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Some Properties of Gamma and Beta Matrix Functions

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Abstract—In this paper, conditions for matrices P, Q so that the Beta matrix function $B(P,Q)$ satisfies $B(P,Q) = B(Q,P)$ and $B(P,Q) = \Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q)$ are given. Counter-examples showing that hypotheses cannot be removed are also included. A limit expression for the Gamma function of a matrix is established.

Keywords—Gamma matrix function, Beta matrix function, Simultaneous diagonalization.

1. INTRODUCTION

It is well known that many of the ordinary special functions of mathematical physics, and most of their properties, can be derived from the theory of group representations. Special functions of a matrix argument appear in the study of spherical functions on certain symmetric spaces and multivariate analysis in statistics, see [1]. Special functions of two diagonal matrix argument have been used in [2]. Beta functions of two matrix arguments, but where one of them is a scalar multiple of the identity has been recently used in [3] in the framework of orthogonal matrix polynomials.

In this paper, some properties of the Gamma and Beta matrix functions are proved. An analogue of the expression of the scalar Gamma function as a limit is given for the Gamma function of a matrix. Conditions for matrices P, Q in $C^{r\times r}$ so that $B(P,Q)$ is well defined and satisfy $B(P,Q) = B(Q,P)$ and $B(P,Q) = \Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q)$ are established. For the sake of clarity, in the presentation we recall some properties of the Riesz-Dunford functional calculus that may be found in [4–6]. If P is a matrix in $C^{r\times r}$, we denote by $\|P\|$ its 2-norm defined in [6, p. 56]. The set of all the eigenvalues of P is denoted by $\sigma(P)$.

If P is a matrix in $C^{r\times r}$ such that $\mathrm{Re}(z) > 0$ for all eigenvalue z of P, then $\Gamma(P)$ is well defined as

$$\Gamma(P) = \int_0^\infty e^{-t} t^{P-I}\, dt, \qquad t^{P-I} = \exp\left((P-I)\ln t\right). \tag{1}$$

Since the reciprocal Gamma function denoted by $\Gamma^{-1}(z) = 1/\Gamma(z)$ is an entire function of the complex variable z, for any matrix P in $C^{r\times r}$, the Riesz-Dunford functional calculus shows that the image of $\Gamma^{-1}(z)$ acting on P, denoted $\Gamma^{-1}(P)$, is a well-defined matrix (see [4, Chapter 7]). Furthermore, if P is a matrix such that

$$P + nI \text{ is invertible for every integer } n \geq 0, \tag{2}$$

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Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX then $\Gamma(P)$ is invertible, its inverse coincides with $\Gamma^{-1}(P)$, and

$$P(P+I)\cdots(P+(n-1)I)\Gamma^{-1}(P+nI) = \Gamma^{-1}(P), \qquad n \geq 1 \tag{3}$$

(see [5, p. 253]). If $f(z)$ and $g(z)$ are homomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and P is a matrix in $C^{r\times r}$ such that $\sigma(P) \subset \Omega$, then from the properties of the matrix functional calculus [4, p. 558], it follows that $f(P)g(P) = g(P)f(P)$. Under condition (2), from that, equation (3) can be written in the form

$$P(P+I)\cdots(P+(n-1)I) = \Gamma(P+nI)\Gamma^{-1}(P), \qquad n \geq 1. \tag{4}$$

If we take into account the scalar factorial function denoted by $(z)_n$ and defined by $(z)_n = z(z+1)\cdots(z+n-1)$, $n \geq 1$, $(z)_0 = 1$, then by application of the matrix functional calculus to this function, for any matrix P in $C^{r\times r}$ one gets

$$(P)_n = P(P+I)\cdots(P+(n-1)I), \qquad n \geq 1, \quad (P)_0 = I. \tag{5}$$

If $f(P)$ is well defined and S is an invertible matrix in $C^{r\times r}$, then [6, p. 541]

$$f\left(SPS^{-1}\right) = Sf(P)S^{-1}. \tag{6}$$

If P lies in $C^{r\times r}$, using its Schur decomposition and denoting $\alpha(P) = \max_{z\in\sigma(P)}\{\mathrm{Re}(z)\}$ for $t \in R$, it follows that [6, pp. 336,556]:

$$\left\|e^{tP}\right\| \leq e^{t\alpha(P)}\left\{\sum_{k=0}^{r-1}\frac{\left(\|P\|\sqrt{r}\,t\right)^k}{k!}\right\}. \tag{7}$$

2. ON THE GAMMA AND THE BETA MATRIX FUNCTIONS

Let M be a matrix in $C^{r\times r}$ such that

$$\mathrm{Re}(z) > 0, \qquad \text{for all } z \text{ in } \sigma(M), \tag{8}$$

and let n be an integer $n \geq 1$. By [7, p. 17], we have

$$g(z) = \int_0^1 (1-s)^n s^{z-1}\,ds = n!\,\left[z(z+1)\cdots(z+n)\right]^{-1}, \qquad \mathrm{Re}(z) > 0, \tag{9}$$

$$f(z) = \int_0^n \left(1-\frac{s}{n}\right)^n s^{z-1}\,ds = n!\,n^z\left[z(z+1)\cdots(z+n)\right]^{-1}, \qquad \mathrm{Re}(z) > 0. \tag{10}$$

As f and g are homomorphic functions in $\mathrm{Re}(z) > 0$, by application of the matrix functional calculus in (9) and (10), one gets

$$g(M) = \int_0^1 (1-s)^n s^{M-I}\,ds = n!\,\left[M(M+I)\cdots(M+nI)\right]^{-1}, \tag{11}$$

$$f(M) = \int_0^n \left(1-\frac{s}{n}\right)^n s^{M-I}\,ds = n!\,n^M\left[M(M+I)\cdots(M+nI)\right]^{-1}. \tag{12}$$

By (1) and (12), we can write

$$\begin{aligned}\Gamma(M) - n!\,n^M&\left[M(M+I)\cdots(M+nI)\right]^{-1}\\ &= \int_0^\infty e^{-t}t^{M-I}\,dt - \int_0^n\left(1-\frac{t}{n}\right)^n t^{M-I}\,dt\\ &= \int_0^n\left[e^{-t}-\left(1-\frac{t}{n}\right)^n\right]t^{M-I}\,dt + \int_n^\infty e^{-t}t^{M-I}\,dt.\end{aligned} \tag{13}$$

Since $\int_0^\infty e^{-t}t^{M-I}\,dt$ is convergent, one gets

$$\lim_{n\to\infty}\int_n^\infty e^{-t}t^{M-I}\,dt = 0. \tag{14}$$

Now we prove that

$$\lim_{n\to\infty}\int_0^n \left[e^{-t} - \left(1-\frac{t}{n}\right)^n\right] t^{M-I}\,dt = 0, \tag{15}$$

By [7, p. 16], one gets $0 \le e^{-t} - (1-t/n)^n \le t^2e^{-t}/n,\ 0\le t\le n$, hence,

$$\left\|\int_0^n \left[e^{-t} - \left(1-\frac{t}{n}\right)^n\right] t^{M-I}\,dt\right\| \le \frac{1}{n}\int_0^n \left\|t^{M+I}\right\| e^{-t}\,dt. \tag{16}$$

By (7) and using $\ln t \le t$ for $t > 0$, we can write

$$\begin{aligned}\left\|t^{M+I}\right\| &\le t^{\alpha(M)+1}\left[\sum_{j=0}^{r-1}\frac{\left[(\|M\|+1)\sqrt{r}\ln t\right]^j}{j!}\right]\\ &\le t^{\alpha(M)+1}\left[\sum_{j=0}^{r-1}\frac{\left[(\|M\|+1)\sqrt{r}\,t\right]^j}{j!}\right], \qquad t>0.\end{aligned} \tag{17}$$

By (16) and (17),

$$\frac{1}{n}\int_0^n \left\|t^{M+I}\right\| e^{-t}\,dt \le \frac{1}{n}\left\{\sum_{j=0}^{r-1}\frac{\left[(\|M\|+1)\sqrt{r}\right]^j}{j!}\int_0^n e^{-t}t^{\alpha(M)+1+j}\,dt\right\}. \tag{18}$$

Since, for $0\le j\le r-1$, we have

$$\int_0^\infty e^{-t}t^{\alpha(M)+1+j}\,dt < +\infty, \tag{19}$$

by (16)–(19) one gets (15). Thus, the following result has been established.

THEOREM 1. *Let M be a matrix satisfying (8) and let $n\ge 1$ be an integer. Then*

$$\Gamma(M) = \lim_{n\to\infty}(n-1)!\,(M)_n^{-1}\,n^M,$$

where $(M)_n$ is defined by (5).

Let P, Q be matrices in $C^{r\times r}$ such that

$$\mathrm{Re}(z) > 0,\quad \mathrm{Re}(w) > 0,\qquad \text{for all } z\in\sigma(P),\ w\in\sigma(Q). \tag{20}$$

By (7) and using $\ln t < t$ and $\ln(1-t) < 1-t$ for $0<t<1$, it follows that

$$\begin{aligned}&\int_0^1 \left\|t^{P-I}\right\|\left\|(1-t)^{Q-I}\right\|\,dt\\ &\le \sum_{j=0}^{r-1}\sum_{k=0}^{r-1}\frac{(\|P\|+1)^j(\|Q\|+1)^k(\sqrt{r})^{j+k}}{j!\,k!}\int_0^1 t^{\alpha(P)-1}(1-t)^{\alpha(Q)-1}\ln^j(t)\ln^k(1-t)\,dt\\ &\le \sum_{j=0}^{r-1}\sum_{k=0}^{r-1}\frac{(\|P\|+1)^j(\|Q\|+1)^k r^{(j+k)/2}}{j!\,k!}\int_0^1 t^{\alpha(P)+j-1}(1-t)^{\alpha(Q)+k-1}\,dt\\ &= \sum_{j=0}^{r-1}\sum_{k=0}^{r-1}\frac{(\|P\|+1)^j(\|Q\|+1)^k r^{(j+k)/2}B(\alpha(P)+j,\alpha(Q)+k)}{j!\,k!} < +\infty.\end{aligned}$$

Thus, we can define

$$B(P,Q) = \int_0^1 t^{P-I}(1-t)^{Q-I}\,dt. \tag{21}$$

In [3], it has been shown that if P, Q are matrices in $C^{r\times r}$ satisfying (20) and P or Q are scalar multiples of the identity matrix, then $B(P,Q) = B(Q,P)$. The next two examples show that if P, Q are not diagonalizable, or if they do not commute, then the property $B(P,Q) = B(Q,P)$ does not hold true.

EXAMPLE 1. Let $P = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ be matrices in $C^{2\times 2}$ with $\sigma(P) = \sigma(Q) = \{1,2\}$. Hence, they are diagonalizable and

$$PQ = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} \neq \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} = QP.$$

Note that $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ for all $n \geq 1$. Hence, for $0 < t < 1$, one gets

$$t^{P-I} = t^{\left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right]} = \begin{bmatrix} 1 & 0 \\ t-1 & t \end{bmatrix}, \qquad (1-t)^{Q-I} = (1-t)^{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]} = \begin{bmatrix} 1 & -t \\ 0 & 1-t \end{bmatrix},$$

$$(1-t)^{P-I} = (1-t)^{\left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right]} = \begin{bmatrix} 1 & 0 \\ -t & 1-t \end{bmatrix}, \qquad t^{Q-I} = t^{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]} = \begin{bmatrix} 1 & t-1 \\ 0 & t \end{bmatrix},$$

$$\begin{aligned} B(P,Q) &= \int_0^1 t^{P-I}(1-t)^{Q-I}\,dt = \int_0^1 \begin{bmatrix} 1 & 0 \\ t-1 & t \end{bmatrix} \begin{bmatrix} 1 & -t \\ 0 & 1-t \end{bmatrix} dt \\ &= \int_0^1 \begin{bmatrix} 1 & -t \\ t-1 & 2t(1-t) \end{bmatrix} dt = \begin{bmatrix} 1 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{3} \end{bmatrix}, \\ B(Q,P) &= \int_0^1 t^{Q-I}(1-t)^{P-I}\,dt = \int_0^1 \begin{bmatrix} 1 & t-1 \\ 0 & t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -t & 1-t \end{bmatrix} dt \\ &= \int_0^1 \begin{bmatrix} -t^2+t+1 & -(1-t)^2 \\ -t^2 & t(1-t) \end{bmatrix} dt = \begin{bmatrix} \frac{7}{6} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{1}{6} \end{bmatrix}. \end{aligned}$$

Hence, $B(P,Q) \neq B(Q,P)$.

The following lemmas are easy to prove. For the sake of brevity, we only prove the first.

LEMMA 1. *Let P, Q be commuting matrices in $C^{r\times r}$ satisfying (20). Then $B(P,Q) = B(Q,P)$.*

PROOF. Since $PQ = QP$, it follows that $(P-I)(\ln t)(Q-I)\ln(1-t) = (Q-I)(\ln(1-t))(P-I)\ln t$ for $0 < t < 1$. Hence, we can write

$$\begin{aligned} B(P,Q) &= \int_0^1 t^{P-I}(1-t)^{Q-I}\,dt = \int_0^1 e^{(P-I)\ln t} e^{(Q-I)\ln(1-t)}\,dt \\ &= \int_0^1 e^{(Q-I)\ln(1-t)} e^{(P-I)\ln t}\,dt = \int_0^1 e^{(Q-I)\ln u} e^{(P-I)\ln(1-u)}\,du = B(Q,P). \end{aligned}$$

LEMMA 2. *Let D, E be diagonal matrices in $C^{r\times r}$ satisfying (20). Then,*

$$B(D,E) = \Gamma(D)\Gamma(E)\Gamma^{-1}(D+E).$$

THEOREM 2. *Let P, Q be diagonalizable matrices in $C^{r\times r}$ such that $PQ = QP$ and satisfy condition (20). Then,*

$$B(P,Q) = \Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q). \tag{22}$$

PROOF. Since P, Q are diagonalizable and commute by [8, Theorem 1.3.12], they are simultaneously diagonalizable. Let S be an invertible matrix in $C^{r\times r}$ such that

$$S^{-1}PS = D, \quad S^{-1}QS = E; \qquad D, E \text{ are diagonal matrices.} \tag{23}$$

In order to prove (22), note that by [8, p. 54], if $\sigma(P) = \{\lambda_1, \dots, \lambda_r\}$ and $\sigma(Q) = \{\mu_1, \dots, \mu_r\}$, then $\sigma(P+Q) = \{\lambda_1 + \mu_{i_j}\}_{j=1}^{r}$, for some permutation $i_1,\ i_2, \dots, i_r$ of $1, 2, \dots, r$. Since matrices P and Q satisfy (20), it follows that

$$\operatorname{Re}(w) > 0, \qquad \text{for all } w \in \sigma(P+Q). \tag{24}$$

By Lemmas 1 and 2 and (23), it follows $P + Q = S(D+E)S^{-1}$ and

$$\Gamma(P+Q) = S\left[\int_0^\infty e^{-t}t^{D+E-I}\,dt\right]S^{-1} = S\Gamma(D+E)S^{-1}, \tag{25}$$

$$\Gamma(P) = S\Gamma(D)S^{-1}, \qquad \Gamma(Q) = S\Gamma(E)S^{-1}, \tag{26}$$

$$B(P,Q) = SB(D,E)S^{-1} = S\left[\Gamma(D)\Gamma(E)\Gamma^{-1}(D+E)\right]S^{-1}. \tag{27}$$

By (25), one gets $\Gamma^{-1}(D+E) = S^{-1}\Gamma^{-1}(P+Q)S$, and by (26) and (27), it follows that

$$\begin{aligned} B(P,Q) &= S\Gamma(D)\Gamma(E)\left[S^{-1}\Gamma^{-1}(P+Q)S\right]S^{-1} \\ &= \left(S\Gamma(D)S^{-1}\right)\left(S\Gamma(E)S^{-1}\right)\Gamma^{-1}(P+Q) = \Gamma(P+Q) = \Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q). \end{aligned}$$

Thus, the result is established.

REMARK. Apart from the commutativity hypothesis, the diagonalizability condition of Theorem 2 guarantees that every eigenvalue z of the matrix $P + Q$ lies in the right half-plane. The following example shows that, in general, if P, Q are matrices satisfying (20), its sum $P + Q$ does not satisfy this condition. Let a, b be positive numbers such that $ab > 1$. Then matrices $P = \begin{bmatrix} 1/2 & 0 \\ a & 1/2 \end{bmatrix}$, $Q = \begin{bmatrix} 1/2 & b \\ 0 & 1/2 \end{bmatrix}$ satisfy $\sigma(P) = \sigma(Q) = \{1/2\}$, but $P + Q = \begin{bmatrix} 1 & b \\ a & 1 \end{bmatrix}$ and $\sigma(P+Q) = \{1 - \sqrt{ab}, 1 + \sqrt{ab}\}$ with $1 - \sqrt{ab} < 0$.

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