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ON J. NAGATA'S UNIVERSAL SPACES

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The main result of this paper is the following extension of an embedding theorem by Nagata: given a sequence of zero-dimensional sets $X_1, X_2, \ldots$ in a metrizable space X of weight $\tau \geq \aleph_0$, the set of homeomorphic embeddings h of X into $S(\tau)^{\aleph_0}$, satisfying $h(X_n) \subset K_{n-1}(\tau)$ for $n = 1, 2, \ldots$, is dense in the function space of all continuous mappings of X into $S(\tau)^{\aleph_0}$, where $K_n(\tau)$ is the n-dimensional universal Nagata's space in the countable product of the star-space $S(\tau)$ of weight τ. This seems to be a new result even in the separable case $\tau = \chi_0$ and provides in particular an answer to a question asked by Kuratowski (see Remark 2.6 for the details).

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Nagata's n-dimensional universal space	zero-dimensional
countable dimensional	generalized Baire space

1. Notation and basic definitions

Our terminology follows [9]. All spaces considered in this paper are assumed to be metrizable. By dimension we understand the covering dimension dim. A space X is countable-dimensional if it is the union of countably many 0-dimensional subsets. We denote by I the unit interval with the usual metric and by I^ω the Hilbert cube with the standard metric $\rho(\{x_i\}, \{y_i\}) = [\sum_{i=1}^{\infty} 2^{-i}|x_i - y_i|^2]^{1/2}$. By $|A|$ we denote the cardinality of the set A.

1.1. The function space $C(X, Y)$. Given spaces X and Y, with a fixed bounded metric $\tilde{\rho}$ in Y, we denote by $C(X, Y)$ the space of all continuous mappings of X to Y, endowed with the supremum metric $d(f, g) = \sup\{\tilde{\rho}(f(x), g(x)) \colon x \in X\}$.

1.2. The universal metrizable space $S(\tau)^{\aleph_0}$. Let $\{I_\alpha \colon \alpha \in A\}$ be a system of intervals $I_\alpha = I$. The star-space with the index set A is the set $S(A)$ obtained by identifying all zeros in $\bigcup\{I_\alpha \colon \alpha \in A\}$, endowed with the metric

$$\rho(x, y) = \begin{cases} |x-y| & \text{if } x, y \text{ belong to the same interval } I_\alpha, \\ x+y & \text{if } x, y \text{ belong to distinct intervals.} \end{cases}$$

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The τ-star space $S(\tau)$ is the space $S(A)$, where $|A| = \tau$ (this space is also called a hedgehog of spininess τ). By a rational point in $S(\tau)$ we mean a point which is at a rational distance from 'origin' O. By a theorem of Kowalsky [2] every metrizable space of weight $\tau \geq \aleph_0$ is homeomorphic with a subspace of the countable power $S(\tau)^{\aleph_0}$ of the τ-star-space. We fix a metrix in the space $S(\tau)^{\aleph_0}$ by putting

$$\tilde{\rho}(\{x_i\}, \{y_i\}) = \left[\sum_{i=1}^{\infty} 2^{-i} \rho(x, y)^2\right]^{1/2}.$$

1.3. The Nagata's n-dimensional and countable-dimensional universal spaces $K_n(\tau)$ and $K_\infty(\tau)$. Let $K_n(\tau)$ (respectively $K_\infty(\tau)$) be the subspace of the space $S(\tau)^{\aleph_0}$ consisting of all points in $S(\tau)^{\aleph_0}$ which have at most n (respectively, only finitely many) rational coordinates distinct from 0. Nagata proved (see [6; 7; 9, Theorems VI. 10 and VI.11) that a metrizable space X of weight $\leq \tau$ has $\dim X \leq n$ (respectively, X is countable-dimensional) if and only if X is homeomorphic to a subset of $K_n(\tau)$ (respectively, $K_\infty(\tau)$).

1.4. The generalized Baire space $B(\tau)$. Let τ be a cardinal number $\geq \aleph_0$. By $B(\tau)$ we denote the generalized Baire space of weight τ, i.e. the Cartesian power $D(\tau)^{\aleph_0}$, where $D(\tau)$ is the discrete space of cardinality τ. We fix a metrix σ in $B(\tau)$ by putting

$$\sigma(\{\alpha_i\}, \{\beta_i\}) = \frac{1}{\min\{i\colon \alpha_i \neq \beta_i\}},$$

$$\text{where } \alpha_i, \beta_i \in D(\tau) \quad \text{for } i = 1, 2, \ldots, \quad \{\alpha_i\} \neq \{\beta_i\}.$$

1.5. Strongly metrizable spaces. A space is strongly metrizable if it has a σ-star-finite open basis (see [1, Ch. 6, § 3]). As was proved by Morita [3], every strongly metrizable space of weight $\tau \geq \aleph_0$ can be embedded into the product $B(\tau) \times I^\omega$. Below we consider the space $B(\tau) \times I^\omega$ with the fixed metric

$$\tilde{\rho}((x_1, y_1), (x_2, y_2)) = [\sigma(x_1, x_2)^2 + \rho(y_1, y_2)^2]^{1/2}$$

for $x_1, x_2 \in B(\tau)$, $y_1, y_2 \in I^\omega$, where σ is a metric defined in 1.4 and ρ is the standard metric in I^ω.

1.6. The Nagata's universal spaces for n-dimensional and countable-dimensional strongly metrizable spaces of given weight. Let N_n^ω (respectively, N_∞^ω) be the subspace of the Hilbert cube I^ω consisting of all points which have at most n (respectively, only finitely many) rational coordinates. J. Nagata proved (see [4; 5; 9, Theorem VI.5]) that a strongly metrizable space X of weight $\tau \geq \aleph_0$ has $\dim X \leq n$ (respectively, X is countable-dimensional) if and only if X is homeomorphic to a subset of the product $B(\tau) \times N_n^\omega$ (respectively, $B(\tau) \times N_\infty^\omega$). In particular, a metrizable separable space is countable-dimensional if and only if it is homeomorphic to a subset of $N_\infty^\omega \subset I^\omega$.

2. The main results

The following theorem is the main result of this paper.

2.1. Theorem. *Let X be a metrizable space of weight $\tau \geqslant \aleph_0$ and $X_1, X_2, \ldots$ a sequence of subspaces such that* $\dim X_n \leqslant 0$ *for* $n = 1, 2, \ldots$. *Then the set*

$$\mathcal{H} = \{h \in C(X, S(\tau)^{\aleph_0}) \colon h \text{ is an embedding and } h(X_n) \subset K_{n-1}(\tau) \text{ for every } n = 1, 2, \ldots\}$$

is dense in the space $C(X, S(\tau)^{\aleph_0})$.

2.2. Corollary. *If X is a metrizable space of weight $\tau \geqslant \aleph_0$ and* $\dim X \leqslant n$ *(respectively, X is countable-dimensional), then the set of all homeomorphic embeddings of X into $K_n(\tau)$ (respectively, $K_\infty(\tau)$) is dense in the space* $C(X, S(\tau)^{\aleph_0})$.

We prove also a similar result about strongly metrizable spaces.

2.3. Theorem. *Let X be a strongly metrizable space of weight $\tau \geqslant \aleph_0$ and $X_1, X_2, \ldots$ a sequence of subspaces such that* $\dim X_n \leqslant 0$ *for* $n = 1, 2, \ldots$. *Then the set*

$$\mathcal{H} = \{h \in C(X, B(\tau) \times I^\omega) \colon h \text{ is an embedding and } h(X_n) \subset B(\tau) \times N^\omega_{n-1} \text{ for every } n = 1, 2, \ldots\}$$

is dense in the space $C(X, B(\tau) \times I^\omega)$.

2.4. Corollary. *If X is a strongly metrizable space of weight $\tau \geqslant \aleph_0$ and* $\dim X \leqslant n$ *(respectively, X is countable-dimensional), then the set of all homeomorphic embeddings of X into $B(\tau) \times N^\omega_n$ (respectively, $B(\tau) \times N^\omega_\infty$) is dense in the space $C(X, B(\tau) \times I^\omega)$.*

2.5. Remark. It seems that Theorems 2.1. and 2.3 are new even for separable spaces, i.e. for $\tau = \aleph_0$. In the special case of separable spaces Corollary 2.4 is a classical theorem for n-dimensional spaces (see [9, IV.4.C] or [1, Ch. 4, Theorem 12]), but gives a new result for countable-dimensional spaces:

For a countable-dimensional metrizable separable space X the set of all homeomorphic embeddings of X into N^ω_∞ is dense in the space $C(X, I^\omega)$.

2.6. Remark. Corollary 2.2 provides in particular a positive answer to the following question stated by Nagata [8] (attributed to K. Kuratowski):

Is the set of all homeomorphic embeddings of an n-dimensional metrizable space X of weight $\tau > \aleph_0$ into $K_n(\tau)$ dense in the space $C(X, S(\tau)^{\aleph_0})$ of all continuous mappings of X into $S(\tau)^{\aleph_0}$ (as in the separable case)?

However, in this case the result can be considerably strengthened (see [11, Theorem 3.1]):

For an n-dimensional metrizable space X of weight $\tau \geq \aleph_0$ the set $\mathscr{H}$ of all homeomorphic embeddings h of X into $S(\tau)^{\aleph_0}$ such that $\overline{h(X)} \subset K_n(\tau)$ is a residual set in $C(X, S(\tau)^{\aleph_0})$ (i.e. $\mathscr{H}$ contains a dense G_δ-subset of $C(X, S(\tau)^{\aleph_0})$).

Such a strengthening is not possible in a more general situation considered in Theorem 2.1. As we show in Remark 5.2, the set $\mathscr{H}$ of all homeomorphic embeddings of I into I^ω such that $h(P) \subset P^{\aleph_0} = N_0^\omega$, where P is the set of irrationals, is of the first category (both in $C(X, I^\omega)$ and as a space). Moreover, the set $\mathscr{H}$ of all homeomorphic embeddings of a countable-dimensional metrizable space X of weight τ into the Nagata's countable-dimensional universal space $K_\infty(\tau)$ need not be residual in $C(X, S(\tau)^{\aleph_0})$. This follows from the following result, obtained recently by the author:

Let X be a complete metric separable space. Then the set of all homeomorphic embeddings of X into the Nagata's universal space N_∞^ω is residual in $C(X, I^\omega)$ if and only if X is strongly countable-dimensional, i.e. X is the union of countably many closed finite-dimensional subsets.

The proof will be published elsewhere.

2.7. Remark. The main idea of the proofs of Theorems 2.1 and 2.3 is to modify the proofs of Nagata (which exploit the idea of the standard proof of Urysohn's Lemma) by applying a method of constructing embeddings as limits of Cauchy sequences of inductively defined functions. This method leads also to a simplification of the original proofs of Nagata (see Section 5; for the simplest case of 0-dimensional spaces, cf. also [10, 6.4]).

3. Auxiliary lemma

Recall that if $\mathscr{A}$ is a family of subsets of X, then the order of $\mathscr{A}$ at a point x, abbreviated $\operatorname{ord}_x\mathscr{A}$, is the number of elements of $\mathscr{A}$ which contain x. For any subset V of a space X let $B(V)$ denote the boundary of V in X.

The following lemma can be extracted from the proof of III.4.A) of [9] or Lemma 1, Ch.10, § 2 of [1].

3.1. Lemma. *Let X be a metrizable space and $X_1, X_2, \ldots$ a sequence of subspaces such that* $\dim X_n \leq 0$ *for* $n = 1, 2, \ldots$ *and let* $\{K_1, K_2, \ldots, K_k\}$ *be a family of closed subsets of X such that*

$$\operatorname{ord}_x\{K_1, K_2, \ldots, K_k\} \leq n-1$$

for each $x \in X_n$. Then for every two disjoint closed subsets C and D of X there exists an open subset V of X such that $C \subset V$, $\bar{V} \cap D = \emptyset$ and

$$\operatorname{ord}_x\{K_1, K_2, \ldots, K_k, B(V)\} \leq n-1$$

for every $x \in X_n$.

4. Proofs

Let $g=\{g_m\}$ belong to $C(X, S(\tau)^{\aleph_0})$. Then every open ball with center g with respect to the metric d described in 1.1, where $\tilde{\rho}$ is a metric in $S(\tau)^{\aleph_0}$ introduced in 1.2, contains an open neighbourhood of the form

$$U(g, \varepsilon, m_0)=\{f=\{f_m\}\in C(X, S(\tau)^{\aleph_0}): \rho(f_m(x), g_m(x))<\varepsilon \text{ for every } x\in X \text{ and } m=1,2,\ldots, m_0\},$$

where m_0 is a natural number and $\varepsilon>0$. Thus Theorem 2.1 can be deduced directly from the following proposition.

4.1. Proposition. *Suppose that X is a metrizable space of weight $\tau\geqslant\aleph_0$ and $X_1, X_2,\ldots$ a sequence of subspaces such that* $\dim X_n\leqslant 0$ *for $n=1,2,\ldots$. Let $S(\tau)^{\aleph_0}=\mathbb{P}_{m=1}^{\infty} S_m(A_m)$, where $S_m(A_m)$ is the star-space with the index set $A_m=A$ for every $m=1,2,\ldots$, and $|A|=\tau$. Let $g=\{g_m\}: X\to S(\tau)^{\aleph_0}$ be any continuous mapping, m_0 any natural number and ε any real number such that $0<\varepsilon<1$. Then there exists an embedding $h=\{h_m\}: X\to S(\tau)^{\aleph_0}$ such that*

$$h(X_n)\subset K_{n-1}(\tau), \text{ for every } n=1,2,\ldots,$$

and

$$\rho(h_m(x), g_m(x))<\varepsilon \text{ for every } x\in X \text{ and } m=1,2,\ldots, m_0.$$

Proof. Let us take a σ-discrete open basis of X, which is the union of discrete families $\mathcal{W}_m=\{W_{m\alpha}: \alpha\in A_m\}$ for $m=m_0+1, m_0+2,\ldots$. We can assume without loss of generality that there exist open sets $V_{m\alpha}$, $\alpha\in A_m$, $m=m_0+1, m_0+2,\ldots$ such that $F_{m\alpha}=\bar{V}_{m\alpha}\subset W_{m\alpha}$ and

$$\text{for every neighbourhood } U(x) \text{ of every point } x\in X \text{ there exist } m \text{ and } \alpha\in A_m \text{ for which } x\in F_{m\alpha}\subset W_{m\alpha}\subset U(x). \tag{1}$$

We can assume that A_m is the same as in the formulation of the Proposition. Putting $W_m=\bigcup\{W_{m\alpha}: \alpha\in A_m\}$ and $F_m=\bigcup\{F_{m\alpha}: \alpha\in A_m\}$ we obtain open sets W_m and closed sets F_m satisfying $F_m\subset W_m$, where $m=m_0+1, m_0+2,\ldots$.

We will construct a sequence $\{f_m\}$ of mappings $f_m: X\to S_m(A_m)$ such that

$$\rho(f_m(x), g_m(x))<\varepsilon \quad \text{for } m=1,2,\ldots, m_0 \text{ and every } x\in X, \tag{2}$$

$$f_m(X\setminus W_m)=0,\ f_m(F_{m\alpha})=1\in I_\alpha \text{ and } f_m(W_{m\alpha})\subset I_\alpha \text{ for } m=m_0+1, m_0+2,\ldots, \tag{3}$$

and

$$|\{m: f_m(x) \text{ is a rational point distinct from 0 and 1}\}|\leqslant n-1 \text{ for every } x\in X_n. \tag{4}$$

Arrange all rational numbers in $(0, 1)$ into a sequence $r_2, r_3, \ldots$ (without repetitions) and put

$$\delta_i = \min\{|r_j - r_k|, |r_j|, |r_j - 1| \colon k, j \leq i\},$$

$$\varepsilon_i = 2^{-(i+3)} \cdot \delta_i \cdot \varepsilon, \qquad a_i = r_i - \varepsilon_i, \qquad b_i = r_i + \varepsilon_i,$$

$$K_i = \{x \in S(A) \colon \rho(x, 0) \leq a_i\}, \qquad L_i = \{x \in S(A) \colon \rho(x, 0) \geq b_i\}$$

and

$$J_i = \{x \in S(A) \colon a_i < \rho(x, 0) < b_i\}.$$

For each m we will define a uniformly convergent sequence $\{f_{mi}\}_{i=1}^{\infty}$ of mappings $f_{mi} : X \to S_m(A_m)$ such that the limit $f_m = \lim_{i\to\infty} f_{mi}$ will be the required mapping.

We will define f_{mj} by induction in the following order: $f_{11}, f_{12}, f_{21}, f_{13}, f_{22}, f_{31}, f_{14}, \ldots$. Suppose that $f_{mj} = f_{11}$ or that we have already defined all f_{kp} before f_{mj} and that for each such f_{kp} we have defined an open set $V_{kp} \subset X$ such that

$$f_{kp}^{-1}(r_p) = B(V_{kp}) \quad \text{for } p \geq 2 \quad \text{and} \quad V_{kp} = \emptyset \quad \text{for } p = 1, \tag{5}$$

$$\operatorname{ord}_x\{B(V_{11}), B(V_{12}), \ldots, B(V_{kp})\} \leq n - 1 \text{ for each } x \in X_n. \tag{6}$$

To define f_{mj} we consider three cases:

(a) Suppose that $m \in \{1, 2, \ldots, m_0\}$ and $j = 1$. Then we put $f_{mj} = g_m$ and $V_{mj} = \emptyset$.

(b) Suppose that $m > m_0$ and $j = 1$. As f_{mj} we take any function such that $f_{mj}(X \setminus W_m) = 0$, $f_{mj}(W_{m\alpha}) \subset I_\alpha$ and $f_{mj}(F_{m\alpha}) = 1 \in I_\alpha$ and we put $V_{mj} = \emptyset$.

(c) Suppose that $m \in \{1, 2, \ldots\}$ and $j > 1$. Let

$$C = f_{mj-1}^{-1}(K_j), \qquad C_\alpha = f_{mj-1}^{-1}(K_j \cap I_\alpha), \qquad D = f_{mj-1}^{-1}(L_j),$$

$$D_\alpha = f_{mj-1}^{-1}(L_j \cap I_\alpha).$$

By Lemma 3.1, there exists an open set V_{mj} of X such that

$$C \subset V_{mj}, \ \bar{V}_{mj} \cap D = \emptyset \quad \text{and} \quad \operatorname{ord}_x\{B(V_{11}), B(V_{12}), \ldots, B(V_{mj})\} \leq n - 1$$

for every $x \in X_n$. We use Tietze's Extension Theorem to define, for every $\alpha \in A_m$, a function $f_\alpha : f_{mj-1}^{-1}(I_\alpha) \to I_\alpha$ such that f_α coincides with f_{mj-1} on $C_\alpha \cup D_\alpha$, $f_\alpha^{-1}((K_j \cup L_j) \cap I_\alpha) = C_\alpha \cup D_\alpha$ and $f_\alpha^{-1}(r_j) = B(V_{mj}) \cap f_{mj-1}^{-1}(I_\alpha)$. As f_{mj} we take the combination of all f_α, where $\alpha \in A_m$.

It is easy to verify that for every $x \in X$

$$\rho(f_{mi+1}(x), f_{mi}(x)) \leq 2 \cdot \varepsilon_i = 2^{-(i+3)} \cdot \delta_{i+1} \cdot \varepsilon.$$

Thus, for every $x \in X$, $\{f_{mi}(x)\}_{i=1}^{\infty}$ is a Cauchy sequence, so the sequence $\{f_{mi}\}_{i=1}^{\infty}$ of functions converges to a function $f_m : X \to S_m(A_m)$.

For every $x \in X$ and all natural numbers m and k

$$\rho(f_m(x), f_{mk}(x)) \leq 2^{-3} \cdot \varepsilon \cdot [2^{-1} \cdot \delta_{k+1} + 2^{-2} \cdot \delta_{k+2} + \cdots] \leq \tfrac{1}{8} \cdot \varepsilon \cdot \delta_k, \tag{7}$$

hence the sequence $\{f_{mk}\}_{k=1}^{\infty}$ is uniformly convergent and f_m is continuous.

In particular, for $m = 1, 2, \ldots, m_0$ and every $x \in X$

$$\rho(f_m(x), g_m(x)) = \rho(f_m(x), f_{m1}(x)) \leq \tfrac{1}{8} \cdot \varepsilon < \varepsilon,$$

hence condition (2) is satisfied.

For every $m > m_0$ we have $f_{m1}(X \setminus W_m) = 0$, $f_{m1}(W_{m\alpha}) \subset I_\alpha$ and $f_{m1}(F_{m\alpha}) = 1 \in I_\alpha$. Since all f_{mj} coincide with f_{m1} on $X \setminus W_m$ and $F_{m\alpha}$, and $f_{mj}^{-1}(I_\alpha) = f_{m1}^{-1}(I_\alpha)$, condition (3) is satisfied.

Now, let $x \in X_n$. Since, by (6), $\operatorname{ord}_x\{B(V_{11}), B(V_{12}), B(V_{21}), \ldots\} \leq n-1$, the set $N_0 = \{k: x \in B(V_{kp}) \text{ for some } p\}$ consists of at most $n-1$ natural numbers. Let $m \notin N_0$ and let r_i be any rational number, where $i \geq 2$. Since $x \notin B(V_{mi})$, then $f_{mi}(x)$ is a point of some I_α different from $r_i \in I_\alpha$. For every $j > i$ we have $f_{mj}(x) \in I_\alpha$. If $f_{mj}(x) = f_{mi}(x)$ for all $j > i$, then $f_m(x) = f_{mi}(x) \neq r_i \in I_\alpha$. In the other case, let k be the smallest integer greater than i, with $f_{mk}(x) \neq f_{mi}(x)$. Then $f_{mk-1}(x) = f_{mi}(x) \in J_k$ and therefore $\rho(f_{mk}(x), r_i) \geq \delta_k - \varepsilon_k \geq \tfrac{7}{8} \cdot \delta_k$. On the other hand, $\rho(f_m(x), f_{mk}(x)) \leq \tfrac{1}{8} \cdot \varepsilon \cdot \delta_k \leq \tfrac{1}{8} \cdot \delta_k$ by (7), hence $\rho(f_m(x), r_i) \geq \tfrac{5}{8} \cdot \delta_k > 0$. Therefore $f_m(x) \neq r_i$ for $m \notin N_0$, so condition (4) is satisfied.

Let us take a positive rational number $a < 1$ such that the functions $h_m : X \to S_m(A_m)$ defined by $h_m(x) = a \cdot f_m(x) \in I_\alpha$ if $x \in I_\alpha$ also satisfy the condition $\rho(h_m(x), g_m(x)) < \varepsilon$ for every $x \in X$ and $m = 1, 2, \ldots, m_0$. Put $h = \{h_m\} : X \to S(\tau)^{\aleph_0}$. Since $h_m(x) < 1$ for every $x \in X$ and every m, then by (4) we have $h(X_n) \subset K_{n-1}(\tau)$ for every $n = 1, 2, \ldots$. The mapping h is an embedding, because by (1) and (3) the family $\{h_m\}$ separates points from closed sets. Thus h has all required properties. $\square$

Corollary 2.2 follows immediately for the finite-dimensional case from an application of the decomposition theorem (see [9, Theorem II.4]). If X is countable-dimensional, we note that the monotonicity of dimension implies that X can be decomposed into a union of countably many disjoint sets of dimension ≤ 0.

The proof of Theorem 2.3 follows easily from the following proposition.

4.2. Proposition. *Suppose that X is a strongly metrizable space of weight $\tau \geq \aleph_0$ and $X_1, X_2, \ldots$ a sequence of subspaces such that* $\dim X_n \leq 0$ *for* $n = 1, 2, \ldots$. *Let* $\varphi = (c, g) : X \to B(\tau) \times I^\omega$ *be any continuous mapping, where* $c = \{c_m\} : X \to B(\tau)$ *and* $g = \{g_m\} : X \to I^\omega = \mathrm{P}_{m=1}^{\infty} I_m$, *where* $I_m = I$ *for* $m = 1, 2, \ldots$. *Let* m_0 *be any natural number and* ε *any positive real number* < 1. *Then there exists an embedding* $\psi = (d, h) : X \to B(\tau) \times I^\omega$, *where* $d = \{d_m\} : X \to B(\tau)$ *and* $h = \{h_m\} : X \to I^\omega$ *such that*

$$\psi(X_n) \subset B(\tau) \times N_{n-1}^{\omega} \quad \textit{for every } n = 1, 2, \ldots,$$

and

$$c_m(x) = d_m(x), \qquad \rho(h_m(x), g_m(x)) < \varepsilon$$

$$\textit{for every } m = 1, 2, \ldots, m_0 \textit{ and every } x \in X.$$

Proof. We modify the proof of Theorem VI.5 of [9] in the following way. Recall that if $A \in \mathscr{A}$, then $S(A, \mathscr{A}) = S^1(A, \mathscr{A}) = \bigcup\{B \in \mathscr{A}: B \cap A \neq \emptyset\}$, $S^n(A, \mathscr{A}) = S$

$(S^{n-1}(A, \mathscr{A}), \mathscr{A})$ for $n>1$ and $S^\infty(A,\mathscr{A})=\bigcup_{n=1}^\infty S^n(A,\mathscr{A})$. Let $\mathcal{N}_1 > \mathcal{N}_2 > \cdots$ be a sequence of star-finite open coverings of X such that $\{S(x,\mathcal{N}_i): i=1,2,\ldots\}$ is a neighbourhood basis at each point $x\in X$. For $i=1,2,\ldots$ let

$$\mathscr{S}_i=\{S^\infty(N,\mathcal{N}_i):\ N\in\mathcal{N}_i\}.$$

Since $\mathcal{N}_i$ is star-finite, we have $\mathscr{S}_i=\{S_\alpha:\ \alpha\in A_i\}$, where $S_\alpha\cap S_\beta=\emptyset$ for $\alpha\neq\beta$ and $S_\alpha=\bigcup\{N_{\alpha j}: j=1,2,\ldots\}$, where $N_{\alpha j}\in\mathcal{N}_i$. Moreover, if A is the discrete space of cardinality τ, we can assume that $A_i=A$ for $i=1,2,\ldots$.

For each $i=1,2,\ldots$ construct an open covering $\mathscr{P}_i$ of X such that

$$\mathscr{P}_i=\{P_{\alpha j}:\ \alpha\in A_i, j=1,2,\ldots\},\quad \text{where } \overline{P_{\alpha j}}\subset N_{\alpha j},$$

and define $U_{ij}=\bigcup\{N_{\alpha j}:\ \alpha\in A_i\}$ and $F_{ij}=\bigcup\{\overline{P_{\alpha j}}:\ \alpha\in A_i\}$. We now arrange all sets U_{ij} and F_{ij} into sequences

$$\{U_m:\ m=m_0+1, m_0+2,\ldots\}\quad\text{and}\quad\{F_m:\ m=m_0+1, m_0+2,\ldots\},$$

so that $F_m\subset U_m$. Next we define a sequence $\{f_m:\ m=1,2,\ldots\}$ of functions from X into I satisfying

$$|\{m: f_m(x) \text{ is a rational number distinct from 0 and 1}\}|\leq n-1$$
$$\text{for every } x\in X_n \text{ and } n=1,2,\ldots,$$

$f_m(X\setminus U_m)\subset\{0\}$, $f_m(F_m)\subset\{1\}$ for $m=m_0+1$, $m_0+2,\ldots$, and $\rho(f_m(x), g_m(x))<\varepsilon$ for $m=1,2,\ldots,m_0$ and every $x\in X$. To do this, we modify the construction of functions f_m in the proof of Proposition 4.1 by replacing $S(A)$ by I and making all necessary changes following from this replacement.

Having functions f_m defined, we take a homeomorphism $j_m: I_m\to I_m$ satisfying $j_m(I_m)\subset I_m\setminus\{0\}\setminus\{1\}$, $j_m(P)\subset P$, and such that the functions $h_m=j_m\circ f_m: X\to I_m$ also satisfy the condition $\rho(h_m(x), g_m(x))<\varepsilon$ for every $x\in X$ and $m=1,2,\ldots,m_0$. Let $h=\{h_m\}: X\to \mathbb{P}_{m=1}^\infty I_m$; then $h(X_n)\subset N_{n-1}^\omega$ for every $n=1,2,\ldots$.

Finally, we define $d: X\to B(\tau)$ by putting $d(x)=(c_1(x),\ldots,c_{m_0}(x),\alpha_1,\alpha_2,\ldots)$ if $x\in S_{\alpha_i}$, $\alpha_i\in A_i$, $i=1,2,\ldots$. Then the function $\psi=(d,h): X\to B(\tau)\times I^\omega$ satisfies $\psi(X_n)\subset B(\tau)\times N_{n-1}^\omega$, for $n=1,2,\ldots$. The same reasoning as in the proof of Theorem VI.5 of [9] shows that ψ is a homeomorphic embedding.

Corollary 2.4 follows immediately from the decomposition theorem and the monotonicity of dimension.

5. Remarks

5.1. Remark. It seems that the approach given in this paper simplifies the original proofs of Nagata's embedding theorems. In fact, the part of Nagata's proofs corresponding to Proposition VI.2.A of [9] can be replaced by the following proposition:

Proposition. *Let X be a metric space and $X_1, X_2,\ldots$ a sequence of subspaces such that* $\dim X_n\leq 0$ *for $n=1,2,\ldots$. Let $\{U_m:\ m=1,2,\ldots\}$ be a family of open subsets* *of X and $\{F_m: m = 1, 2, \ldots\}$ a family of closed subsets of X with $F_m \subset U_m$ for $m = 1, 2, \ldots$. Then there exists a function $f = \{f_m\}: X \to I^\omega = \mathbb{P}_{m=1}^{\infty} I_m$, where $I_m = I$ for every $m = 1, 2, \ldots$, such that $f(X_n) \subset N_{n-1}^\omega$ for every $n = 1, 2, \ldots$ and $f_m(X \setminus U_m) \subset \{0\}$, $f_m(F_m) \subset \{1\}$ for $m = 1, 2, \ldots$.*

The construction of the sequence $\{f_m\}$ is analogous to the construction of $\{f_m\}$ given in the proof of Proposition 4.1 (where we consider the simpler case for which $m_0 = 0$ and $S(A)$ is replaced by I) and seems to be simpler than the construction of a special open collection $\mathcal{V}$ given in VI.2.A) of [9].

5.2. Remark. We will show that the set $\mathcal{H} = \{f \in C(I, I^\omega): f$ is an embedding and $f(P) \subset P^{\aleph_0}\}$ is of the first category in $C(I, I^\omega)$ and as a space. First observe that the set $\mathcal{F}$ of all continuous mappings of I into I^ω such that $f(P) \subset P^{\aleph_0}$ and $f(I)$ contains more than one point is of the first category in $C(I, I^\omega)$. Indeed, for every r from the set of rationals Q, every $j \in N$ and every $q \in I$ let

$$L_{rj} = \{x = \{x_n\} \in I^\omega: x_j = r\} \quad \text{and} \quad \mathcal{F}_{qrj} = \{f \in C(I, I^\omega): f(q) \in L_{rj}\}.$$

It is easy to see that each $\mathcal{F}_{qrj}$ is closed and nowhere dense in $C(I, I^\omega)$. If $f \in \mathcal{F}$, then $f(P) \subset P^{\aleph_0}$, but $f(I) \not\subset P^{\aleph_0}$ as a non-one-point continuum, so there exists $q \in Q$ such that $f(q) \notin P^{\aleph_0}$. This means that $f(q) \in L_{rj}$ for some $r \in Q$ and $j \in N$, i.e. $f \in \mathcal{F}_{qrj}$. Thus $\mathcal{F} \subset \bigcup \{\mathcal{F}_{qrj}: q \in Q, r \in Q, j \in N\}$.

It follows that $\mathcal{F}$ and $\mathcal{H} \subset \mathcal{F}$ are of the first category in $C(I, I^\omega)$. Since $\mathcal{H}$ is dense in $C(I, I^\omega)$ by Theorem 2.2, then it is also of the first category as a space.

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