\nu \leqslant T$,

$$x_\nu^1 \geqslant \operatorname*{ess\,sup}_{\pi \in \widetilde{\Pi}} \mathbb{E}\big(\Gamma_{\nu,T}^\pi \overline{\Lambda} \big| \mathcal{F}_\nu\big). \tag{23}$$

Hence $\sup_{\pi \in \Pi} \mathbb{E}(\Gamma_{0,T}^\pi \overline{\Lambda}) \leqslant x_0^1 < +\infty$.

By using inequalities (22) and (23) we get for all stopping times $\nu \leqslant T$,

$$\bar{x}_\nu = \operatorname*{ess\,sup}_{\pi \in \Pi} \mathbb{E}\big(\Gamma_{\nu,T}^\pi \overline{\Lambda} \big| \mathcal{F}_\nu\big) = \operatorname*{ess\,sup}_{\pi \in \widetilde{\Pi}} \mathbb{E}\big(\Gamma_{\nu,T}^\pi \overline{\Lambda} \big| \mathcal{F}_\nu\big).$$

This completes the proof. □

The following remark plays a crucial role in our results.

Remark 6.1. Let $(x^1, z^1) \in \mathcal{C} \times \mathcal{L}^{2,d}$ be a solution of Eq. (17).

1. By using Fatou's lemma, one can see that x^1 satisfies the following inequality

$$x_t^1 \geqslant \mathbb{E}(\overline{\Lambda} | \mathcal{F}_t) = \mathbb{E}\big(F\big(H^{-1}\big(H(\Lambda) + \eta_T\big), C_T\big) \big| \mathcal{F}_t\big) \geqslant F\big(H^{-1}(\eta_t), C_t\big) \geqslant 0, \quad \forall t \in [0, T].$$

   This means that $(x_t^1, C_t, \eta_t) \in \mathcal{G}$, for all $(t, \omega) \in [0, T] \times \Omega$, where $\mathcal{G}$ is defined by (16).
2. For all $t \in [0, T]$, let us set

$$x_t := G\big(x_t^1, C_t, \eta_t\big). \tag{24}$$

   It is easy to see that

(a) $$\frac{\partial G}{\partial x}(x, c, \eta) = \frac{\phi(G(x, c, \eta)) e^{-c \int_D^{F^{-1}(x,c)} \psi(r)\, dr}}{\phi(F^{-1}(x, c))},$$

(b) $$\frac{\partial^2 G}{\partial x^2}(x, c, \eta) = \frac{(\frac{\partial G}{\partial x}(x, c, \eta))^2}{\phi(G(x, c, \eta))} \big[\phi'\big(G(x, c, \eta)\big) - \phi'\big(F^{-1}(x, c)\big) - c\phi\big(F^{-1}(x, c)\big)\psi\big(F^{-1}(x, c)\big)\big],$$

(c) $$\frac{\partial G}{\partial c}(x, c, \eta) = -\frac{\partial G}{\partial x}(x, c, \eta) \int_D^{F^{-1}(x,c)} e^{c \int_D^t \psi(r)\, dr} \int_D^t \psi(r)\, dr\, dt,$$

(d) $$\frac{\partial G}{\partial \eta}(x, c, \eta) = -\phi\big(G(x, c, \eta)\big).$$

Therefore, by using Itô's formula, one can see that x satisfies the following BSDE

$$x_t = \Lambda + \int_t^T \phi(x_s)\,d\eta_s + \int_t^T \frac{C_s\psi(x_s)}{2}|z_s|^2\,ds + \int_t^T R_s|z_s|\,ds + \int_t^T dk_s - \int_t^T z_s\,dB_s, \tag{25}$$

where (z,k) is given by:

$$z_s = \frac{\phi(x_s)e^{-C_s\int_0^{F^{-1}(x_s^1,C_s)}\psi(r)\,dr}}{\phi(F^{-1}(x_s^1,C_s))}z_s^1, \tag{26}$$

$$dk_s = -\frac{\partial G}{\partial c}\left(x_s^1,C_s,\eta_s\right)dC_s + \frac{1}{2}\frac{\phi(G(x_s^1,C_s,\eta_s))e^{-2C_s\int_0^{F^{-1}(x_s^1,C_s)}\psi(r)\,dr}}{(\phi(F^{-1}(x_s^1,C_s)))^2}\left|z_s^1\right|^2 M_s\,ds \tag{27}$$

with

$$M_s = \varphi\left(F^{-1}\left(x_s^1,C_s\right),C_s\right) - \varphi\left(G\left(x_s^1,C_s,\eta_s\right),C_s\right) \quad \text{and}$$
$$\varphi(x,c) = \phi'(x) + c\phi(x)\psi(x). \tag{28}$$

We can now formulate our main results of this section.

6.1. Main results

The following results give sufficient conditions for the solvability of $E^+(\Lambda,\phi(x)\,d\eta_s + \frac{C_s\psi(x)}{2}|z|^2\,ds + R_s|z|\,ds)$. Their proofs follow easily by using Remark 6.1.

Theorem 6.1. *Suppose that the following conditions hold*:

1. $\sup_{\pi\in\Pi}\mathbb{E}\Gamma_{0,T}^{\pi}\overline{\Lambda} < +\infty$.
2. *There exists a solution* (x^1,z^1) *to Eq.* (17) *such that,* dk *defined by* (27)*, is a positive measure.*

Then equation $E^+(\Lambda,\phi(x)\,d\eta_s + \frac{C_s\psi(x)}{2}|z|^2\,ds + R_s|z|\,ds)$ *has a solution* (x,z,k) *given by* (24), (26) *and* (27).

In particular, since $-\frac{\partial G}{\partial c}(x_s^1,C_s,\eta_s)\,dC_s$ is a positive measure, we have the following corollary.

Corollary 6.1. *Assume that*

1. $\sup_{\pi\in\Pi}\mathbb{E}\Gamma_{0,T}^{\pi}\overline{\Lambda} < +\infty$.
2. *There exists a solution* (x^1,z^1) *to Eq.* (17) *such that the process* M*, defined by* (28)*, is positive.*

Then equation $E^+(\varLambda, \phi(x)\,d\eta_s + \frac{C_s\psi(x)}{2}|z|^2\,ds + R_s|z|\,ds)$ *has a solution* (x, z, k) *given by* (24), (26) *and* (27).

An interesting corollary of Theorem 6.1 is the following.

Corollary 6.2. *Suppose that the following assumptions hold*:

1. $\sup_{\pi\in\varPi} \mathbb{E}\varGamma^{\pi}_{0,T}\overline{\varLambda} < +\infty$.
2. *The function* $x \mapsto \varphi(x, C_s(\omega))$, *given by* (28), *is nondecreasing on* $[D, +\infty[\; ds\,dP$ *a.e. for* (s, ω).

Then equation $E^+(\varLambda, \phi(x)\,d\eta_s + \frac{C_s\psi(x)}{2}|z|^2\,ds + R_s|z|\,ds)$ *has a solution* (x, z, k) *given by* (24), (26) *and* (27).

Remark 6.2. It follows from Hölder's inequality that, for all stopping times $\nu \leqslant T$,

$$\operatorname*{ess\,sup}_{\pi\in\varPi} \mathbb{E}\big(\varGamma^{\pi}_{\nu,T}\overline{\varLambda}\big|\mathcal{F}_\nu\big) \leqslant \Delta_\nu,$$

where

$$\Delta_\nu := \operatorname*{ess\,sup}_{n}\operatorname*{ess\,inf}_{q>1}\Big(\mathbb{E}\Big(e^{\frac{q}{2(q-1)}\int_\nu^T R_s^2\,ds}(\overline{\varLambda})^q 1_{\{\overline{\varLambda}+\int_0^T R_s^2\,ds\leqslant n\}}\Big|\mathcal{F}_\nu\Big)\Big)^{\frac{1}{q}}.$$

Indeed, for all $\pi \in \varPi$, $n \in \mathbb{N}$ and $q > 1$, we have

$$\begin{aligned}
&\mathbb{E}\big(\varGamma^{\pi}_{\nu,T}\overline{\varLambda}1_{\{\overline{\varLambda}+\int_0^T R_s^2\,ds\leqslant n\}}\big|\mathcal{F}_\nu\big)\\
&\quad\leqslant \Big(\mathbb{E}\Big(e^{\int_\nu^T \frac{q}{q-1}R_u\pi_u\,dB_u - \frac{1}{2}\int_\nu^T \frac{q^2}{(q-1)^2}R_u^2|\pi_u|^2\,du}\Big|\mathcal{F}_\nu\Big)\Big)^{\frac{q-1}{q}}\\
&\qquad\times \Big(\mathbb{E}\Big(e^{\frac{q}{2(q-1)}\int_\nu^T R_s^2\,ds}(\overline{\varLambda})^q 1_{\{\overline{\varLambda}+\int_0^T R_s^2\,ds\leqslant n\}}\Big|\mathcal{F}_\nu\Big)\Big)^{\frac{1}{q}}\\
&\quad\leqslant \Big(\mathbb{E}\Big(e^{\frac{q}{2(q-1)}\int_\nu^T R_s^2\,ds}(\overline{\varLambda})^q 1_{\{\overline{\varLambda}+\int_0^T R_s^2\,ds\leqslant n\}}\Big|\mathcal{F}_\nu\Big)\Big)^{\frac{1}{q}}.
\end{aligned}$$

Hence $\Delta_0 < +\infty$ is a sufficient condition to have $\sup_{\pi\in\varPi} \mathbb{E}\varGamma^{\pi}_{0,T}\overline{\varLambda} < +\infty$.

Remark 6.3. By taking into account the results of Corollary 6.2 and Remark 6.2, assumptions 1 and 2 of Theorem 6.1 can be replaced by the following strong assumptions:

1. $\Delta_0 < +\infty$.
2. The function $x \mapsto \varphi(x, C_s(\omega))$, given by (28), is nondecreasing on $[D, +\infty[\; ds\,dP$ a.e. for (s, ω).

In order to justify the assumptions we introduce to prove the existence of solutions for both one barrier GBSDE and GBSDE we give the following consequences.

6.2. Second consequences of Theorems 4.1 and 4.2: the unbounded case

In this subsection, we apply the results from the above sections to study the problem of existence of solutions to the GRBSDE (10) and GBSDE (14). We give various existence results dealing with the case of unbounded terminal condition ξ and unbounded barrier L.

6.2.1. One barrier GBSDE

The following corollary follows from Theorems 4.1 and 6.1.

Corollary 6.3. *Suppose that the following assumptions hold*:

1. $\sup_{\pi\in\Pi}\mathbb{E}\Gamma^{\pi}_{0,T}\overline{\Lambda}<+\infty$.
2. *There exists a solution* (x^1,z^1) *to Eq.* (17) *such that* dk*, defined by* (27)*, is a positive measure.*
3. $\xi\vee\sup_{t\leqslant T}L_t\leqslant\Lambda$.
4. *For all* $(s,\omega)\in[0,T]\times\Omega$
$$f(s,\omega,x_s,z_s)\leqslant\alpha_s\phi(x_s)+\frac{C_s\psi(x_s)}{2}|z_s|^2+R_s|z_s|,$$
$$g(s,\omega,x_s)\leqslant\beta_s\phi(x_s).$$
5. *There exist two nonnegative predictable processes* $\bar{\alpha}$ *and* $\bar{\beta}$ *such that* $\int_0^T\bar{\alpha}_s\,ds+\int_0^T\bar{\beta}_s\,dA_s<+\infty$ P*-a.s., and* $\overline{\psi}\in\mathcal{C}$ *such that* $\forall(s,\omega)$ *and* $\forall(y,z)$ *satisfying* $L_s\leqslant y\leqslant x_s$
$$\big|f(s,\omega,y,z)\big|\leqslant\bar{\alpha}_s+\frac{\overline{\psi}_s}{2}|z|^2\quad\text{and}\quad\big|g(s,\omega,y)\big|\leqslant\bar{\beta}_s,$$
where x_t *and* z_t *are given respectively by relations* (24) *and* (26).

Then the GRBSDE (10) *has a solution such that* $L_t\leqslant Y_t\leqslant x_t$.

The following corollaries are direct and interesting applications of Corollaries 6.2–6.3 and Remark 6.3, since all the required assumptions are obviously satisfied.

Corollary 6.4. *Suppose that there exists nonnegative real number* D *such that*:

(i) $R=0$, $\phi(x)=x$ *on* $[D,+\infty[$, $\psi(x)=1$ *on* $[D,+\infty[$ *and* $C\in\mathbb{R}_++\mathcal{K}$.

(ii) $\mathbb{E}\overline{\Lambda}<+\infty$, *where* $\overline{\Lambda}=\frac{e^{C_T(\Lambda e^{\eta T}-D)}-1}{C_T}1_{\{C_T>0\}}+(\Lambda e^{\eta T}-D)1_{\{C_T=0\}}$ *and* $\Lambda=\xi\vee\sup_{t\leqslant T}L_t\vee D$.

(iii) *There exist two nonnegative predictable processes* $\bar{\alpha}$ *and* $\bar{\beta}$ *satisfying* $\int_0^T\bar{\alpha}_s\,ds+\int_0^T\bar{\beta}_s\,dA_s<+\infty$ P*-a.s., and* $\overline{\psi}\in\mathcal{C}$ *such that* $\forall(s,\omega)$ *and* $\forall(y,z)$ *satisfying* $L_s\leqslant y\leqslant x_s$
$$-\bar{\alpha}_s-\frac{\overline{\psi}_s}{2}|z|^2\leqslant f(s,y,z)\leqslant\alpha_s\phi\big(|y|\big)+\frac{C_s\psi(|y|)}{2}|z|^2,$$
$$-\bar{\beta}_s\leqslant g(s,\omega,y)\leqslant\beta_s\phi\big(|y|\big),$$
where
$$x_s=G\big(\mathbb{E}(\overline{\Lambda}|\mathcal{F}_s),C_s,\eta_s\big)$$
$$=e^{-\eta_s}\left[D+\frac{\ln(1+C_s\mathbb{E}(\overline{\Lambda}|\mathcal{F}_s))}{C_s}1_{\{C_s>0\}}+\mathbb{E}(\overline{\Lambda}|\mathcal{F}_s)1_{\{C_s=0\}}\right].$$

Then the GRBSDE (10) *has a solution such that* $L_t\leqslant Y_t\leqslant x_t$.

Corollary 6.5. *Suppose that there exist two real numbers $D > 1$ and $m > 0$ such that*:

(i) $R = 0$, $\phi(x) = x\ln(x)$ *on* $[D, +\infty[$, $\psi(x) = 1$ *on* $[D, +\infty[$ *and* $C_s = m$, $\forall s \in [0, T]$.
(ii) $\mathbb{E}e^{me^{\ln(\Lambda)e^{\eta_T}}} < +\infty$, *where* $\Lambda = \xi \vee \sup_{t\leqslant T} L_t \vee D$ *and* $\eta_t := \int_0^t \alpha_s\,ds + \int_0^t \beta_s\,dA_s$.
(iii) *There exist two nonnegative predictable processes* $\bar\alpha$ *and* $\bar\beta$ *satisfying* $\int_0^T \bar\alpha_s\,ds + \int_0^T \bar\beta_s\,dA_s < +\infty$ *P-a.s., and* $\overline{\psi} \in \mathcal{C}$ *such that* $\forall(s,\omega)$ *and* $\forall(y,z)$ *satisfying* $L_s \leqslant y \leqslant x_s$

$$-\bar\alpha_s - \frac{\overline{\psi}_s}{2}|z|^2 \leqslant f(s,y,z) \leqslant \alpha_s\phi\big(|y|\big) + \frac{m\psi(|y|)}{2}|z|^2,$$
$$-\overline{\beta}_s \leqslant g(s,\omega,y) \leqslant \alpha_s\phi\big(|y|\big),$$

where $x_s = G(\mathbb{E}(e^{m\ln(\Lambda)e^{\eta_T}} - \frac{1}{m}|\mathcal{F}_s), C_s = m, \eta_s) = e^{e^{-\eta_s}\ln[D+\frac{1}{m}\ln(\mathbb{E}(e^{me^{\ln(\Lambda)e^{\eta_T}}}|\mathcal{F}_s))]}$.

Then the GRBSDE (10) *has a solution such that* $L_t \leqslant Y_t \leqslant x_t$.

Corollary 6.6. *Suppose that there exist two positive real numbers D and m such that*:

(i) $R = 0$, $\phi(x) = x$ *on* $[D, +\infty[$, $\psi(x) = x$ *on* $[D, +\infty[$ *and* $C_s = m$, $\forall s \in [0, T]$.
(ii) $\mathbb{E}(\int_0^{\Lambda e^{\eta_T}} e^{\frac{m}{2}t^2}\,dt) < +\infty$, *where* $\Lambda = \xi \vee \sup_{t\leqslant T} L_t \vee D$.
(iii) *There exist two nonnegative predictable processes* $\bar\alpha$ *and* $\bar\beta$ *satisfying* $\int_0^T \bar\alpha_s\,ds + \int_0^T \bar\beta_s\,dA_s < +\infty$ *P-a.s., and* $\overline{\psi} \in \mathcal{C}$ *such that* $\forall(s,\omega)$ *and* $\forall(y,z)$ *satisfying* $L_s \leqslant y \leqslant x_s$,

$$-\bar\alpha_s - \frac{\overline{\psi}_s}{2}|z|^2 \leqslant f(s,y,z) \leqslant \alpha_s\phi\big(|y|\big) + \frac{m\psi(|y|)}{2}|z|^2,$$
$$-\overline{\beta}_s \leqslant g(s,\omega,y) \leqslant \beta_s\phi\big(|y|\big),$$

where $x_s = e^{-\eta_s}F_0^{-1}(\mathbb{E}(F_0(\Lambda e^{\eta_T})|\mathcal{F}_t))$ *where the function* F_0 *is defined by*: $F_0(x) = \int_D^x e^{\frac{m}{2}(t^2-D^2)}\,dt$ *and* F_0^{-1} *its inverse.*

Then the GRBSDE (10) *has a solution such that* $L_t \leqslant Y_t \leqslant x_t$.

Corollary 6.7. *Suppose that there exist two positive real numbers D and m such that*:

(i) $\phi(x) = x$ *on* $[D, +\infty[$, $\psi(x) = 0$ *on* $[D, +\infty[$, $R \in \mathcal{L}^{2,1}$ *and* $C \in \mathbb{R}_+ + \mathcal{K}$.
(ii) *There exists* $q > 1$ *such that* $\mathbb{E}(e^{\frac{q}{2(q-1)}\int_0^T R_s^2\,ds}(\Lambda e^{\eta_T} - D)^q) < +\infty$ *where* $\Lambda = \xi \vee \sup_{t\leqslant T} L_t \vee D$.
(iii) *There exist two nonnegative predictable processes* $\bar\alpha$ *and* $\bar\beta$ *such that* $\int_0^T \bar\alpha_s\,ds + \int_0^T \bar\beta_s\,dA_s < +\infty$ *P-a.s., and* $\overline{\psi} \in \mathcal{C}$ *such that* $\forall(s,\omega)$ *and* $\forall(y,z)$ *satisfying* $L_s \leqslant y \leqslant x_s$

$$-\bar\alpha_s - \frac{\overline{\psi}_s}{2}|z|^2 \leqslant f(s,\omega,y,z) \leqslant \alpha_s\phi\big(|y|\big) + \frac{C_s\psi(|y|)}{2}|z|^2 + R_s|z|,$$
$$-\overline{\beta}_s \leqslant g(s,\omega,y) \leqslant \beta_s\phi\big(|y|\big),$$

where $x_s = \operatorname{ess\,sup}_{\pi\in\Pi}(e^{-\eta_s}(\mathbb{E}(\Gamma^{\pi}_{s,T}\Lambda e^{\eta_T}|\mathcal{F}_s)))$.

Then the GRBSDE (10) *has a solution such that* $L_t \leqslant Y_t \leqslant x_t$.

6.2.2. GBSDE without reflection

By combining Theorems 4.2 and 6.1 we obtain the following.

Corollary 6.8. *Assume that the following hold*:

1. $\sup_{\pi\in\Pi}\mathbb{E}\Gamma^{\pi}_{0,T}\overline{\Lambda}<+\infty$.
2. *There exists a solution* (x^1,z^1) *to Eq.* (17) *such that* dk*, defined by* (27)*, is a positive measure.*
3. $|\xi|\leqslant\Lambda$.
4. $\forall(s,\omega)$ *and* $\forall(y,z)$ *satisfying* $|y|\leqslant x_s$

$$\begin{aligned}&\left|f(s,\omega,y,z)\right|\leqslant\alpha_s\phi\big(|y|\big)+\frac{\psi(|y|)}{2}|z|^2+R_s|z|,\\&\left|g(s,\omega,y)\right|\leqslant\beta_s\phi\big(|y|\big),\end{aligned}$$

where x_s *is given by* (24).

Then the GRBSDE (14) *has a solution such that* $|Y_t|\leqslant x_t$.

Corollary 6.9. *Suppose that there exists nonnegative real number D such that*:

(i) $R=0$, $\phi(x)=x$ *on* $[D,+\infty[$, $\psi(x)=1$ *on* $[D,+\infty[$ *and* $C\in\mathbb{R}_+ +\mathcal{K}$.
(ii) $\mathbb{E}\overline{\Lambda}<+\infty$, *where* $\overline{\Lambda}=\frac{e^{C_T(\Lambda e^{\eta_T}-D)}-1}{C_T}1_{\{C_T>0\}}+(\Lambda e^{\eta_T}-D)1_{\{C_T=0\}}$ *and* $\Lambda=|\xi|\vee D$.
(iii) $\forall(s,\omega)$ *and* $\forall(y,z)$ *satisfying* $|y|\leqslant x_s$

$$\begin{aligned}&\left|f(s,y,z)\right|\leqslant\alpha_s\phi\big(|y|\big)+\frac{C_s\psi(|y|)}{2}|z|^2,\\&\left|g(s,\omega,y)\right|\leqslant\beta_s\phi\big(|y|\big),\end{aligned}$$

where

$$\begin{aligned}x_s&=G\big(\mathbb{E}(\overline{\Lambda}|\mathcal{F}_s),C_s,\eta_s\big)\\&=e^{-\eta_s}\left[D+\frac{\ln(1+C_s\mathbb{E}(\overline{\Lambda}|\mathcal{F}_s))}{C_s}1_{\{C_s>0\}}+\mathbb{E}(\overline{\Lambda}|\mathcal{F}_s)1_{\{C_s=0\}}\right].\end{aligned}$$

Then the GRBSDE (14) *has a solution such that* $|Y_t|\leqslant x_t$.

The following remark gives a sufficient condition for the existence of solution for the BSDE (14) when $f(s,y,z)=\frac{\gamma_s}{2}|z|^2$ and $g(s,y)=0$.

Remark 6.4. Let γ be a nonnegative process which is $\mathcal{F}_t$-adapted and $C_t=\sup_{0\leqslant s\leqslant t}\gamma_s$, $\forall t\in[0,T]$. We consider the following BSDE

$$Y_t=\xi+\int_t^T\frac{\gamma_s}{2}|Z_s|^2\,ds-\int_t^T Z_s\,dB_s.\tag{29}$$

It follows from Corollary 6.9 that if

$$\mathbb{E}\left[\frac{e^{C_T|\xi|}-1}{C_T}1_{\{C_T>0\}}+|\xi|1_{\{C_T=0\}}\right]<+\infty,$$

then the BSDE (29) has a solution satisfying

$$|Y_t| \leqslant \frac{\ln(1 + C_t \mathbb{E}(\overline{\Lambda}|\mathcal{F}_t))}{C_t} 1_{\{C_t>0\}} + \mathbb{E}(\overline{\Lambda}|\mathcal{F}_t) 1_{\{C_t=0\}},$$

where $\overline{\Lambda} = \frac{e^{C_T|\xi|}-1}{C_T} 1_{\{C_T>0\}} + |\xi| 1_{\{C_T=0\}}$.

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