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A DISCRIMINATION ALGORITHM INSIDE λ-β-CALCULUS

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Abstract. A finite set $\{F_1, \dots, F_n\}$ of λ-terms is said to be discriminable if, given n arbitrary λ-terms $X_1, \dots, X_n$, there exists a λ-term Δ such that:

$$\Delta F_i \geqslant X_i \quad \text{for } 1 \leqslant i \leqslant n.$$

In the present paper each finite set of normal combinators which are pairwise non α-η-convertible is proved to be discriminable. Moreover a discrimination algorithm is given.

Introduction

The aim of the present paper is, given n distinct λ-β-normal combinators[1] (briefly ncs) $C_1, \dots, C_n$ which are pairwise non α-η-convertible and n λ-terms $X_1, \dots, X_n$, to build a λ-term Δ such that $\Delta C_i \geqslant X_i$[2] for $1 \leqslant i \leqslant n$.

Obviously the construction of Δ may be reduced to that of a nc D such that: $DC_i \geqslant \lambda y_1 \cdots y_n . y_i (1 \leqslant i \leqslant n)$. In fact $\Delta = \lambda t . DtX_1 \cdots X_n$.[2] We will say that D is a *discriminator* for the set $\{C_1, \dots, C_n\}$. The similarity to an interpolation problem is clear.

We notice that D, as any nc, cannot be a total function on the set of normal forms (briefly nfs), i.e. there exist a minimum h and h ncs $F_1, \dots, F_h$ such that the application $DF_1 \cdots F_h$ is without nf. The evaluation of h and the construction of $F_1, \dots, F_h$ for a specified D are given in [6].

[1] Combinator is used synonymously with closed λ-term, i.e., term without free variables.
[2] $\geqslant$ denotes α-β-reducibility and $=$ denotes α-β-convertibility.

The discrimination problem, for $n = 1$, has the trivial solution $D = \underset{\sim}{K}\underset{\sim}{I}$.[3,4] For $n = 2$ this problem has been solved in [4]. It is impossible to generalize this result in a trivial way. In fact, given three ncs C_1, C_2, C_3 and three λ-terms X_1, X_2, X_3, using [4] we are able to build three λ-terms $\Delta_{12}, \Delta_{13}, \Delta_{23}$ such that:

$$\Delta_{12}C_1 = X_1, \qquad \Delta_{13}C_1 = X_1, \qquad \Delta_{23}C_2 = X_2,$$

$$\Delta_{12}C_2 = X_2, \qquad \Delta_{13}C_3 = X_3, \qquad \Delta_{23}C_3 = X_3.$$

But we are unable to say anything about the application $\Delta_{ij}C_k$ when $1 \leq i, j, k \leq 3$ and $k \neq i, k \neq j$. Here we solve the problem for every finite set Φ of ncs, and we prove that the existence of a discriminator implies the existence of Kronecker's δ, i.e. a nc δ_Φ such that:

$$\delta_\Phi MN = \underset{\sim}{I} \quad \text{if } M = N,$$

$$\delta_\Phi MN = \underset{\sim}{K}\underset{\sim}{I} \quad \text{if } M \neq N,$$

where $M, N \in \Phi$.

In [2] it is proved that Kronecker's δ is not definable as a λ-term when M, N range on the whole set of ncs.

The method given here fails in the case of an infinite set of ncs. Nevertheless for each infinite set of λ-terms, which is a numerical system, a Kronecker's δ can be built by means of the recursion combinator [8, p. 220].

H. Barendregt presents in [3] a proof of the discriminability of the finite sets of ncs. His proof is based on the sketch presented by Böhm and Peretti at the Logic Colloquium 72 in Orleans.

H. B. Curry pointed out [8, p. 157], that the discriminators built in [4], for the case $n = 2$, work correctly also if we allow only combinatory weak reductions. This is no longer true in the present generalization, since the proof Lemma 3 requires strong reductions, as exemplified in Remark 1. R. Hindley has proved (in a private communication) that if discrimination is possible for λ-β-normal forms and λ-β-reductions, then it becomes automatically possible for combinatory strong normal forms and weak reductions.

1. Key notions and definitions

It is known that a nf N has the following shape:

$$N \equiv \lambda x_1 \cdots x_n . x_j N_1 \cdots N_m{}^{5} \quad (m, n \geq 0)$$

where $N_i (1 \leq i \leq m)$ are nfs.

We call x_j the *head variable* of N, $N_i (1 \leq i \leq m)$ the *i*th *component* of N and $\lambda x_1 \cdots x_n$ the *initial abstractions* of N.

[3] $\underset{\sim}{K} \equiv \lambda xy . x$.

[4] $\underset{\sim}{I} \equiv \lambda x . x$.

[5] $\equiv$ denotes α-convertibility.

We assume that, in a nf, variables bound in different abstractions have different labels. This is always feasible by α-reducing.

We call *order* of a variable x in a nf N the maximum number of components of subterms of N whose head variable is x.

If X, Y are two ncs then $X \circ Y$ denotes

$$\mathbf{B}XY = \lambda z . X(Yz) \quad (\textit{composition} \text{ of } X \text{ and } Y).$$

X^r abbreviates $\underbrace{X \circ X \circ \cdots \circ X}_{r}$.

In what follows *set of ncs* will denote a *finite*, non-empty set of distinct ncs which are pairwise non-α-η-convertible. $|\Phi|$ will denote the cardinality of the set Φ.

If X is a nc, and Φ is a set of ncs, then $X[\Phi]$ denotes the set of combinators obtained by applying X to each element of Φ. Since we will always choose X taking care that all elements of $X[\Phi]$ are reducible to ncs, we will think of $X[\Phi]$ as a set of ncs. We introduce next an equivalence relation between ncs as in [4]. Our discrimination algorithm is based on this equivalence relation.

Definition 1. If $C_1 \equiv \lambda x_1 \cdots x_{n_1} . x_{j_1} C_1^{(1)} \cdots C_{m_1}^{(1)}$, $C_2 \equiv \lambda x_1 \cdots x_{n_2} . x_{j_2} C_1^{(2)} \cdots C_{m_2}^{(2)}$ are ncs, then C_1 is *equivalent* to $C_2 (C_1 \sim C_2)$ iff:

$$j_1 = j_2 \quad \text{and} \quad n_1 - m_1 = n_2 - m_2.$$

Example 1. $C_1 \equiv \lambda x_1 x_2 x_3 . x_1 (\lambda x_4 x_5 . x_5)(x_2 x_3)$ and $C_2 \equiv \lambda x_1 x_2 . x_1 (x_1 x_2)$ are equivalent, since $j_1 = j_2 = 1$, and $n_1 = 3$, $m_1 = 2$, $n_2 = 2$, $m_2 = 1$, i.e. $n_1 - m_1 = n_2 - m_2 = 1$.

Let us extend the notion of equivalence to a set of ncs in an obvious way.

Definition 2. A set $\Phi = \{C_i \mid i \leq n\}$ of ncs is said to be:
- *an equivalent set* (e.s.) iff $n > 1$ and for all $1 \leq i, l \leq n$: $C_i \sim C_l$
- *a non-equivalent set* (n.e.s.) iff $n = 1$ or for all $1 \leq i, l \leq n, i \neq l$: $C_i \not\sim C_l$.

An arbitrary set of ncs may be always split into disjoint equivalent (or non-equivalent) subsets.

Example 2.

$$\Phi = \{\lambda x_1 x_2 x_3 . x_2 (\lambda x_4 x_5 . x_5) x_2 x_3, \lambda x_1 x_2 . x_2 (\lambda x_3 x_4 . x_3) x_2, \lambda x_1 x_2 . x_2 (\lambda x_3 x_4 . x_3) x_1\}$$

is an e.s. while

$$\Psi = \{\lambda x_1 x_2 x_3 . x_2 (\lambda x_4 x_5 . x_5) x_3, \lambda x_1 x_2 . x_2 (\lambda x_3 x_4 . x_3) x_2, \lambda x_1 x_2 . x_1 (\lambda x_3 x_4 . x_3) x_1\}$$

is a n.e.s.

The set

$$\Omega = \{\lambda x_1x_2x_3 \,.\, x_2(\lambda x_4x_5 \,.\, x_5)x_2x_3,\ \lambda x_1x_2 \,.\, x_2(\lambda x_3x_4 \,.\, x_3)x_2,$$
$$\lambda x_1x_2 \,.\, x_1(\lambda x_3x_4 \,.\, x_3)x_1\}$$

is neither an e.s. nor a n.e.s.

We define now three properties of sets of ncs which will be used later.

Definition 3. A set $\Phi = \{C_i \mid i \leq n\}$ of ncs is *varied* iff for all $1 \leq i, l \leq n$

$$C_i \equiv \lambda x_1 \cdots x_{n_i} \,.\, x_{j_i}C_1^{(i)} \cdots C_{m_i}^{(i)} \not\sim C_l \equiv \lambda x_1 \cdots x_{n_l} \,.\, x_{j_l}C_1^{(l)} \cdots C_{m_l}^{(l)}$$

implies that their head variables x_{j_i} and x_{j_l} are different, i.e. $j_i \neq j_l$.

Example 3. The set $\Phi = \{C_i \mid i \leq 5\}$ where:

$C_1 \equiv \lambda x_1x_2 \,.\, x_1(\lambda x_3x_4 \,.\, x_3x_3)$,

$C_2 \equiv \lambda x_1x_2 \,.\, x_1(\lambda x_3 \,.\, x_3)$,

$C_3 \equiv \lambda x_1x_2x_3 \,.\, x_2(\lambda x_4x_5 \,.\, x_5)x_3x_2$,

$C_4 \equiv \lambda x_1x_2 \,.\, x_2(\lambda x_3x_4 \,.\, x_4)(\lambda x_3 \,.\, x_1)$,

$C_5 \equiv \lambda x_1x_2 \,.\, x_2(\lambda x_3x_4 \,.\, x_4)(\lambda x_3 \,.\, x_3x_3)$

is varied since $C_1 \sim C_2 \not\sim C_3 \sim C_4 \sim C_5$.

In contrast, the set $\Psi = \{G_i \mid i \leq 5\}$ where:

$G_1 \equiv C_1$,

$G_2 \equiv \lambda x_1x_2 \,.\, x_1$,

$G_3 \equiv \lambda x_1x_2x_3 \,.\, x_2(\lambda x_4x_5 \,.\, x_5)x_3x_1$,

$G_4 \equiv C_4$,

$G_5 \equiv C_5$

is non-varied since $G_1 \not\sim G_2$ but their head variable is x_1.

Definition 4. A set $\Phi = \{C_i \mid i \leq n\}$ of ncs is *easy* iff the head variable of C_i occurs only once in C_i for $1 \leq i \leq n$.

Example 4. The set Ψ of example 3 is easy since $x_1(x_2)$ don't occur in the components of $G_1, G_2(G_3, G_4, G_5)$. At the contrary, the set Φ of example 3 is non-easy since x_2 occurs twice in C_3.

Definition 5. A set $\Phi = \{C_i \mid i \leq n\}$ of ncs is *homogeneous* iff all C_i for $1 \leq i \leq n$ have the same number of initial abstractions.

2. Discriminators and their properties

Here is the formal definition of discriminable set of ncs.

Definition 6. A set $\Phi = \{C_i \mid i \leq n\}$ of ncs is discriminable iff there exists a nc D_Φ (*discriminator for* Φ)[6] such that:

$$D_\Phi C_i \geq \lambda y_1 \cdots y_n . y_i \overset{\text{df}}{=} U_n^{(i)} \quad (1 \leq i \leq n).$$

Example 6. The set $\Phi = \{C_i \mid i \leq 3\}$ where:

$$C_1 \equiv \lambda x_1 x_2 x_3 . x_1 x_1,$$

$$C_2 \equiv \lambda x_1 x_2 x_3 . x_2(\lambda x_4 x_5 . x_4)(\lambda x_4 . x_4),$$

$$C_3 \equiv \lambda x_1 x_2 x_3 . x_3(\lambda x_4 . x_4)(\lambda x_4 . x_4)$$

is discriminable. In fact the nc

$$D \equiv \lambda t y_1 y_2 y_3 . t(\lambda y_4 . y_1)(\lambda y_4 y_5 . y_2)(\lambda y_4 y_5 . y_3)$$

is such that:

$$DC_1 \geq \lambda y_1 y_2 y_3 . y_1$$

$$DC_2 \geq \lambda y_1 y_2 y_3 . y_2$$

$$DC_3 \geq \lambda y_1 y_2 y_3 . y_3$$

as may be easily verified.

We notice that, if $X_1, \ldots, X_n$ are n arbitrary λ-terms, and D_Φ discriminates the set $\Phi = \{C_i \mid i \leq n\}$, then

$$D_\Phi C_i X_1 \cdots X_n \geq X_i \quad (1 \leq i \leq n).$$

Given an arbitrary set Φ of ncs we will prove in a constructive way that it is discriminable. We build a discriminator for Φ as composition and application of ncs which we call transformers. Each transformer maps a set of ncs into another one with the same cardinality enjoying some properties peculiar to that transformer. More formally we give the following:

Definition 7. A nc T is a *transformer* for a given set Φ of ncs iff $\Psi = T[\Phi]$ is a set of ncs and $|\Psi| = |\Phi|$.

Clearly each discriminator is a transformer too.

[6] We will see that a discriminator for Φ will generally depend on Φ. The writing D_Φ expresses this dependence.

The construction of a discriminator for a given set is split into seven Lemmas and one Theorem. Lemmas 1–3 present properties implying discriminability. Lemmas 4–7 give construction rules for special transformers. Finally, the Main Theorem, using the results of the Lemmas, proves that each set Φ of ncs is discriminable.

Lemma 1. *If Φ and Ψ are two sets of ncs such that:*
- Φ is discriminable,
- there exists a transformer T such that $T[\Psi] = \Phi$,
then Ψ is discriminable.

Proof. A suitable discriminator for Ψ is clearly:

$$D_\Psi = D_\Phi \circ T.$$

Lemma 2. *An homogeneous and varied n.e.s. Φ is discriminable.*

Proof. Let $\Phi = \{C_i \mid i \leqslant n\}$ where

$$C_i \equiv \lambda x_1 \cdots x_w \,.\, x_{j_i} C_1^{(i)} \cdots C_{m_i}^{(i)}.$$

From the hypothesis on Φ it follows that $i \neq l$ implies $j_i \neq j_l$. Then, a discriminator for Φ is:

$$D_\Phi \equiv \lambda t y_1 \cdots y_n \,.\, t R_1 \cdots R_w$$

where

$$R_s \equiv \begin{cases} \text{the nf of } K^{m_i} y_i & \text{if } \exists i (s = j_i) \quad (1 \leqslant i \leqslant n) \\ \text{an arbitrary nc} & \text{otherwise} \end{cases} \qquad (1 \leqslant s \leqslant w).$$

Lemma 3. *If Φ is an homogeneous and varied set and each equivalence class of Φ is discriminable, then Φ is discriminable.*

Proof. Let $\Phi = \bigcup_{l=1}^{q} \Phi_l$ where

$$\Phi_l = \{C_i^{(l)} \mid i \leqslant n_l\} \quad (n_l \geqslant 1, 1 \leqslant l \leqslant q)$$

are the equivalence classes of Φ.

If $C_i^{(l)} \equiv \lambda x_1 \cdots x_w \,.\, x_{j_l} C_1^{(l,i)} \cdots C_{m_l}^{(l,i)}$ are the ncs belonging to Φ_l, we define:

$$v_l = n_1 + \cdots + n_{l-1} \quad (1 \leqslant l \leqslant q),$$

$$n = v_{q+1} = n_1 + \cdots + n_q.$$

By hypothesis $\Phi_l (1 \leqslant l \leqslant q)$ is discriminable, i.e., there exists a discriminator D_{Φ_l} such that:

$$D_{\Phi_l}[\Phi_l] = \{U_{n_l}^{(i)} \mid i \leqslant n_l\}.$$

Then a discriminator for Φ is:

$$D_\Phi \equiv \lambda t\,.\,tR_1 \cdots R_w$$

where:

$$R_s \equiv \begin{cases} \text{the nf of } \underset{\wedge}{K}^{m_l+v_l}(\lambda z_1 \cdots z_{n_l}\,.\,\underset{\wedge}{K}^{n-v_{l+1}}(D_{\Phi_l} t z_1 \cdots z_{n_l})) \\ \qquad\qquad \text{if } \exists l[s=j_l] \quad (1 \leq l \leq q), \\ \text{an arbitrary nc} \quad \text{otherwise} \end{cases} \qquad (1 \leq s \leq w).$$

In fact:

$$D_\Phi C_i^{(l)} = (\lambda t\,.\,tR_1 \cdots R_w)C_i^{(l)} \geq C_i^{(l)}\bar{R}_1 \cdots \bar{R}_w$$
$$= (\lambda x_1 \cdots x_w\,.\,x_{j_l} C_1^{(l,i)} \cdots C_{m_l}^{(l,i)})\bar{R}_1 \cdots \bar{R}_w \geq \bar{R}_{j_l}\bar{C}_1^{(l,i)} \cdots \bar{C}_{m_l}^{(l,i)}$$

where:

$$\bar{R}_s \equiv \begin{cases} \text{the nf of } \underset{\wedge}{K}^{m_l+v_l}(\lambda z_1 \cdots z_{n_l}\,.\,\underset{\wedge}{K}^{n-v_{l+1}}(D_{\Phi_l} C_1^{(l)} z_1 \cdots z_{n_l})) \\ \qquad \text{if } \exists l[s=j_l] \quad (1 \leq l \leq q), \\ R_s \quad \text{otherwise} \end{cases} \qquad (1 \leq s \leq w).$$

$$\bar{C}_t^{(l,i)} = C_t^{(l,i)}[x_s/\bar{R}_s] \quad (1 \leq t \leq m_l, 1 \leq s \leq w).$$

Since by hypothesis D_{Φ_l} is a discriminator for Φ_l, we have:

$$D_\Phi C_i^{(l)} \geq \underset{\wedge}{K}^{m_l+v_l}(\lambda z_1 \cdots z_{n_l}\,.\,\underset{\wedge}{K}^{n-v_{l+1}}(U_{n_l}^{(i)} z_1 \cdots z_{n_l}))\bar{C}_1^{(l,i)} \cdots \bar{C}_{m_l}^{(l,i)}$$
$$\geq \underset{\wedge}{K}^{v_l}(\lambda z_1 \cdots z_{n_l}\,.\,\underset{\wedge}{K}^{n-v_{l+1}} z_i) \geq \underset{\wedge}{K}^{v_l} U_{n-v_l}^{(i)} \geq U_n^{(i+v_l)}.$$

Remark 1. We notice that the proof of this lemma is no longer true inside weak combinatory logic. For example

let $\Phi = \bigcup_{i=1}^{2} \Phi_l$, where: $\Phi_1 = \{C_1, C_2\}$, $\Phi_2 = \{C_3, C_4\}$ and:

$$C_1 \equiv \lambda x_1 x_2\,.\,x_1(\lambda x_3\,.\,x_3),$$

$$C_2 \equiv \lambda x_1 x_2\,.\,x_1(\lambda x_3 x_4\,.\,x_4),$$

$$C_3 \equiv \lambda x_1 x_2\,.\,x_2(\lambda x_3\,.\,x_3),$$

$$C_4 \equiv \lambda x_1 x_2\,.\,x_2(\lambda x_3 x_4\,.\,x_1).$$

If the symbols have the same meaning as in the proof of Lemma 3, we have that:

$$q=2, \quad v_1=0, \quad v_3=n=4, \qquad n_1=n_2=2,$$

$$w=2, \quad v_2=2, \quad m_1=m_2=1, \quad j_1=1, \quad j_2=2.$$

Φ_1 and Φ_2 are discriminable, since their discriminators are respectively:

$$D_{\Phi_1} \equiv \lambda t y_1 y_2 . t(\lambda y_3 . y_3(\lambda y_4 . y_1) y_2) Z_1,$$

$$D_{\Phi_2} \equiv \lambda t y_1 y_2 . t(\lambda y_3 . y_2)(\lambda y_3 . y_3(\lambda y_4 y_5 . y_1) Z_2 y_3)$$

where Z_1, Z_2 are arbitrary ncs.

Then we have that a discriminator for Φ is the nf of:

$$\lambda t . t(\underset{\wedge}{K}(\lambda y_1 y_2 . \underset{\wedge}{K}^2(D_{\Phi_1} t y_1 y_2)))(\underset{\wedge}{K}^3(\lambda y_1 y_2 . D_{\Phi_2} t y_1 y_2))$$

i.e.:

$$D_\Phi \equiv \lambda t . t(\lambda y_1 y_2 y_3 y_4 y_5 . t(\lambda y_6 . y_6(\lambda y_7 . y_2) y_3) Z_1)$$
$$(\lambda y_1 y_2 y_3 y_4 y_5 . t(\lambda y_6 . y_5)(\lambda y_6 . y_6(\lambda y_7 y_8 . y_4) Z_2 y_6)).$$

It may be easily verified that $D_\Phi C_i$ does not weak reduce to $U_i^{(4)}$ for $1 \leq i \leq 4$.

Remark 2. We notice that this lemma permits us to build a discriminator for an homogeneous and varied set Φ by applying a suitable nc P_Φ to discriminators $D_{\Phi_1}, \ldots, D_{\Phi_q}$ for the equivalence classes of Φ. In fact if $w, n, v_l, n_l, m_l, j_l (1 \leq l \leq q)$ are defined as in the proof of Lemma 3, it is sufficient to choose:

$$P_\Phi \equiv \lambda d_1 \cdots d_q t . t P_1 \cdots P_w$$

where

$$P_s \equiv \begin{cases} \text{the nf of } \underset{\wedge}{K}^{m_l + v_l}(\lambda z_1 \cdots z_{n_l} . \underset{\wedge}{K}^{n - v_{l+1}}(d_l t z_1 \cdots z_{n_l})) \\ \qquad\qquad \text{if } \exists l [s = j_l] \quad (1 \leq l \leq q), \\ \text{an arbitrary nc} \quad \text{otherwise} \end{cases} \qquad (1 \leq s \leq w),$$

to obtain

$$D_\Phi = P_\Phi D_{\Phi_1} \cdots D_{\Phi_q}.$$

Lemma 4. *If Φ is a set of ncs, then there exists a transformer H_Φ such that $H_\Phi[\Phi]$ is an homogeneous set of ncs.*

Proof. Let $\Phi = \{C_i \mid i \leq n\}$ and $C_i \equiv \lambda x_1 \cdots x_{w_i} . x_{j_i} C_1^{(i)} \cdots C_{m_i}^{(i)}$.

If $w = \max_{1 \leq i \leq n}(w_i)$, a suitable transformer for Φ is:

$$H_\Phi \equiv \lambda t y_1 \cdots y_w . t y_1 \cdots y_w.$$

We observe that $H_\Phi =_\eta \underset{\wedge}{I}$[7] for each set Φ. The obtained set $H_\Phi[\Phi]$ is then a set of n ncs each one having w initial abstractions.

In the particular case $w_i = w (1 \leq i \leq n)$, $H_\Phi[\Phi] = \Phi$.

[7] $=_\eta$ denotes α-η-convertibility.

Lemma 5. *If Φ is an homogeneous set of ncs, then there exists a transformer V_Φ such that $V_\Phi[\Phi]$ is an homogeneous, varied and easy set of ncs.*

Proof. The desired transformer will be constructed by using the ncs $\pi_{[r]} = \lambda z_1 \cdots z_{r+1} . z_{r+1} z_1 \cdots z_r$ [7, p. 171] in the same way as in [4], [8, p. 160], and [5].

For sufficiently large values of r the combinators $\pi_{[r]}$ bring into head position variables such that:

- non-equivalent ncs have different head variables (condition on varied sets)
- the head variable of a nc occurs only once in this nc (condition on easy sets).

Let $\Phi = \{C_i \mid i \leq n\}$, $C_i \equiv \lambda x_1 \cdots x_w . \bar{C}_i$ $(1 \leq i \leq n)$ and $\Theta = \{x_{j_1}, \ldots, x_{j_p}\}$ $(p > 0)$ be the set of head variables of all the $C_i (1 \leq i \leq n)$. For $l = 1, \ldots, p$ we build a transformer V_l such that the desired V_Φ is the composition of these transformers and more precisely:

$$V_\Phi = V_p \circ V_{p-1} \circ \cdots \circ V_1.$$

Construction of V_l. Let $\Phi^{(0)} = \Phi$ and $\Phi^{(r)} = V_r \Phi^{(r-1)} (1 \leq r \leq l-1)$. Then we may choose:

$$V_l \equiv \lambda t y_1 \cdots y_{a_l + w_l + 1} . t y_1 \cdots y_{j_l - 1} \pi_{[a_l]} y_{j_l + 1} \cdots y_{a_l + w_l + 1}$$

where:

- w_l is the number of initial abstractions of the ncs of $\Phi^{(l-1)}$ (w_l is defined unambiguously, since $\Phi^{(l-1)}$ will be proved to be homogeneous).
- a_l is one plus the maximum of the orders of variables in the ncs of $\Phi^{(l-1)}$.

We notice that $j_l \leq w_l$ since by construction $w_l \geq w$.

If $G_i \equiv \lambda y_1 \cdots y_{w_l} . \zeta_i G_1^{(i)} \cdots G_{m_i}^{(i)}$[8] belongs to $\Phi^{(l-1)}$ since $m_i \leq a_l$, we obtain:

$$V_l G_i \geq \begin{cases} \lambda y_1 \cdots y_{a_l + w_l + 1} . y_{a_l + w_l - m_i + 1} \bar{G}_1^{(i)} \cdots \bar{G}_{m_i}^{(i)} y_{w_l + 1} \cdots \\ \qquad \cdots y_{a_l + w_l - m_i} y_{a_l + w_l - m_i + 2} \cdots y_{a_l + w_l + 1} & \text{if } \zeta_i = y_{j_l} \\ \lambda y_1 \cdots y_{a_l + w_l + 1} . \zeta_i \bar{G}_1^{(i)} \cdots \bar{G}_{m_i}^{(i)} y_{w_l + 1} \cdots y_{a_l + w_l + 1} & \text{otherwise} \end{cases}$$

where $\bar{G}_t^{(i)} = G_t^{(i)}[y_{j_l} / \pi_{[a_l]}] (1 \leq t \leq m_i)$.

Let's observe that the number of components of $V_l G_i$ exceeds that of G_i by either $a_l + 1$ or a_l, according as either $\zeta_i \neq y_{j_l}$ or $\zeta_i = y_{j_l}$. Then, if $C_i \in \Phi$ has head variable $x_{j_r} (1 \leq r \leq p)$ and m components, $(V_l \circ V_{l-1} \circ \cdots \circ V_1) C_i$ has $m + s_l$ components if $r > l$, $m + s_l - 1$ components otherwise, where $s_l = \sum_{u=1}^{l} (a_u + 1)$.

It is easy to verify that:

(i) $V_l \Phi^{(l-1)}$ is an homogeneous set, if $\Phi^{(l-1)}$ is homogeneous.

(ii) if $C_i \in \Phi$ has head variable $x_{j_r} (1 \leq r \leq p)$ and m components, then the head variable of $(V_l \circ \cdots \circ V_1) C_i$ is $y_{a_r + w_r - m - s_{r-1} + 1}$ if $r \leq l$, and is y_{j_r} otherwise.

(iii) each variable $y_u (u \geq w_l + 1)$ occurs only once in $(V_l \circ \cdots \circ V_1) C_i$.

In the Appendix, V_Φ is proved to be a transformer for Φ. Now we must prove that $V_\Phi[\Phi]$ is an homogeneous, varied and easy set.

[8] ζ_i represents a variable bound in the initial abstractions of G_i.

Since Φ is homogeneous, from observation (i) it follows that $V_\Phi[\Phi]$ is homogeneous too.

To prove that $V_\Phi[\Phi]$ is a varied set, let us observe that, if j_r and m are defined as before, any nc of $V_\Phi[\Phi]$ with $y_{a_r+w_r-m-s_{r-1}+1}$ as head variable has $m+s_p-1$ components, as required by the varied condition.

Moreover from observation (iii) it follows that $V_\Phi[\Phi]$ is an easy set.

Example 6. Let $\Phi=\{C_i \mid i\leq 4\}$ where

$$C_1\equiv\lambda x_1x_2\,.\,x_2(\lambda x_3x_4\,.\,x_4x_3),\qquad C_2\equiv\lambda x_1x_2\,.\,x_2(\lambda x_3\,.\,x_3),$$

$$C_3\equiv\lambda x_1x_2\,.\,x_2(\lambda x_3x_4\,.\,x_3x_2),\qquad C_4\equiv\lambda x_1x_2\,.\,x_2(\lambda x_3x_4\,.\,x_3)x_1.$$

Φ is a non-varied and non-easy set, since x_2 is the head variable of non-equivalent ncs and it occurs twice in C_3.

If the symbols have the same meaning as in the proof of Lemma 5, we have that:

$$\Theta=\{x_2\},\ a_1=3,\ w_1=2.$$

Then

$$V_\Phi\equiv V_1\equiv\lambda ty_1\cdots y_6\,.\,ty_1\pi_{[3]}y_3\cdots y_6.$$

It may be easily verified that:

$$V_\Phi C_1\geq\lambda x_1\cdots x_6\,.\,x_5(\lambda x_7x_8\,.\,x_8x_7)x_3x_4x_6$$

$$V_\Phi C_2\geq\lambda x_1\cdots x_6\,.\,x_5(\lambda x_7\,.\,x_7)x_3x_4x_6$$

$$V_\Phi C_3\geq\lambda x_1\cdots x_6\,.\,x_5(\lambda x_7x_8\,.\,x_7\pi_{[3]})x_3x_4x_6$$

$$V_\Phi C_4\geq\lambda x_1\cdots x_6\,.\,x_4(\lambda x_7x_8\,.\,x_7)x_1x_3x_5x_6.$$

In order to prove that our discrimination algorithm terminates, a measure $d[\Phi]$ is defined. $d[\Phi]=l$ means intuitively that there exist at least two ncs of Φ having different subterms at nesting level l of parenthesis, but this is not true for all levels less than l.

More formally we have the following:

Definition 8. If $\Phi=\{C_i \mid i\leq n\}$ is a set of ncs, the *depth* of Φ is $d[\Phi]$ so defined:
- if $n=1$, then $d[\Phi]=\infty$;
- if there exist h,k such that $1\leq h,k\leq n$ and $C_h\not\sim C_k$, then $d[\Phi]=0$;
- otherwise let $H_\Phi[\Phi]=\{G_i \mid i\leq n\}$, $G_i\equiv\lambda x_1\cdots x_w\,.\,x_jG_1^{(i)}\cdots G_m^{(i)}$ and

$$\Phi_p=\{\lambda x_1\cdots x_w\,.\,G_p^{(i)} \mid i\leq n\}\quad\text{for } 1\leq p\leq m.$$

Then $d[\Phi]=1+\min_{1\leq p\leq m}[d[\Phi_p]]$.

Example 7. Let $\Phi = \{C_i \mid i \leq 2\}$, where:

$$C_1 \equiv \lambda x_1 x_2 . x_1(\lambda x_3 x_4 . x_3 x_3),$$

$$C_2 \equiv \lambda x_1 x_2 . x_1(\lambda x_3 . x_3).$$

$d[\Phi]$ is different from 0 since $C_1 \sim C_2$. $H_\Phi[\Phi] \equiv \Phi$ since Φ is an homogeneous set. Then $d[\Phi] = 1 + d[\Phi_1]$, where $\Phi_1 = \{G_1, G_2\}$ and:

$$G_1 \equiv \lambda x_1 x_2 x_3 x_4 . x_3 x_3,$$

$$G_2 \equiv \lambda x_1 x_2 x_3 . x_3.$$

Since $G_1 \sim G_2$, and Φ_1 is non-homogeneous, we must compute H_{Φ_1}.

$$H_{\Phi_1} \equiv \lambda t y_1 y_2 y_3 y_4 . t y_1 y_2 y_3 y_4$$

and

$$H_{\Phi_1}[\Phi_1] = \{G_1', G_2'\}$$

where:

$$G_1' \equiv \lambda x_1 x_2 x_3 x_4 . x_3 x_3,$$

$$G_2' \equiv \lambda x_1 x_2 x_3 x_4 . x_3 x_4.$$

Then $d[\Phi_1] = 1 + d[\Phi_2]$, where $\Phi_2 = \{F_1, F_2\}$ and:

$$F_1 \equiv \lambda x_1 x_2 x_3 . x_3,$$

$$F_2 \equiv \lambda x_1 x_2 x_3 x_4 . x_4.$$

$d[\Phi_2] = 0$ since $F_1 \not\sim F_2$. Therefore $d[\Phi_1] = 1$ and $d[\Phi] = 2$.

We say that a transformer T is *inoffensive* for a set Φ when it preserves the depth of Φ, i.e., $d[\Phi] = d[T[\Phi]]$.

Since H_Φ preserves the equivalence relation between ncs belonging to Φ, then from definition 8 it follows immediately that H_Φ is inoffensive for Φ.

We need to prove that also V_Φ is inoffensive for Φ. This is intuitively true if we take into account the structure of V_Φ. The Appendix contains a formal proof of this fact.

Lemma 6. *If Φ is an homogeneous and easy e.s., then there exists a transformer L_Φ such that $\Psi = L_\Phi[\Phi]$ is an homogeneous set and $d[\Psi] = d[\Phi] - 1$.*

Proof. Let $\Phi = \{C_i \mid i \leq n\}$, $C_i \equiv \lambda x_1 \cdots x_w . x_j C_1^{(i)} \cdots C_m^{(i)}$ and $\Phi_p = \{\lambda x_1 \cdots x_w . C_p^{(i)} \mid i \leq n\}$ for $1 \leq p \leq m$. By Definition 8: $d[\Phi] = 1 + \min_{1 \leq p \leq m} [d[\Phi_p]]$. Let q be the minimum integer such that $d[\Phi_q] = \min_{1 \leq p \leq m} [d[\Phi_p]]$. If

$$C_q^{(i)} \equiv \lambda x_{w+1} \cdots x_{w+v_i} . x_{j_i} G_1^{(i)} \cdots G_{t_i}^{(i)} \quad \text{and } v = \max_{1 \leq i \leq n} v_i,$$

then a suitable transformer for Φ is:

$$L_\Phi \equiv \lambda t y_1 \cdots y_{w+v} \, . \, t y_1 \cdots y_{j-1}(\lambda z_1 \cdots z_q \, . \, z_q y_{w+1} \cdots y_{w+v} z_1 \cdots z_q)$$
$$y_{j+1} \cdots y_w.$$

In fact we have that:

$$L_\Phi C_i \geqslant \lambda x_1 \cdots x_{w+v} \, . \, C_q^{(i)} x_{w+1} \cdots x_{w+v} C_1^{(i)} \cdots C_m^{(i)}$$
$$\geqslant \lambda x_1 \cdots x_{w+v} \, . \, x_{j_i} G_1^{(i)} \cdots G_{t_i}^{(i)} x_{w+v_i+1} \cdots x_{w+v} C_1^{(i)} \cdots C_m^{(i)} \overset{\mathrm{Df}}{=} C_i'.$$

Clearly C_i' is a nc $(1 \leqslant i \leqslant n)$, $C_h' \neq_\eta C_k'$ for $h \neq k$ $(1 \leqslant h, k \leqslant n)$ and moreover $\Psi = \{C_i' \mid i \leqslant n\}$ is an homogeneous set.

We must prove that $d[\Psi] = d[\Phi] - 1$. We split this proof according to $d[\Phi] = 1$ or $d[\Phi] > 1$.

$d[\Phi] = 1$ means $d[\Phi_q] = 0$, i.e. there exist h, k such that

$$\lambda x_1 \cdots x_w \, . \, C_q^{(h)} \not\sim \lambda x_1 \cdots x_w \, . \, C_q^{(k)},$$

i.e., $w + v_h - t_h \neq w + v_k - t_k$ or $x_{j_h} \neq x_{j_k}$.
In this case we prove that $C_h' \not\sim C_k'$, i.e. $d[\Psi] = 0$. C_h', C_k' have both $w + v$ initial abstractions, respectively $t_h + v - v_h + m$, $t_k + v - v_k + m$ components and their head variables are respectively x_{j_h}, x_{j_k}. Then $t_h - v_h \neq t_k - v_k$ or $x_{j_h} \neq x_{j_k}$ implies $C_h' \not\sim C_k'$.

$d[\Phi] = u + 2$ with $u \geqslant 0$ means

$$\min_{1 \leqslant p \leqslant m} [d[\Phi_p]] = d[\Phi_q] = u + 1.$$

Φ_q is in general a non-homogeneous set. By Definition 8 we must consider the set $\bar{\Phi}_q = H_{\Phi_q}[\Phi_q]$. By Lemma 4, $H_{\Phi_q} \equiv \lambda t y_1 \cdots y_{w+v} \, . \, t y_1 \cdots y_{w+v}$ and therefore $\bar{\Phi}_q = \{G_i \mid i \leqslant n\}$, where $G_i \equiv \lambda x_1 \cdots x_{w+v} \, . \, x_{j_i} G_1^{(i)} \cdots G_{t_i}^{(i)} x_{w+v_i+1} \cdots x_{w+v} \overset{\mathrm{Df}}{=} \lambda x_1 \cdots x_{w+v} \, . \, x_{j_i} G_1^{(i)} \cdots G_{v+r}^{(i)}$ and $r = t_i - v_i$ for $1 \leqslant i \leqslant n$. If we define

$$\Psi_p^* = \{\lambda x_1 \cdots x_{w+v} \, . \, G_p^{(i)} \mid i \leqslant n\} \quad \text{for } 1 \leqslant p \leqslant v + r,$$

then

$$d[\Phi_q] = d[\bar{\Phi}_q] = 1 + \min_{1 \leqslant p \leqslant v+r} [d[\Psi_p^*]],$$

i.e., $\min_{1 \leqslant p \leqslant v+r} [d[\Psi_p^*]] = u$.

By Definition 8, $d[\Psi] = 1 + \min_{1 \leqslant p \leqslant v+r+m} [d[\Psi_p]]$ where the sets Ψ_p are so defined:

- $\Psi_p = \Psi_p^*$ for $1 \leqslant p \leqslant r + v$,
- $\Psi_{r+v+p} = \Phi_p$ for $1 \leqslant p \leqslant m$.

Then since $\min_{1\le p\le r+v}[d[\Psi_p^*]] = u$ and $\min_{1\le p\le m}[d[\Phi_p]] = u+1$, we have that $d[\Psi] = u+1$

Example 8. Let Φ be the same as in example 7. Φ is an homogeneous and easy e.s., and $d[\Phi] = 2$. If the symbols have the same meaning as in the proof of Lemma 6, we have:

$$w = 2, \qquad v = 2, \qquad j = 1, \qquad q = 1.$$

Then the transformer L_Φ is:

$$L_\Phi \equiv \lambda t y_1 y_2 y_3 y_4 \,.\, t(\lambda y_5 \,.\, y_5 y_3 y_4 y_5) y_2.$$

It may be easily verified that $L_\Phi[\Phi] = \Psi = \{G_1, G_2\}$, where:

$$G_1 \equiv \lambda x_1 x_2 x_3 x_4 \,.\, x_3 x_3(\lambda x_5 x_6 \,.\, x_5 x_5),$$

$$G_2 \equiv \lambda x_1 x_2 x_3 x_4 \,.\, x_3 x_4(\lambda x_5 \,.\, x_5).$$

Clearly Ψ is an homogeneous set and $d[\Psi] = 1$.

Remark 3. The condition that Φ is an easy set is needed to prevent the creation of λ-terms not possessing nf in the application of L_Φ. The following example may clarify this. Let $\Phi = \{C_1, C_2\}$ where:

$$C_1 \equiv \lambda x_1 x_2 \,.\, x_1 x_1, \qquad C_2 \equiv \lambda x_1 x_2 \,.\, x_1 x_2.$$

Φ is an homogeneous e.s. which is non-easy since x_1 occurs twice in C_1.
By a (wrong) application of Lemma 6 we would obtain:

$$L_\Phi \equiv \lambda t y_1 y_2 \,.\, t(\lambda y_3 \,.\, y_3 y_3) y_2.$$

It may be easily verified that:

$$L_\Phi C_1 \geqslant \lambda x_1 \,.\, (\lambda x_2 \,.\, x_2 x_2)(\lambda x_2 \,.\, x_2 x_2),$$

i.e. Ψ contains one combinator not possessing nf!

Lemma 7. *If Φ is an homogeneous and easy set of ncs, then there exists a transformer N_Φ such that $N_\Phi[\Phi]$ contains at least two non-equivalent ncs.*

Proof. We make this proof by induction on $d[\Phi]$.
First step. $d[\Phi] = 0$ implies by definition that there are $C_h, C_k \in \Phi$ such that $C_h \not\sim C_k$. Therefore Φ itself satisfies the desired condition.
Inductive step. Let this Lemma be true for $d[\Phi] = l$, we prove it for $d[\Phi] = l+1$. If $d[\Phi] = l+1$, by Lemma 6 there exists a transformer L_Φ such that $\Psi = L_\Phi[\Phi]$ and $d[\Psi] = l$. Moreover Ψ is an homogeneous set. If Ψ is an easy set, then N_Ψ exists by inductive hypothesis and $N_\Phi = N_\Psi \circ L_\Phi$. Otherwise we must consider $\Sigma = V_\Psi[\Psi]$. $d[\Sigma] = l$ since V_Ψ is inoffensive for Ψ. N_Σ exists by inductive hypothesis and then we

have: $N_\Phi = N_\Sigma \circ V_\Psi \circ L_\Phi$. Lastly we remark that N_Φ as composition of transformers is again a transformer for Φ.

Example 9. Let Φ be the same as in Example 7. As shown in Example 8, $d[\Phi]=2$ and the transformer L_Φ is:

$$L_\Phi \equiv \lambda t y_1 y_2 y_3 y_4 . t(\lambda y_5 . y_5 y_3 y_4 y_5) y_2.$$

The obtained set is $\Psi = \{G_1, G_2\}$, where

$$G_1 \equiv \lambda x_1 x_2 x_3 x_4 . x_3 x_3(\lambda x_5 x_6 . x_5 x_5),$$

$$G_2 \equiv \lambda x_1 x_2 x_3 x_4 . x_3 x_4(\lambda x_5 . x_5)$$

and $d[\Psi]=1$. We notice that Ψ is a non-easy set. The transformer V_Ψ is:

$$V_\Psi = V_1 = \lambda t y_1 \cdots y_8 . t y_1 y_2 \pi_{[3]} y_4 \cdots y_8$$

and the obtained set is $\Sigma = \{F_1, F_2\}$, where:

$$F_1 \equiv \lambda x_1 \cdots x_8 . x_6 \pi_{[3]}(\lambda x_9 x_{10} . x_9 x_9) x_5 x_7 x_8,$$

$$F_2 \equiv \lambda x_1 \cdots x_8 . x_6 x_4(\lambda x_9 . x_9) x_5 x_7 x_8$$

and $d[\Sigma]=1$. Now we must compute $N_\Sigma \equiv L_\Sigma$ since here $d[\Sigma]=1$.

$$L_\Sigma \equiv \lambda t y_1 \cdots y_{12} . t y_1 \cdots y_5(\lambda y_{13} . y_{13} y_9 \cdots y_{13}) y_7 y_8$$

and the obtained set is $\Omega = \{E_1, E_2\}$, where:

$$E_1 \equiv \lambda x_1 \cdots x_{12} . x_{12} x_9 x_{10} x_{11} \pi_{[3]}(\lambda x_{13} x_{14} . x_{13} x_{13}) x_5 x_7 x_8;$$

$$E_2 \equiv \lambda x_1 \cdots x_{12} . x_4 x_9 x_{10} x_{11} x_{12} x_4(\lambda x_{13} . x_{13}) x_5 x_7 x_8$$

and $d[\Omega]=0$, because $E_1 \not\sim E_2$.
Then the desired transformer N_Φ is the nf of: $L_\Sigma \circ V_\Psi \circ L_\Phi$, i.e.:

$$N_\Phi \equiv \lambda t y_1 \cdots y_{12} . t(\lambda y_{13} . y_{13} \pi_{[3]} y_4 y_{13}) y_2 y_5 (\lambda y_{13} . y_{13} y_9 \cdots y_{13}) y_7 y_8.$$

Main Theorem. *Any set of ncs is discriminable.*

Proof. Let Φ be the current set and $|\Phi| = n$. We give a constructive proof, i.e., a method to build a nc D_Φ such that $D_\Phi[\Phi] = \{U_i^{(n)} \mid i \leq n\}$.

This construction is based on the possibility of composition which is assured by Lemma 1.

As first step we build $\Psi = H_\Phi[\Phi]$. Now the problem is to discriminate Ψ. If Ψ is a non-varied set we build $\Sigma = V_\Psi[\Psi]$. Otherwise we assume $\Sigma = \Psi$.

If Σ is a n.e.s. it is discriminable by Lemma 2.

If $d[\Sigma]=0$ we assume $\chi=\Sigma$. If Σ is an easy e.s. we build $\chi=N_\Sigma[\Sigma]$. Lastly, if Σ is a non-easy e.s. we build first $\Omega=V_\Sigma[\Sigma]$ and then $\chi=N_\Omega[\Omega]$. In all cases we have $d[\chi]=0$.

If χ is a varied set, then we assume $\Gamma=\chi$. Otherwise we build $\Gamma=V_\chi(\chi)$.

Now the problem is to discriminate Γ. Γ is a varied set which is the union of (at least 2) equivalence classes $\Gamma_1,\ldots,\Gamma_q$. Lemma 3 assures us that if D_{Γ_l} is a discriminator for Γ_l for $1\leq l\leq q$, then a discriminator for Γ is $P_\Gamma D_{\Gamma_1}\cdots D_{\Gamma_q}$. Hence the problem is to discriminate $\Gamma_1,\ldots\Gamma_q$. Since the cardinality of Γ_l is less than n for $1\leq l\leq q$ this procedure recursively applied to each Γ_l terminates.

Example 10. Let $\Phi=\{C_i\mid i\leq 4\}$, where:

$$C_1\equiv\lambda x_1x_2\,.\,x_1(\lambda x_3x_4\,.\,x_3x_3),$$

$$C_2\equiv\lambda x_1x_2\,.\,x_1(\lambda x_3\,.\,x_3),$$

$$C_3\equiv\lambda x_1x_2\,.\,x_2(\lambda x_3\,.\,x_3),$$

$$C_4\equiv\lambda x_1x_2\,.\,x_2(\lambda x_3x_4\,.\,x_1).$$

Since Φ is homogeneous, varied and $d[\Phi]=0$, we assume $\Gamma=\chi=\Sigma=\Psi=\Phi$. Γ is the union of 2 equivalence classes Γ_1, Γ_2, where

$$\Gamma_1=\{C_1,C_2\},\qquad \Gamma_2=\{C_3,C_4\}.$$

Then, for Lemma 3, a discriminator D_Φ for Φ is the nf of $P_\Phi D_{\Gamma_1}D_{\Gamma_2}$ where

$$P_\Phi\equiv\lambda y_1y_2t\,.\,t(\lambda y_3y_4y_5y_6y_7\,.\,y_1ty_4y_5)$$
$$(\lambda y_3y_4y_5y_6y_7\,.\,y_2ty_6y_7).$$

Now we must build D_{Γ_1} and D_{Γ_2}. Γ_1 is an homogeneous and easy e.s. Therefore we must build the set $\Delta=N_{\Gamma_1}[\Gamma_1]$. As shown in the Example 9, the transformer N_{Γ_1} is:

$$N_{\Gamma_1}\equiv\lambda ty_1\cdots y_{12}\,.\,t(\lambda y_{13}\,.\,y_{13}\pi_{[3]}y_4y_{13})y_2y_5$$
$$(\lambda y_{13}\,.\,y_{13}y_9\cdots y_{13})y_7y_8$$

and the obtained set is $\Sigma_1=\{E_1,E_2\}$ where:

$$E_1\equiv\lambda x_1\cdots x_{12}\,.\,x_{12}x_9x_{10}x_{11}\pi_{[3]}(\lambda x_{13}x_{14}\,.\,x_{13}x_{13})x_5x_7x_8,$$

$$E_2\equiv\lambda x_1\cdots x_{12}\,.\,x_4x_9x_{10}x_{11}x_{12}x_4(\lambda x_{13}\,.\,x_{13})x_5x_7x_8.$$

The set Σ_1 is discriminable, by Lemma 2, and a discriminator D_{Σ_1} is:

$$D_{\Sigma_1}\equiv\lambda ty_1y_2\,.\,tZ_1Z_2Z_3(\lambda y_3\cdots y_{11}\,.\,y_2)Z_4\cdots Z_{10}(\lambda y_3\cdots y_{10}\,.\,y_1)$$

where $Z_i (1 \leq i \leq 10)$ are arbitrary ncs. By Lemma 1, a discriminator D_{Γ_1} is the nf of $D_{\Sigma_1} \circ N_{\Gamma_1}$, i.e.:

$$D_{\Gamma_1} \equiv \lambda t y_1 y_2 . t(\lambda y_3 . y_3 \pi_{[3]}(\lambda y_4 \cdots y_{12} . y_2) y_3) Z_2 Z_4$$
$$(\lambda y_3 . y_3 Z_8 Z_9 Z_{10}(\lambda y_4 \cdots y_{11} . y_1) y_3) Z_6 Z_7$$

where indexed Z are arbitrary ncs.

Now we must discriminate the set Γ_2. Since Γ_2 is an homogeneous and easy e.s., we must build the transformer N_{Γ_2}. By Lemma 7, $N_{\Gamma_2} \equiv L_{\Gamma_2}$, where

$$L_{\Gamma_2} \equiv \lambda t y_1 y_2 y_3 y_4 . t y_1 (\lambda y_5 . y_5 y_3 y_4 y_5).$$

In fact $L_{\Gamma_2}[\Gamma_2] = \Omega_2 = \{F_1, F_2\}$, where:

$$F_1 \equiv \lambda x_1 x_2 x_3 x_4 . x_3 x_4 (\lambda x_5 . x_5),$$

$$F_2 \equiv \lambda x_1 x_2 x_3 x_4 . x_1 (\lambda x_5 x_6 . x_1)$$

and $d[\Omega_2] = 0$. Ω_2 is an homogeneous and varied n.e.s. and therefore, by Lemma 2, a discriminator for Ω_2 is:

$$D_{\Omega_2} \equiv \lambda t y_1 y_2 . t(\lambda y_3 . y_2) Z_{11} (\lambda y_3 y_4 . y_1) Z_{12}$$

(where Z_{11}, Z_{12} are arbitrary ncs).
Then a discriminator for Γ_2 is, by Lemma 1, the nf of $D_{\Omega_2} \circ L_{\Gamma_2}$, i.e.:

$$D_{\Gamma_2} \equiv \lambda t y_1 y_2 . t(\lambda y_3 . y_2)(\lambda y_3 . y_3 (\lambda y_4 y_5 . y_1) Z_{11} y_3).$$

Therefore a discriminator D_Φ is the nf of:

$$\lambda t . t(\lambda y_1 y_2 y_3 y_4 y_5 . D_{\Gamma_1} t y_2 y_3)(\lambda y_1 y_2 y_3 y_4 y_5 . D_{\Gamma_2} t y_4 y_5),$$

i.e.:

$$D_\Phi \equiv \lambda t . t(\lambda y_1 \cdots y_5 . t(\lambda y_6 . y_6 \pi_{[3]}(\lambda y_7 \cdots y_{15} . y_3) y_6) Z_2 Z_5)$$
$$(\lambda y_6 . y_6 Z_8 Z_9 Z_{10} (\lambda y_7 \cdots y_{14} . y_2) y_6) Z_6 Z_7)$$
$$(\lambda y_1 \cdots y_5 . t(\lambda y_6 . y_5)(\lambda y_6 . y_6 (\lambda y_7 y_8 . y_4) Z_{12} y_6))$$

where indexed Z are arbitrary ncs.

The following corollary proves, in a constructive way, the existence of Kronecker's δ for each finite set of ncs.

Corollary. *For every finite set of ncs $\Phi = \{C_i \mid i \leq n\}$ there exists a nc δ_Φ (Kronecker's δ), such that:*

$$\delta_\Phi C_i C_j = \begin{cases} I & \text{if } C_i \equiv C_j, \\ KI & \text{otherwise.} \end{cases}$$

Proof. The existence of a discriminator D_Φ, proved in the Main Theorem, assures the existence of n ncs $\Delta_1, \ldots, \Delta_n$ such that:

$$\Delta_i C_k = \begin{cases} \underset{\sim}{I} & \text{if } C_i \equiv C_k \\ \underset{\sim}{K}\underset{\sim}{I} & \text{otherwise} \end{cases} \qquad \text{for } 1 \leqslant k \leqslant n.$$

In fact Δ_i is the nf of

$$\lambda x \,.\, D_\Phi x \underbrace{(\underset{\sim}{K}\underset{\sim}{I}) \cdots (\underset{\sim}{K}\underset{\sim}{I})}_{i-1} \underset{\sim}{I} \underbrace{(\underset{\sim}{K}\underset{\sim}{I}) \cdots (\underset{\sim}{K}\underset{\sim}{I})}_{n-i}.$$

Then the desired δ_Φ is the nf of $\lambda x \,.\, D_\Phi x \Delta_1 \cdots \Delta_n$; in fact

$$(\lambda x \,.\, D_\Phi x \Delta_1 \cdots \Delta_n) C_i C_k \geqslant \Delta_i C_k.$$

3. Conclusion

The present paper treats the discrimination of finite sets of ncs. Clearly any set of nfs with free variables may be transformed into a set of ncs simply by abstracting all the nfs relative to the free variables (in an arbitrary order). So the discrimination of a set of nfs may be reduced to the discrimination of a set of ncs. A different approach is given in [3] where the discrimination of a set of nfs is realized by replacing the free variables. An uniform treatment for both open and closed nfs succeeds by means of the notion of context, as was first suggested by Wadsworth (see for example [9]).

For the whole set of λ-terms, Wadsworth [9] extends the notion of discrimination in the following sense: "two λ-terms X, Y are semi-separable iff there exists a context $\mathscr{C}[\;]$ such that $\mathscr{C}[X] \geqslant \underset{\sim}{I}$ and $\mathscr{C}[Y]$ does not possess head normal form, or viceversa". Moreover in [9] two λ-terms are proved to be semi-separable iff their values in Scott's D_∞-models are different. No corresponding statement for $n > 2$ is known.

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Appendix

Since we need here a precise notion of nesting level of parentheses, we assume to write each nf recursively as follows:

$$N \equiv \lambda x_1 \cdots x_n \,.\, x_j (N_1) \cdots (N_m).$$

To simplify the proof that V_Φ is an inoffensive transformer for Φ, we look first at two auxiliary properties of the depth and of the cardinality of sets of ncs with more than one element (any Φ with $|\Phi| = 1$ is varied).

Property 1. *If $\Phi = \{C_i \mid i \leq n\}$, $\Psi = \{G_i \mid i \leq n\}$ are two sets of ncs with $n > 1$ such that $C_i \equiv \lambda x_1 \cdots x_t . \bar{C}_i$ and $G_i \equiv \lambda x_1 \cdots x_t y_1 \cdots y_s . \bar{C}_i$ for $1 \leq i \leq n$ and $t > 0$, $s \geq 0$ are independent of i, then $d[\Phi] = d[\Psi]$.*

Proof. We prove this property by induction on the sum v of nesting levels of parentheses of $C_1, \ldots, C_n$.
First step. $v = 0$ means that all ncs of Φ have no component and therefore $d[\Phi] = d[\Psi] = 0$.
Inductive step. Let this property be true for $v \leq u$, we prove it for $v = u + 1$.
We distinguish the cases $d[\Phi] = 0$ and $d[\Phi] > 0$.
Case 1. $d[\Phi] = 0$ means that there exist h, k such that $C_h \not\sim C_k$. Then also $G_h \not\sim G_k$ by construction and therefore $d[\Psi] = 0$.
Case 2. If $d[\Phi] > 0$ we may assume that Φ, Ψ be homogeneous sets of ncs, since otherwise we would instead consider $H_\Phi[\Phi]$, $H_\Psi[\Psi]$. Let

$$\bar{C}_i \equiv \lambda x_{t+1} \cdots x_w . x_j (C_1^{(i)}) \cdots (C_m^{(i)}) \quad \text{for } 1 \leq i \leq n.$$

From Definition 8 it follows that $d[\Phi] = 1 + \min_{1 \leq p \leq m} [d[\Phi_p]]$ where

$$\Phi_p = \{\lambda x_1 \cdots x_w . C_p^{(i)} \mid i \leq n\} \quad \text{and} \quad d[\Psi] = 1 + \min_{1 \leq p \leq m} [d[\Psi_p]]$$

where

$$\Psi_p = \{\lambda x_1 \cdots x_t y_1 \cdots y_S x_{t+1} \cdots x_w . C_p^{(i)} \mid i \leq n\}.$$

Since the sum of nesting levels of parentheses of the ncs belonging to Φ_p is less than $u + 1$ then by inductive hypothesis $d[\Phi_p] = d[\Psi_p]$ for $1 \leq p \leq m$. Therefore we obtain $d[\Phi] = d[\Psi]$.

Property 2. *Let $\Phi = \{C_i \mid i \leq n\}$ be a set of ncs with $n > 1$ and $C_i \equiv \lambda x_1 \cdots x_{w_i} . \bar{C}_i (1 \leq i \leq n)$, a be greater than the maximum order of variables in C_i for $1 \leq i \leq n$ and j be less or equal to w_i for $1 \leq i \leq n$. If $\bar{C}'_i \equiv \bar{C}_i[x_j/\pi_{[a]}]$, $C'_i \equiv \lambda x_1 \cdots x_{w_i} . C'_i$, then C'_i possesses nf and if $\Phi' = \{C'_i \mid i \leq n\}$, then $|\Phi| = |\Phi'|$* and *$d[\Phi] = d[\Phi']$.*

Proof. We make this proof by induction on the sum v of nesting levels of parentheses of $C_1, \ldots, C_n$.

First step. $v = 0$ means that all ncs of Φ have no component, i.e. $C_i \equiv \lambda x_1 \cdots x_{w_i} . \zeta_i$ with $w_i \geq j (1 \leq i \leq n)$ and $a \geq 1$. Obviously

$$C'_i \equiv C_i \text{ if } \quad \zeta_i \neq x_j$$

and

$$C_i' \equiv \lambda x_1 \cdots x_{w_i} y_1 \cdots y_{a+1} \,.\, y_{a+1}(y_1) \cdots (y_a) \quad \text{otherwise.}$$

The property follows immediately.

Inductive step. Let this property be true for $v \leq u$, we prove it for $v = u + 1$. Let $C_i \equiv \lambda x_1 \cdots x_{w_i} \,.\, \zeta_i(C_1^{(i)}) \cdots (C_{m_i}^{(i)})$ with $w_i \geq j$ and $a > m_i$ for $1 \leq i \leq n$. By inductive hypothesis each $C_p^{(i)\prime} = C_p^{(i)}[x_j/\pi_{[a]}]$ (for $1 \leq i \leq n$ and $1 \leq p \leq m_i$) possesses nf. Then we have

$$C_i' \equiv \lambda x_1 \cdots x_{w_i} \,.\, \zeta_i(C_1^{(i)\prime}) \cdots (C_{m_i}^{(i)\prime}) \text{ (shape a)} \quad \text{if } \zeta_i \neq x_j$$

and

$$C_i' \equiv \lambda x_1 \cdots x_{w_i} y_{m_i+1} \cdots y_{a+1} \,.\, y_{a+1}(C_1^{(i)\prime}) \cdots (C_{m_i}^{(i)\prime})(y_{m_i+1}) \cdots (y_a)$$

$$\text{(shape b)} \qquad \text{otherwise.}$$

Clearly each C_i' possesses nf.

We define $r_i = w_i - m_i (1 \leq i \leq n)$.

Proof of $|\Phi'| = |\Phi|$. It is sufficient to prove that $C_h \neq_\eta C_k$ implies $C_h' \neq_\eta C_k'$ for all h, k such that $1 \leq h, k \leq n$. We distinguish the cases $C_h \not\sim C_k$ and $C_h \sim C_k$.

Case 1. $C_h \not\sim C_k$. We prove a stronger result, i.e. $C_h' \not\sim C_k'$. We split the proof according to three possible subcases:

(i) $\zeta_h \neq x_j$, $\zeta_k \neq x_j$. In this case, both C_h' and C_k' are of shape a and therefore $C_h' \not\sim C_k'$.

(ii) $r_h \neq r_k$, $\zeta_h = \zeta_k = x_j$. In this case both C_h' and C_k' are of shape b. $C_h' \not\sim C_k'$ since $a + r_h + 1 - a \neq a + r_k + 1 - a$.

(iii) $\zeta_h \neq x_j$, $\zeta_k = x_j$[9]. In this case C_h' is of shape a and C_k' is of shape b. To obtain $C_h' \sim C_k'$ it must be $r_h = r_k + 1$ and $\zeta_h = x_{a+r_k+1}$. $\zeta_h = x_{a+r_k+1}$ implies $w_h \geq a + r_k + 1$. $r_h = r_k + 1$ together with $w_h \geq a + r_k + 1$ imply $m_h \geq a$, which is contrary to our choice of a.

Case 2. $C_h \sim C_k$. We consider $H_\Phi[\Phi] = \{G_i / i \leq n\}$ where $G_i \equiv \lambda x_1 \cdots x_w$.

$$\zeta_i(C_1^{(i)}) \cdots (C_{m_i}^{(i)})(x_{w_i+1}) \cdots (x_w) \overset{\text{Df}}{=}$$

$$= \lambda x_1 \cdots x_w \,.\, \zeta_i(G_1^{(i)}) \cdots (G_{u_i}^{(i)}) \quad (1 \leq i \leq n).$$

We define

$$G_p^{(i)\prime} = G_p^{(i)}[x_j/\pi_{[a]}] \quad (1 \leq p \leq u_i)$$

and

$$G_i' = \lambda x_1 \cdots x_w \,.\, \zeta_i[x_j/\pi_{[a]}](G_1^{(i)\prime}) \cdots (G_{u_i}^{(i)\prime}) \quad (1 \leq i \leq n).$$

[9] The symmetric case $\zeta_h = x_j$, $\zeta_k \neq x_j$ may be proved by the same argument.

Clearly $G_i =_\eta C_i$ and $G'_i =_\eta C'_i$. $C_h \sim C_k$ and $C_h \neq_\eta C_k$ means that there exists an integer t $(1 \leq t \leq u_h = u_k)$ such that $G_t^{(h)} \neq_\eta G_t^{(k)}$. By inductive hypothesis $G_t^{(h)\prime} \neq_\eta G_t^{(k)\prime}$ and therefore $G'_h \neq_\eta G'_k$. Then we obtain $C'_h \neq_\eta C'_k$.
Proof of $d[\Phi'] = d[\Phi]$. We distinguish the cases $d[\Phi] = 0$ and $d[\Phi] > 0$.
Case 1. $d[\Phi] = 0$ means that there exist h, k such that $C_h \not\sim C_k$, i.e., we are in case 1 of the proof of $|\Phi'| = |\Phi|$. Then we have $C'_h \not\sim C'_k$, i.e., $d[\Phi'] = 0$.
Case 2. Let G_i, $G_p^{(i)}$, G'_i and $G_p^{(i)\prime}$ for $1 \leq i \leq n$ and $1 \leq p \leq u_i$ be defined as in case 2 of the proof of $|\Phi'| = |\Phi|$.

$d[\Phi] > 0$ means that all G_i have the same head variable and the same component number, i.e., $\zeta_i = \zeta$ and $u_i = m$ for $1 \leq i \leq n$. Moreover we define

$$\Phi_p = \{\lambda x_1 \cdots x_w . G_p^{(i)} \mid i \leq n\} \quad \text{and} \quad \Phi_p^* = \{\lambda x_1 \cdots x_w . G_p^{(i)\prime} \mid i \leq n\}$$

for $1 \leq p \leq m$. By inductive hypothesis $d[\Phi_p] = d[\Phi_p^*]$ for $1 \leq p \leq m$. We split the proof according to two possible subcases:

(i) $\zeta \neq x_j$. In this case $G'_i \equiv \lambda x_1 \cdots x_w . \zeta(G_1^{(i)\prime}) \cdots (G_m^{(i)\prime})$ and therefore

$$d[\Phi'] = 1 + \min_{1 \leq p \leq m} [d[\Phi_p^*]] = 1 + \min_{1 \leq p \leq m} [d[\Phi_p]] = d[\Phi].$$

(ii) $\zeta = x_j$. In this case

$$G'_i \equiv \lambda x_1 \cdots x_w y_{m+2} \cdots y_{a+1} . y_{a+1}(G_1^{(i)\prime}) \cdots (G_m^{(i)\prime})(y_{m+1}) \cdots (y_a).$$

By Definition 8, $d[\Phi'] = 1 + \min_{1 \leq p \leq a} [d[\Phi'_p]]$ where the sets Φ'_p are so defined:

$$\Phi'_p = \{\lambda x_1 \cdots x_w y_{m+1} \cdots y_{a+1} . G_p^{(i)\prime} \mid i \leq n\} \quad \text{for } 1 \leq p \leq m,$$

$$\Phi'_p = \{\lambda x_1 \cdots x_w y_{m+1} \cdots y_{a+1} . y_p\} \quad \text{for } m+1 \leq p \leq a.$$

From Definition 8 $d[\Phi'_p] = \infty$ for $m+1 \leq p \leq a$, i.e., $d[\Phi'] = 1 + \min_{1 \leq p \leq m} [d[\Phi'_p]]$. From Property 1 it follows that $d[\Phi'_{p+1}] = d[\Phi_p^*]$ for $1 \leq p \leq m$ and therefore $d[\Phi'] = d[\Phi]$.

Property 3. *V_Φ is an inoffensive transformer for an homogeneous set Φ with $|\Phi| > 1$.*

Proof. Let p, $\Phi^{(l-1)}$ and V_l $(1 \leq l \leq p)$ be defined as in the proof of Lemma 5. It is clearly sufficient to prove that v_l is an inoffensive transformer for $\Phi^{(l-1)} (1 \leq l \leq p)$, i.e., if $\Phi^{(l-1)} = \{C_i \mid i \leq n\}$, $\Psi = V_l[\Phi^{(l-1)}]$ and $C'_i = V_l C_i$, then:

(a) C'_i reduces to a nc $(1 \leq i \leq n)$,
(b) $|\Psi| = n$,
(c) $d[\Psi] = d[\Phi]$.

Let's notice that if $C_i \equiv \lambda y_1 \cdots y_{w_i} . \bar{C}_i$, then $C'_i \equiv \lambda y_1 \cdots y_{a_l + w_i + 1} . \bar{C}_i[y_{i_l}/\pi_{[a_l]}] (y_{w_i+1}) \cdots (y_{a_l + w_i + 1}) =_\eta \lambda y_1 \cdots y_{w_i} . \bar{C}_i[y_{i_l}/\pi_{[a_l]}]$ and so conditions *a*, *b* and *c* follow immediately from Property 2.

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