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Stable Equivalence of Dualizing R-Varieties*

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In order to explain the background of this and succeeding papers in this series, it is useful to review some basic facts concerning artin algebras.

Throughout this discussion R denotes a commutative artin ring with radical r. An R-algebra Λ is said to be an artin R-algebra if it is finitely generated as an R-module. Suppose Λ is an artin R-algebra. Then Λ^{op}, the opposite ring of Λ, is also an artin R-algebra. It is well known that mod(Λ) and mod(Λ^{op}), the categories of finitely generated Λ and Λ^{op}-modules are dual categories. This duality is given as follows. Let $E(R/\mathrm{r})$ be an injective envelope for the R-module R/r. Then the functor [1] D: mod(Λ) $\to$ mod(Λ^{op}) given by $D(M) = \mathrm{Hom}_R(M, E(R/\mathrm{r}))$ is our desired duality. The first part of this paper is devoted to developing a generalization of the notion of an artin R-algebra which we call a dualizing R-variety. Before giving a formal definition, we try to explain the origins of the notion of a dualizing R-variety.

In recent work on the representation and module theory of artin R-algebras (see [5] and [6] for example) the abelian category $(\mathrm{mod}(\Lambda)^{\text{op}}, Ab)$ of all additive functors from the opposite category of mod(Λ) to abelian groups, has played a significant role. Following [4], we usually denote the category $(\mathrm{mod}(\Lambda)^{\text{op}}, Ab)$ by Mod(mod(Λ)), which we call the category of modules on mod(Λ). The full subcategory mod(mod(Λ)) of Mod(mod(Λ)) consisting of the finitely presented mod(Λ)-modules plays a particularly significant role. We recall that a mod(Λ)-module M is finitely presented if there is an exact sequence $\mathrm{Hom}_\Lambda(\cdot, A) \to \mathrm{Hom}_\Lambda(\cdot, B) \to M \to 0$ of mod(Λ)-modules with A and B in mod(Λ) (see [4] for further details). Of critical importance in studying both mod(Λ) and mod(mod(Λ)) is the fact that the duality D: mod(Λ) $\to$ mod(Λ^{op}) can be extended to give a duality D: mod(mod(Λ)) $\to$ mod(mod($\Lambda)^{\text{op}}$) defined by $D(M)(X) = \mathrm{Hom}_R(M(X), E(R/\mathrm{r}))$ for all X in mod(Λ).

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This fact serves as the primary motivation for the definition of a dualizing R-variety.

We recall that a variety of annuli, or more simply, a variety is a skeletally small additive category in which idempotents split. An R-variety is a variety $\mathbf{C}$ together with an R-module structure on the abelian group $\mathbf{C}(A, B)$ for each pair of objects A and B in $\mathbf{C}$, such that the composition maps $\mathbf{C}(A, B) \times \mathbf{C}(B, C) \to \mathbf{C}(A, C)$ in $\mathbf{C}$ are bilinear R-module maps. An R-variety $\mathbf{C}$ is said to be a finite R-variety if $\mathbf{C}(A, B)$ is a finitely generated R-module for each pair of objects A, B in $\mathbf{C}$. Clearly if Λ is an artin R-algebra, then $\text{mod}(\Lambda)$ is a finite R-variety. Also if $\mathbf{C}$ is a finite R-variety, then $\mathbf{C}^{\text{op}}$ is a finite R-variety.

Suppose $\mathbf{C}$ is a finite R-variety. If $M\colon \mathbf{C}^{\text{op}} \to Ab$ is a $\mathbf{C}$-module (i.e., an additive functor from $\mathbf{C}^{\text{op}}$ to abelian groups), then for each X in $\mathbf{C}$, the abelian group $M(X)$ also has a structure as an $\text{End}_{\mathbf{C}}(X)^{\text{op}}$-module and hence as an R-module since $\text{End}_{\mathbf{C}}(X)^{\text{op}}$ is an R-algebra. Further, if $f\colon M \to M'$ is a morphism of $\mathbf{C}$-modules, i.e., a morphism of functors, then $f_X\colon M(X) \to M'(X)$ is an R-module morphism for each X in $\mathbf{C}$. We denote by $(\mathbf{C}^{\text{op}}, \text{mod}(R))$, the full subcategory of $(\mathbf{C}^{\text{op}}, Ab)$ consisting of all $\mathbf{C}$-modules M such that $M(X)$ is a finitely generated R-module. Because R is an artin ring, it is easily seen that $(\mathbf{C}^{\text{op}}, \text{mod}(R))$ is an abelian category with the property that the inclusion morphism $(\mathbf{C}^{\text{op}}, \text{mod}(R)) \to (\mathbf{C}^{\text{op}}, Ab)$ is exact. It is also easily checked that $\mathbf{C}$ being a finite R-variety implies that $(\mathbf{C}^{\text{op}}, \text{mod}(R))$ contains $\text{mod}(\mathbf{C})$, the full subcategory of finitely presented $\mathbf{C}$-modules.

In general, the functors $D\colon (\mathbf{C}^{\text{op}}, \text{mod}(R)) \to (\mathbf{C}, \text{mod}(R))$ and $D\colon (\mathbf{C}, \text{mod}(R)) \to (\mathbf{C}^{\text{op}}, \text{mod}(R))$ given by $D(M)(X) = \text{Hom}_R(M(X), E(R/\mathrm{r}))$ for all X in $\mathbf{C}^{\text{op}}$ and all X in $\mathbf{C}$, define a duality between $(\mathbf{C}^{\text{op}}, \text{mod}(R))$ and $(\mathbf{C}, \text{mod}(R))$. However it is not always the case that if M is a finitely presented $\mathbf{C}$-module ($\mathbf{C}^{\text{op}}$-module), then $D(M)$ is a finitely presented $\mathbf{C}^{\text{op}}$-module ($\mathbf{C}$-module) even though as observed earlier this is always the case when $\mathbf{C} = \text{mod}(\Lambda)$ with Λ an artin algebra. This fact leads us to define a dualizing R-variety to be a finite R-variety $\mathbf{C}$ with the property that for each finitely presented $\mathbf{C}$-module ($\mathbf{C}^{\text{op}}$-module) M, the $\mathbf{C}^{\text{op}}$-module ($\mathbf{C}$-module) $D(M)$ is also finitely presented. Hence if $\mathbf{C}$ is a dualizing R-variety, we have the duality $D\colon \text{mod}(\mathbf{C}) \to \text{mod}(\mathbf{C}^{\text{op}})$ given by $D(M)(X) = \text{Hom}_R(M(X), E(R/\mathrm{r}))$ for all X in $\text{mod}(\mathbf{C})$. From our previous discussion it follows that if $\mathbf{C}$ is the category of finitely generated projective modules over an artin ring, then $\mathbf{C}$ is a dualizing R-variety.

One of the advantages gained by introducing the notion of dualizing

R-varieties is that very often constructions on dualizing R-varieties again yield dualizing R-varieties. For example, it is shown in Section 2 that mod(**C**) is a dualizing R-variety if **C** is a dualizing R-variety. Hence if Λ is an artin R-algebra, then mod(mod(Λ)) is a dualizing R-variety.

The first part of this paper is devoted to establishing this as well as other basic formal properties of dualizing R-varieties. The rest of the paper is devoted to discussing a particular type of dualizing R-variety which we now describe.

Let **C** be a dualizing R-variety and **D** the dualizing R-variety mod(**C**). We denote by $\mathbf{D}/P$ the category of **C**-modules modulo projectives. That is, the objects of $\mathbf{D}/P$ are the same as the objects of **D** and $\mathbf{D}/P(A, B)$ for each pair of objects A, B in **D**, is the group $\mathbf{D}(A, B)/P(A, B)$, where $P(A, B)$ is the subgroup of those morphisms from A to B which factor through projective objects in **D**. In Section 6 it is shown that $\mathbf{D}/P$ is a dualizing R-variety. In Section 8 the notion of stable equivalence is introduced. Namely, two dualizing R-varieties **C** and **C**′ are said to be stably equivalent if the dualizing R-varieties $\mathbf{D}/P$ and $\mathbf{D}'/P$ are equivalent where $\mathbf{D} = \text{mod}(\mathbf{C})$ and $\mathbf{D}' = \text{mod}(\mathbf{C}')$. A basic question is to describe what it means for two dualizing R-varieties to be stably equivalent. Some preliminary results along these lines are obtained in Section 9. While they are in no sense definitive, they do suffice to derive quite satisfactory descriptions of when a dualizing R-variety **C** has the property that gl dim mod($\mathbf{D}/P$) $= 0, 1$, or 2 where $\mathbf{D} = \text{mod}(\mathbf{C})$ as is shown in Section 10.

In this connection, it is pointed out in Section 10 that if gl dim mod(**C**) $\leqslant 1$, then gl dim($\mathbf{D}/P$) $\leqslant 2$. Hence if **C** is stably equivalent to **C**′ with gl dim($\mathbf{D}'/P$) $\leqslant 2$ ($\mathbf{D}' = \text{mod}(\mathbf{C}')$), then gl dim($\mathbf{D}/P$) $\leqslant 2$. If **C** is the category of finitely generated projective modules over an artin R-algebra Λ, it was shown in [6], that if gl dim($\mathbf{D}/P$) $\leqslant 2$ then **C** is stably equivalent to a hereditary artin algebra. While this result is true for a larger class of dualizing R-varieties than the categories of finitely generated projective modules over artin R-algebras, it is not true for arbitrary dualizing R-varieties. The problem of describing precisely which dualizing R-varieties **C** have the property that gl dim($\mathbf{D}/P$) $\leqslant 2$ implies **C** is stably equivalent to a hereditary dualizing R-variety will be taken up in [7], another paper in this series.

The rest of the papers in this series will deal with rather specific calculations and examples.

In general, we follow the notation and conventions established in [4].

In particular, the unadorned word module always means a left module. Right modules will generally be viewed as left modules over the opposite ring.

1. R-Categories

Let R be a commutative ring. By an *R-category* we mean a preadditive category $\mathbf{A}$ together with an R-module structure on each abelian group $\mathbf{A}(A_1, A_2)$ such that the composition maps $\mathbf{A}(A_1, A_2) \times \mathbf{A}(A_2, A_3) \to \mathbf{A}(A_1, A_3)$ are bilinear maps of R-modules, where $\mathbf{A}(X, Y)$ denotes the group of morphisms in $\mathbf{A}$ from the object X to the object Y. It should be noted that preadditive categories are the same thing as Z-categories where Z is the ring of integers.

Suppose $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are R-categories. An *R-functor* $F: \mathbf{A} \to \mathbf{B}$ is a functor $F: \mathbf{A} \to \mathbf{B}$ of preadditive categories such that $F: \mathbf{A}(A_1, A_2) \to \mathbf{B}(B_1, B_2)$ is an R-module morphism, not just a morphism of abelian groups, for all A_1, A_2 in $\mathbf{A}$. Clearly if $F: \mathbf{A} \to \mathbf{B}$ and $G: \mathbf{B} \to \mathbf{C}$ are R-functors, then $GF: \mathbf{A} \to \mathbf{C}$ is an R-functor. Suppose $F, G: \mathbf{A} \to \mathbf{B}$ are two R-functors. Given a morphism $f: F \to G$ and an element r in R, it is easy to see that $rf: F \to G$ given by $(rf)_A = r(f_A)$ for all A in $\mathbf{A}$ is also a morphism of functors. We say that $\mathbf{A}$ is skeletally small if the collection of isomorphism classes of objects is a set. If we assume that $\mathbf{A}$ is skeletally small, then the operation just described of R on the abelian group (F, G), the group of morphisms from F to G, makes (F, G) an R-module in such a way that $R - (\mathbf{A}, \mathbf{B})$, the category of all R-functors from $\mathbf{A}$ to $\mathbf{B}$ is an R-category. Because the R-functors are the only functors we will ever consider between R-categories, we will usually refer to them simply as functors and denote $R - (\mathbf{A}, \mathbf{B})$ by $(\mathbf{A}, \mathbf{B})$ when $\mathbf{A}$ and $\mathbf{B}$ are R-categories with $\mathbf{A}$ skeletally small.

Suppose A is an object in the R-category $\mathbf{A}$. The fact that $\mathbf{A}(A, A)$ is an R-module enables us to define the map $R \to \mathbf{A}(A, A)$ given by $r \to r \cdot 1_A$ for each r in R and 1_A the identity on A. It is easily checked that this map is a ring morphism $R \to \mathrm{End}_{\mathbf{A}}(A)$ with the property that $\mathrm{Im}(R \to \mathrm{End}_{\mathbf{A}}(A))$ is contained in the center of $\mathrm{End}_{\mathbf{A}}(A)$. This means that $R \to \mathrm{End}_{\mathbf{A}}(A)$ is an R-algebra and it is the only way we consider $\mathrm{End}_{\mathbf{A}}(A)$ an R-algebra.

Suppose now that $F: \mathbf{A} \to Ab$ is an additive functor. Then for each A in $\mathbf{A}$, the abelian group $F(A)$ is an $\mathrm{End}_{\mathbf{A}}(A)$-module and hence an R-module by means of the standard ring morphism. Hence associated with each functor $F: \mathbf{A} \to Ab$ is the functor $F': \mathbf{A} \to \mathrm{Mod}(R)$ where

$F'(A)$ is the R-module $F(A)$ we just described for each A in $\mathbf{A}$. It is not difficult to check that $F'\colon \mathbf{A} \to \mathrm{Mod}(R)$ is an R-functor where $\mathrm{Mod}(R)$, the category of R-modules is considered an R-category in the usual fashion. Therefore we obtain the functor $(\mathbf{A}, Ab) \to R-(\mathbf{A}, \mathrm{Mod}(R))$ given by $F \to F'$ for all additive functors F in $(\mathbf{A}, Ab)$. It is not difficult to check that this functor is an isomorphism of categories. We generally view this isomorphism as an identification and use the notations $(\mathbf{A}, Ab)$ and $R - (\mathbf{A}, \mathrm{Mod}(R))$, or more simply, $(\mathbf{A}, \mathrm{Mod}(R))$ interchangeably.

Suppose $\mathbf{A}$ is an R-category. Then we consider $\mathbf{A}^{\mathrm{op}}$ an R-category by letting $\mathbf{A}^{\mathrm{op}}(A_1, A_2)$ be the R-module $\mathbf{A}(A_2, A_1)$. We recall (see [4]) that if $\mathbf{A}$ is skeletally small, then $(\mathbf{A}^{\mathrm{op}}, Ab)$ is called the category of $\mathbf{A}$-*modules* and is usually denoted by $\mathrm{Mod}(\mathbf{A})$. Hence our previous remarks show that $\mathrm{Mod}(\mathbf{A}) = (\mathbf{A}^{\mathrm{op}}, \mathrm{Mod}(R))$. We now recall the notion of a relation on an additive category.

Let $\mathbf{A}$ be an R-category. A relation S on $\mathbf{A}$ is a collection of R-submodules $S(A_1, A_2)$ of $\mathbf{A}(A_1, A_2)$, one for each pair of objects A_1, A_2 in $\mathbf{A}$ satisfying:

(a) if f is in $S(A_1, A_2)$ and g is in $\mathbf{A}(A_2, A_3)$, then gf is in $S(A_1, A_3)$;

(b) if f is in $S(A_1, A_2)$ and g is in $\mathbf{A}(A_0, A_1)$, then fg is in $S(A_0, A_2)$.

Suppose S is a relation on $\mathbf{A}$. It is easily seen that for A_1, A_2, A_3 in $\mathbf{A}$, there is a unique bilinear map of R-modules

$$\mathbf{A}(A_1, A_2)/S(A_1, A_2) \times \mathbf{A}(A_2, A_3)/S(A_2, A_3) \to \mathbf{A}(A_1, A_3)/S(A_1, A_3)$$

which makes the diagram

$$\begin{array}{ccccc}
\mathbf{A}(A_1, A_2) & \times & \mathbf{A}(A_2, A_3) & \longrightarrow & \mathbf{A}(A_1, A_3) \\
\downarrow & \times & \downarrow & & \downarrow \\
\mathbf{A}(A_1, A_2)/S(A_1, A_2) & \times & \mathbf{A}(A_2, A_3)/S(A_2, A_3) & \longrightarrow & \mathbf{A}(A_1, A_3)/S(A_1, A_3)
\end{array}$$

commute, where the vertical maps are given by the canonical morphisms of a module onto a factor module. It is now easily checked that the following data define a preadditive category $\mathbf{A}/S$.

(a) The objects of $\mathbf{A}/S$ are the objects of $\mathbf{A}$;

(b) $(\mathbf{A}/S)(A_1, A_2) = \mathbf{A}(A_1, A_2)/S(A_1, A_2)$ for all A_1, A_2 in $\mathbf{A}/S$;

(c) the composition maps $(\mathbf{A}/S)(A_1, A_2) \times (\mathbf{A}/S)(A_2, A_3) \to (\mathbf{A}/S)(A_1, A_3)$ are the unique bilinear maps described above.

The category $\mathbf{A}/S$ is called the *category* $\mathbf{A}$ *modulo* S. Clearly $\mathbf{A}/S$ is an R-category with the property that the functor $\mathbf{A} \to \mathbf{A}/S$ given by $A \to A$ for each object A in $\mathbf{A}$ and $\mathbf{A}(A_1, A_2) \to (\mathbf{A}/S)(A_1, A_2)$ is the canonical morphism, is a full R-functor which is an isomorphism on objects. It is the only functor from $\mathbf{A}$ to $\mathbf{A}/S$ we shall ever consider.

It is well known and not difficult to check that the functor $\mathbf{A} \to \mathbf{A}/S$ has the following properties.

PROPOSITION 1.1. *Let* $\mathbf{B}$ *be an R-category.*

(a) *A functor* $F: \mathbf{A} \to \mathbf{B}$ *has the property that* $S(A_1, A_2)$ *is contained in* $\operatorname{Ker}(F: \mathbf{A}(A_1, A_2) \to \mathbf{B}(F(A_1), F(A_2))$ *for all* A_1 *and* A_2 *in* $\mathbf{A}$ *if and only if there is a functor* $G: \mathbf{A}/S \to \mathbf{B}$ *such that* F *is the composition* $\mathbf{A} \to \mathbf{A}/S \to^G \mathbf{B}$. *Further if such a* G *exists it is unique.*

(b) *The functor* $(\mathbf{A}/S, \mathbf{B}) \to (\mathbf{A}, \mathbf{B})$ *induced by* $A \to A/S$ *is a fully faithful functor.*

Stated in other words, Proposition 1.1 simply says that the morphism $(\mathbf{A}/S, \mathbf{B}) \to (\mathbf{A}, \mathbf{B})$ induces an isomorphism of categories between $(\mathbf{A}/S, \mathbf{B})$ and the full subcategory of $(\mathbf{A}, \mathbf{B})$ consisting of those functors which vanish on the relation S. We will often view this equivalence as an identification.

We end this section by extending the following notions for pre-additive categories to R-categories.

A category $\mathbf{A}$ is said to be an annulus if $\mathbf{A}$ is a skeletally small additive category in which idempotents split and which has an additive generator G, that is, every object in $\mathbf{A}$ is a summand of a finite sum of copies of G. An R-category $\mathbf{A}$ is said to be an R-annulus if $\mathbf{A}$ as an additive category is an annulus. A category $\mathbf{A}$ is said to be a *variety of annuli* or more simply a *variety* if $\mathbf{A}$ is a skeletally small additive category in which idempotents split. An R-category $\mathbf{A}$ is said to be an *R-variety* or *R-variety of annuli* if $\mathbf{A}$, as an additive category, is a variety or variety of annuli.

2. DESCRIPTION OF DUALIZING R-VARIETIES

Throughout this section we assume that R is a commutative ring. In studying the category of finitely generated modules over an R-algebra Λ which is a finitely generated R-module, not only do the R-categories $\mathbf{p}(\Lambda)$. the category of finitely generated projective Λ-modules and mod(Λ), the category of finitely presented Λ-modules play an important role, but also the finitely presented modules over mod(Λ). In this section we describe a certain class of R-varieties which provides a common framework in which to discuss these and other R-varieties which occur in studying artin algebras.

Let $\mathbf{C}$ be an R-variety and mod(R) the category of finitely presented R-modules. The category of all functors $(\mathbf{C}^{\text{op}}, \text{mod}(R))$ is obviously the same thing as the full subcategory of Mod($\mathbf{C}$) consisting of all $\mathbf{C}$-modules M such that $M(C)$ is a finitely generated R-module for all C in $\mathbf{C}$. Clearly, if $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of $\mathbf{C}$-modules, then M_2 is in $(\mathbf{C}^{\text{op}}, \text{mod}(R))$ if and only if M_1 and M_2 are in $(\mathbf{C}^{\text{op}}, \text{mod}(R))$. Hence $(\mathbf{C}^{\text{op}}, \text{mod}(R))$ is an abelian category and the inclusion functor $(\mathbf{C}^{\text{op}}, \text{mod}(R)) \to \text{Mod}(\mathbf{C})$ is exact.

We recall that since R is a commutative artin ring, the injective envelope E of R/r, where r is the radical of R, is a finitely generated R-module with the property that the contravariant functor D: mod(R) $\to$ mod(R) defined by $D(X) = \text{Hom}_R(M, E)$ is a duality. We recall that the isomorphism $\alpha: I_{\text{mod}(R)} \to D^2$ giving this duality is defined by $\alpha_X: X \to \text{Hom}_R(\text{Hom}_R(X, E), E)$ where α_X is the usual R-morphism $\alpha_X(x)(f) = f(x)$ for all x in X and f in $\text{Hom}_R(X, E)$.

Now let M be a $\mathbf{C}$-module in $(\mathbf{C}^{\text{op}}, \text{mod}(R))$. Then the composition of functors $\mathbf{C}^{\text{op}} \xrightarrow{M} \text{mod}(R) \xrightarrow{D} \text{mod}(R)$ is a contravariant functor from $\mathbf{C}^{\text{op}}$ to mod(R), or what is the same thing, DM is a contravariant functor from $\mathbf{C}$ to mod(R). Thus we have the functor D: $(\mathbf{C}^{\text{op}}, \text{mod}(R)) \to (\mathbf{C}, \text{mod}(R))$ given by $D(M) = DM$. Clearly $D(M)(X) = D(M(X))$ for each X in $\mathbf{C}$. In a similar way one obtains a functor D: $(\mathbf{C}, \text{mod}(R)) \to (\mathbf{C}^{\text{op}}, \text{mod}(R))$. These functors establish a duality between $(\mathbf{C}^{\text{op}}, \text{mod}(R))$ and $(\mathbf{C}, \text{mod}(R))$ given by the isomorphisms $\alpha: I_{(\mathbf{C}^{\text{op}}, \text{mod}(R))} \to D^2$ and $\beta: I_{(\mathbf{C}, \text{mod}(R))} \to D^2$. The morphism $\alpha: M \to D^2(M)$ is given by the standard morphism of R-modules $\alpha_C: M(C) \to D^2(M(C))$ giving the duality on mod(R) for each C in $\mathbf{C}^{\text{op}}$, and β is defined similarly. It should be noted that if $\mathbf{C}'$ is a full R-subcategory of $\mathbf{C}$, then we get commutative diagrams

$$\begin{array}{ccccc}
(\mathbf{C}^{\text{op}}, \text{mod}(R)) & \xrightarrow{D} & (\mathbf{C}, \text{mod}(R)) & \xrightarrow{D} & (\mathbf{C}^{\text{op}}, \text{mod}(R)) \\
\downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \text{res}} \\
(\mathbf{C}'^{\text{op}}, \text{mod}(R)) & \xrightarrow{D} & (\mathbf{C}', \text{mod}(R)) & \xrightarrow{D} & (\mathbf{C}'^{\text{op}}, \text{mod}(R)).
\end{array}$$

This duality is of particular interest in the following circumstance.

We say that $\mathbf{C}$ is a finite R-variety if f.g.($\mathbf{C}$), the category of finitely generated $\mathbf{C}$-modules, is contained in ($\mathbf{C}^{\mathrm{op}}$, mod(R)). Since a $\mathbf{C}$-module M is finitely generated if and only if there is an epimorphism $\mathbf{C}(\cdot, C) \to M \to 0$ for some C in $\mathbf{C}$, it follows that $\mathbf{C}$ is a finite R-variety if and only if $\mathbf{C}(\cdot, C)$ is in ($\mathbf{C}^{\mathrm{op}}$, mod(R)) for all C in $\mathbf{C}$. Obviously $\mathbf{C}$ is a finite R-variety if and only if $\mathbf{C}^{\mathrm{op}}$ is a finite R-variety. The rest of this section is devoted to describing the finite R-varieties $\mathbf{C}$ which have the property that a $\mathbf{C}$-module M in ($\mathbf{C}^{\mathrm{op}}$, mod(R)) is finitely presented if and only if the $\mathbf{C}^{\mathrm{op}}$-module $D(M)$ in ($\mathbf{C}$, mod(R)) is finitely presented. Stated differently, we want to find which finite R-varieties $\mathbf{C}$ have the property that the duality D: ($\mathbf{C}^{\mathrm{op}}$, mod(R)) $\to$ ($\mathbf{C}$, mod(R)) induces a duality D: mod($\mathbf{C}^{\mathrm{op}}$) $\to$ mod($\mathbf{C}$) where mod($\mathbf{C}^{\mathrm{op}}$) and mod($\mathbf{C}$) are the categories of finitely presented $\mathbf{C}$ and $\mathbf{C}^{\mathrm{op}}$-modules respectively. It is these types of finite R-varieties which will be our major concern throughout the rest of this paper.

We begin with the following characterization of the finitely presented $\mathbf{C}$-modules.

PROPOSITION 2.1. *Let* $\mathbf{C}$ *be a finite* R*-variety and* M *a* $\mathbf{C}$*-module in* ($\mathbf{C}^{\mathrm{op}}$, mod(R)). *Then the following statements are equivalent*:

(a) M *is a finitely presented* $\mathbf{C}$*-module.*

(b) *There is a* C *in* $\mathbf{C}$ *such that* M *has a finite projective presentation over* $\mathbf{V}(C)$ *where* $\mathbf{V}(C)$ *is the annulus generated by* C, *or equivalently, there is an exact sequence* $\mathbf{C}(\cdot, C_1) \to \mathbf{C}(\cdot, C_0) \to M \to 0$ *with the* C_i *summands of finite sums of copies of* C *for* $i = 0$ *and* 1.

(c) *There is a* C *in* $\mathbf{C}$ *such that the morphism res*: $\mathrm{Hom}_{\mathbf{C}}(M, N) \to \mathrm{Hom}_{\mathbf{V}(C)}(M \mid \mathbf{V}(C), N \mid \mathbf{V}(C))$ *is an isomorphism for all* $\mathbf{C}$*-modules* N *in* ($\mathbf{C}^{\mathrm{op}}$, mod(R)).

Proof. (a) implies (b) and (b) implies (c) were proven in [4, Proposition 3.2].

(c) implies (a). Since $\mathbf{C}$ is a finite R-variety, we know that $\mathrm{End}_{\mathbf{C}}(C)^{\mathrm{op}}$ is artinian since it is a finitely generated R-module. Also $M(C)$ is a finitely generated R-module and hence a finitely generated $\mathrm{End}_{\mathbf{C}}(C)^{\mathrm{op}}$-module. Therefore $M \mid \mathbf{V}(C)$ is a finitely presented $\mathbf{V}(C)$-module. Hence there is an exact sequence $\mathbf{V}(C)(\cdot, C_1) \to \mathbf{V}(C)(\cdot, C_0) \to M \mid \mathbf{V}(C) \to 0$ with C_i in $\mathbf{V}(C)$. Let M' be a $\mathbf{C}$-module such that the sequence of $\mathbf{C}$-modules $\mathbf{C}(\cdot, C_1) \to \mathbf{C}(\cdot, C_0) \to M' \to 0$ is exact. Since M' is finitely projectively presented over $\mathbf{V} \mid \mathbf{C}$, it follows that res: $\mathrm{Hom}_{\mathbf{C}}(M', N) \to$

$\text{Hom}_{\mathbf{V}(C)}(M' \mid \mathbf{V}(C), N \mid \mathbf{V}(C))$ is an isomorphism for all $\mathbf{C}$-modules N and so in particular for all N in $(\mathbf{C}^{\text{op}}, \text{mod}(R))$. But $M' \mid \mathbf{V}(C)$ is isomorphic to $M \mid \mathbf{V}(C)$. Hence the functors $\text{Hom}_{\mathbf{V}(C)}(M' \mid \mathbf{V}(C), \cdot)$ and $\text{Hom}_{\mathbf{V}(C)}(M \mid \mathbf{V}(C), \cdot)$ from $(\mathbf{V}(C)^{\text{op}}, \text{mod}(R))$ to abelian groups are isomorphic. Therefore the functors $\text{Hom}_{\mathbf{C}}(M, \cdot)$ and $\text{Hom}_{\mathbf{C}}(M', \cdot)$ from $(\mathbf{C}^{\text{op}}, \text{mod}(R))$ to abelian groups are isomorphic. Since M and M' are in $(\mathbf{C}^{\text{op}}, \text{mod}(R))$, it follows that M and M' are isomorphic in $(\mathbf{C}^{\text{op}}, \text{mod}(R))$ and therefore in $\text{Mod}(\mathbf{C})$. Because M' is a finitely presented $\mathbf{C}$-module, it follows that M is a finitely presented $\mathbf{C}$-module.

This characterization of when a $\mathbf{C}$-module in $(\mathbf{C}^{\text{op}}, \text{mod}(R))$ is finitely presented gives the following useful result.

PROPOSITION 2.2. *Let* $\mathbf{C}$ *be a finite* R*-variety and let* $D: (\mathbf{C}^{\text{op}}, \text{mod}(R)) \to (\mathbf{C}, \text{mod}(R))$ *be the usual duality functor. Then the following statements are equivalent for a finitely presented* $\mathbf{C}$*-module* M.

(a) $D(M)$ *is a finitely presented* $\mathbf{C}^{\text{op}}$*-module.*

(b) *There is a* C *in* $\mathbf{C}$ *such that* res: $\text{Hom}_{\mathbf{C}}(N, M) \to \text{Hom}_{\mathbf{V}(C)}(N \mid \mathbf{V}(C), M \mid \mathbf{V}(C))$ *is an isomorphism for all* N *in* $(\mathbf{C}^{\text{op}}, \text{mod}(R))$.

(c) *There is a* C *in* $\mathbf{C}$ *such that the natural morphism* $((\cdot, X), M) \to ((\cdot, X) \mid \mathbf{V}(C), M \mid \mathbf{V}(C))$ *is an isomorphism for all* X *in* $\mathbf{C}$.

Proof. (a) implies (b). Since $D(M)$ is a finitely presented $\mathbf{C}^{\text{op}}$-module, we can find a C in $\mathbf{C}$ such that $(D(M), N) \to (D(M) \mid \mathbf{V}(C), N \mid \mathbf{V}(C))$ is an isomorphism for all N in $(\mathbf{C}, \text{mod}(R))$. Hence applying the duality functor D, we have that $(D(N), M) \to (D(N) \mid \mathbf{V}(C), M \mid \mathbf{V}(C))$ is an isomorphism since $D(N \mid \mathbf{V}(C)) = D(N) \mid \mathbf{V}(C)$ for all N in $(\mathbf{C}, \text{mod}(R))$. Since $D(N)$ runs through all of $(\mathbf{C}^{\text{op}}, \text{mod}(R))$ as N runs through $(\mathbf{C}, \text{mod}(R))$ we have shown that the object C in $\mathbf{C}$ has the property that $(X, M) \to (X \mid \mathbf{V}(C), M \mid \mathbf{V}(C))$ is an isomorphism for all X in $(\mathbf{C}^{\text{op}}, \text{mod}(R))$ if $D(M)$ is a finitely presented $\mathbf{C}^{\text{op}}$-module.

(b) implies (a). This can be shown by reversing the steps of the proof given for (a) implies (b).

(b) implies (c). Trivial.

(c) implies (b). Assume that C in $\mathbf{C}$ has the property that $((\cdot, X), M) \to ((\cdot, X) \mid \mathbf{V}(C), M \mid \mathbf{V}(C))$ is an isomorphism for all X in $\mathbf{C}$. Let N be an arbitrary object in $(\mathbf{C}^{\text{op}}, \text{mod}(R))$ and let

$$\coprod_{i \in I} (\cdot, B_i) \to \coprod_{j \in J} (\cdot, D_j) \to N \to 0$$

be a projective presentation for N in $(\mathbf{C}^{\mathrm{op}}, \mathrm{Mod}(R))$. We then have the exact commutative diagram

$$\begin{array}{ccccccc}
0 & \to & (N, M) & \to & \coprod_{i\in I} ((\cdot, B_i), M) & \to & \coprod_{j\in J} ((\cdot, D_j), M) \\
 & & \| & & \downarrow & & \downarrow \\
0 & \to & (N, M) & \to & \prod_{i\in I} ((\cdot, B_i), M) & \to & \prod_{j\in J} ((\cdot, D_j), M) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & (N \mid \mathbf{V}(C), M \mid \mathbf{V}(C)) & \to & \prod_{i\in I} ((\cdot, B_i) \mid \mathbf{V}(C), M \mid \mathbf{V}(C)) & \to & \prod_{j\in J} ((\cdot, D_j) \mid \mathbf{V}(C), M \mid \mathbf{V}(C))
\end{array}$$

Since the last two bottom vertical morphisms are isomorphisms, it follows that $(N, M) \to (N \mid \mathbf{V}(C), M \mid \mathbf{V}(C))$ is an isomorphism.

Since the notion of a pseudokernel is involved in the statement of the next result, we recall the definition and some basic properties of pseudokernels [3].

Let $\mathbf{C}$ be an arbitrary variety. A morphism $C_2 \xrightarrow{f} C_1$ is said to be *pseudokernel* for a morphism $C_1 \xrightarrow{g} C_0$ if the sequence of functors

$$\mathbf{C}(\cdot, C_2) \xrightarrow{\mathbf{C}(\cdot, f)} \mathbf{C}(\cdot, C_1) \xrightarrow{\mathbf{C}(\cdot, g)} \mathbf{C}(\cdot, C_0)$$

is exact. It is well known that the following statements are equivalent for the category $\mathbf{C}$.

(a) $\mathbf{C}$ has pseudokernels, i.e., every morphism in $\mathbf{C}$ has a pseudokernel.

(b) Every morphism in $\mathrm{mod}(\mathbf{C})$ has a kernel.

(c) $\mathrm{mod}(\mathbf{C})$ is abelian.

The reader should have no difficulty defining for himself the dual notion of pseudocokernel and seeing that the duals of (a), (b), and (c) hold.

PROPOSITION 2.3. *Let $\mathbf{C}$ be a finite R-variety with pseudocokernels. Then the following statements are equivalent.*

(a) *If M is a finitely presented $\mathbf{C}$-module, then $D(M)$ is a finitely presented $\mathbf{C}^{\mathrm{op}}$-module.*

(b) *For each object B in $\mathbf{C}$, there is an object C in $\mathbf{C}$ such that* res: $\mathrm{Hom}_{\mathbf{C}}(\mathbf{C}(\cdot, X), \mathbf{C}(\cdot, B)) \to \mathrm{Hom}_{\mathbf{V}(C)}(\mathbf{C}(\cdot, X) \mid \mathbf{V}(C), \mathbf{C}(\cdot, B) \mid \mathbf{V}(C))$ *is an isomorphism for all X in $\mathbf{C}$.*

Proof. (a) implies (b) follows from Proposition 2.2.

(b) implies (a). Proposition 2.2 shows that the hypothesis of (b) implies that $D(\mathbf{C}(\cdot, X))$ is a finitely presented $\mathbf{C}^{op}$-module for each X in $\mathbf{C}$. Suppose M is a finitely presented $\mathbf{C}$-module and $\mathbf{C}(\cdot, C_1) \to \mathbf{C}(\cdot, C_0) \to M \to 0$ is exact. Then $0 \to D(M) \to D(\mathbf{C}(\cdot, C_0)) \to D(\mathbf{C}(\cdot, C_1))$ is an exact sequence in $(\mathbf{C}, \text{mod}(R))$. Since $\mathbf{C}$ has pseudocokernels, it follows that $\mathbf{C}^{op}$ has pseudokernels so that $\text{mod}(\mathbf{C}^{op})$ is abelian and the inclusion $\text{mod}(\mathbf{C}^{op}) \to (\mathbf{C}, \text{mod}(R))$ is exact. Consequently $D(M)$ is finitely presented since $D(\mathbf{C}(\cdot, C_0))$ and $D(\mathbf{C}(\cdot, C_1))$ are finitely presented.

Before stating the main result of this section, we make the following definition. We say that an R-variety $\mathbf{C}$ is a *dualizing R-variety* if it is a finite R-variety such that the duality $D: (\mathbf{C}^{op}, \text{mod}(R)) \to (\mathbf{C}, \text{mod}(R))$ induces a duality $D: \text{mod}(\mathbf{C}) \to \text{mod}(\mathbf{C}^{op})$. Obviously a finite R-variety $\mathbf{C}$ being a dualizing R-variety is equivalent to a $\mathbf{C}$-module M in $(\mathbf{C}^{op}, \text{mod}(R))$ being finitely presented if and only if $D(M)$ is a finitely presented $\mathbf{C}^{op}$-module. We now give a complete characterization of dualizing R-varieties.

THEOREM 2.4. *A finite R-variety $\mathbf{C}$ is a dualizing R-variety if and only if it satisfies the following conditions.*

(a) *$\mathbf{C}$ and $\mathbf{C}^{op}$ have pseudokernels.*

(b) *Given any B in $\mathbf{C}$, there is a C in $\mathbf{C}$ such that the morphism*

$$\mathbf{C}(X, B) \to \text{Hom}_{\text{End}_{\mathbf{C}}(C)^{op}}(\mathbf{C}(C, X), \mathbf{C}(C, B))$$

is an isomorphism for all X in $\mathbf{C}$.

(c) *Given any B in $\mathbf{C}^{op}$, there is a D in $\mathbf{C}^{op}$ such that the morphism*

$$\mathbf{C}^{op}(X, B) \to \text{Hom}_{\text{End}_{\mathbf{C}^{op}}(D)^{op}}(\mathbf{C}^{op}(D, B), \mathbf{C}^{op}(D, X))$$

is an isomorphism for all X in $\mathbf{C}^{op}$.

Proof. Suppose that $\mathbf{C}$ is a dualizing R-variety. Then the duality $D: (\mathbf{C}^{op}, \text{mod}(R)) \to (\mathbf{C}, \text{mod}(R))$ induces a duality $D: \text{mod}(\mathbf{C}) \to \text{mod}(\mathbf{C}^{op})$. Since $\text{mod}(\mathbf{C})$ and $\text{mod}(\mathbf{C}^{op})$ both have cokernels, $\text{mod}(\mathbf{C})$ and $\text{mod}(\mathbf{C}^{op})$ both have kernels, which is equivalent to both $\mathbf{C}$ and

$\mathbf{C}^{op}$ having pseudokernels. The fact that $\mathbf{C}$ also satisfies (b) and (c) if $\mathbf{C}$ is a dualizing R-variety follows from Proposition 2.3. The fact that $\mathbf{C}$ is a dualizing R-variety if it satisfies (a), (b), and (c) also follows from Proposition 2.3.

We now apply Theorem 2.4 to obtain some examples of dualizing R-varieties.

PROPOSITION 2.5. *Let Λ be an artin algebra, i.e., the center R of Λ is an artin ring and Λ is a finitely generated module over R. Let $\mathbf{p}(\Lambda)$ be the category of finitely generated projective Λ-modules. Then $\mathbf{p}(\Lambda)$ is a dualizing R-variety.*

Proof. The fact that Λ is an artin algebra shows that $\mathbf{p}(\Lambda)$ is a finite R-variety. Since Λ is an artin ring, we know that $\text{mod}(\Lambda)$ is an abelian category. Hence the equivalent category $\text{mod}(\mathbf{p}(\Lambda))$ is abelian and so $\mathbf{p}(\Lambda)$ has pseudokernels. Next, suppose B is in $\mathbf{p}(\Lambda)$. Then the object Λ in $\mathbf{p}(\Lambda)$ obviously has the property that $\text{Hom}_\Lambda(X, B) \to \text{Hom}_{\Lambda^{op}}(\text{Hom}_\Lambda(\Lambda, X), \text{Hom}_\Lambda(\Lambda, B))$ is an isomorphism for all X in $\mathbf{p}(\Lambda)$.

In order to finish the proof we observe that $\mathbf{p}(\Lambda)^{op}$ is equivalent to $\mathbf{p}(\Lambda^{op})$ by means of the duality $\mathbf{p}(\Lambda) \to \mathbf{p}(\Lambda^{op})$ given by $P \to \text{Hom}_\Lambda(P, \Lambda)$ for all P in $\mathbf{p}(\Lambda)$. Since Λ^{op} is also an artin algebra with center R, it follows that $\mathbf{p}(\Lambda^{op}) = \mathbf{p}(\Lambda)^{op}$ has the properties just derived for $\mathbf{p}(\Lambda)$. Therefore $\mathbf{p}(\Lambda)$ satisfies the conditions of Theorem 2.4 and so $\mathbf{p}(\Lambda)$ is a dualizing R-variety.

PROPOSITION 2.6. *Suppose $\mathbf{C}$ is a dualizing R-variety. Then* $\text{mod}(\mathbf{C})$ *is a dualizing R-variety.*

Proof. We have already seen in Section 1 that $\text{Mod}(\mathbf{C})$ and hence $\text{mod}(\mathbf{C})$ are R-categories. Hence $\text{mod}(\mathbf{C})$ is an R-variety. The fact that $\mathbf{C}$ is a finite R-variety obviously implies that $\text{mod}(\mathbf{C})$ is a finite R-variety. Also the fact that $\mathbf{C}$ is a dualizing R-variety implies that $\mathbf{C}$ has pseudokernels, or equivalently $\text{mod}(\mathbf{C})$ is abelian. Hence $\text{mod}(\mathbf{C})$ has pseudokernels since it has kernels.

Let M be a finitely presented $\mathbf{C}$-module. Since $\mathbf{C}$ is a dualizing R-variety, we know that $D(M)$ is a finitely presented $\mathbf{C}^{op}$-module. Hence by Proposition 2.2, there is a C in $\mathbf{C}$ such that $\text{Hom}_{\mathbf{C}}(N, M) \to \text{Hom}_{\mathbf{V}(C)}(N \mid \mathbf{V}(C), M \mid \mathbf{V}(C))$ is an isomorphism for all N in $\text{mod}(\mathbf{C})$.

From this it follows that the object $\mathbf{C}(\cdot, C)$ in $\text{mod}(\mathbf{C})$ has the property that

$$\text{Hom}_{\mathbf{C}}(N, M) \to \text{Hom}_{\text{End}_{\text{mod}(\mathbf{C})}(\mathbf{C}(\cdot,C))^{\text{op}}}(\text{mod}(\mathbf{C})(\mathbf{C}(\cdot, C), N), \text{mod}(\mathbf{C})(\mathbf{C}(\cdot, C), M))$$

is an isomorphism for all N in $\text{mod}(\mathbf{C})$. The fact that $\mathbf{C}$ is a dualizing R-variety implies that $\mathbf{C}^{\text{op}}$ is also a dualizing R-variety. Hence $\text{mod}(\mathbf{C}^{\text{op}})$ satisfies the types of conditions we just established for $\mathbf{C}$. But $\text{mod}(\mathbf{C})^{\text{op}}$ is equivalent to $\text{mod}(\mathbf{C}^{\text{op}})$ because $\mathbf{C}$ is a dualizing R-variety. This completes the proof that $\text{mod}(\mathbf{C})$ satisfies the conditions of Theorem 2.4 and is therefore a dualizing R-variety.

Later on we shall give other important examples of dualizing R-varieties.

3. Properties of Dualizing R-Varieties

Throughout this section we assume that R is a commutative ring and $\mathbf{C}$ is a dualizing R-variety. Our purpose now is to develop certain properties of dualizing R-varieties which are reminiscent of some of the properties of artin algebras. We begin by looking at the simple $\mathbf{C}$-modules.

Since the endomorphism ring of each object in $\mathbf{C}$ is an artin algebra, it follows that $\mathbf{C}$ is a Krull–Schmidt category, i.e., every nonzero object in $\mathbf{C}$ is a finite sum of endecomposable objects and the endomorphism ring of each indecomposable object is a local ring. It was shown in [4] that under these circumstances the $\mathbf{C}$-module $\mathbf{C}(\cdot, C)$ has a unique maximal submodule which we denote by $\text{r}\mathbf{C}(\cdot, C)$ for each indecomposable object C in $\mathbf{C}$ and that $\mathbf{C}(\cdot, C)$ is a projective cover for the simple $\mathbf{C}$-module $\mathbf{C}(\cdot, C)/\text{r}\mathbf{C}(\cdot, C)$. Also it was shown that given any simple $\mathbf{C}$-module S, there is an indecomposable object C in $\mathbf{C}$ such that S is isomorphic to $\mathbf{C}(\cdot, C)/\text{r}\mathbf{C}(\cdot, C)$. We now show that each simple $\mathbf{C}$-module is finitely presented. To this end we prove the following.

Proposition 3.1. *Let* $\mathbf{C}$ *be a dualizing R-variety and* $D\colon (\mathbf{C}^{\text{op}}, \text{mod}(R)) \to (\mathbf{C}, \text{mod}(R))$ *the usual duality functor. A* $\mathbf{C}$*-module* M *in* $(\mathbf{C}^{\text{op}}, \text{mod}(R))$ *is finitely presented if and only if* M *and* $D(M)$ *are finitely generated* $\mathbf{C}$*- and* $\mathbf{C}^{\text{op}}$*-modules, respectively.*

Proof. Since $\mathbf{C}$ is a dualizing R-variety, we know that if M is a finitely presented $\mathbf{C}$-module, then $D(M)$ is a finitely presented $\mathbf{C}^{\text{op}}$-module. Hence if M is a finitely presented $\mathbf{C}$-module, then M is a finitely generated $\mathbf{C}$-module and $D(M)$ is a finitely generated $\mathbf{C}^{\text{op}}$-module.

Suppose now that M is a finitely generated $\mathbf{C}$-module and $D(M)$ is a finitely generated $\mathbf{C}^{\text{op}}$-module. The fact that $D(M)$ is a finitely generated $\mathbf{C}^{\text{op}}$-module means that there is an exact sequence $\mathbf{C}(C, \cdot) \to D(M) \to 0$. Applying D to this exact sequence gives the exact sequence $0 \to D^2(M) \to D(\mathbf{C}(C, \cdot))$. Since $D^2(M) = M$ and $\mathbf{C}$ is a dualizing R-variety, we obtain an exact sequence $0 \to M \to D(\mathbf{C}(C, \cdot))$ with $D(\mathbf{C}(C, \cdot))$ a finitely presented $\mathbf{C}$-module. The fact that M is a finitely generated $\mathbf{C}$-module means that there is an epimorphism $\mathbf{C}(\cdot, X) \to M \to 0$ for some X in $\mathbf{C}$. Thus we have the morphism of finitely presented $\mathbf{C}$-modules $\mathbf{C}(\cdot, X) \xrightarrow{f} D(\mathbf{C}(C, \cdot))$ with $\operatorname{Im} f = M$. Since $\operatorname{mod}(\mathbf{C})$ is abelian we know that $\operatorname{Ker} f$ is also finitely presented. Hence we have the exact sequence $0 \to \operatorname{Ker} f \to \mathbf{C}(\cdot, X) \to M \to 0$ which shows that M is finitely presented since $\operatorname{Ker} f$ and $\mathbf{C}(\cdot, X)$ are both finitely presented.

As a consequence of this description of finitely presented $\mathbf{C}$-modules we have the following proposition.

PROPOSITION 3.2. *Let S be a simple $\mathbf{C}$-module.*

(a) *S is in* $(\mathbf{C}^{\text{op}}, \operatorname{mod}(R))$.

(b) *$D(S)$ is a simple $\mathbf{C}^{\text{op}}$-module.*

(c) *S is a finitely presented $\mathbf{C}$-module.*

Proof. (a) Since S is simple it is finitely generated and so is in $(\mathbf{C}^{\text{op}}, \operatorname{mod}(R))$.

(b) Follows easily from the fact that $D\colon (\mathbf{C}^{\text{op}}, \operatorname{mod}(R)) \to (\mathbf{C}, \operatorname{mod}(R))$ is a duality.

(c) Follows from Proposition 3.1 since S and $D(S)$ are finitely generated.

As an immediate consequence of this proposition we have the following.

COROLLARY 3.3. *Each finite $\mathbf{C}$-module (i.e., module of finite length) is finitely presented.*

We now want to describe the radical of a finitely presented $\mathbf{C}$-module M. We define the *radical* of M, which we denote by $\mathrm{r}M$, to be the intersection of the maximal submodules of M. Obviously if C is an indecomposable object in $\mathbf{C}$, then the radical of $\mathbf{C}(\cdot, C)$ is the unique maximal submodule $\mathrm{r}\mathbf{C}(\cdot, C)$ of $\mathbf{C}(\cdot, C)$. We now determine the radical of arbitrary finitely generated projective $\mathbf{C}$-modules.

PROPOSITION 3.3. *Let C be a nonzero object in $\mathbf{C}$ and let $C = \coprod_{i \in I} C_i$ be a representation of C as a finite sum of indecomposable objects in $\mathbf{C}$. Then the epimorphism $\mathbf{C}(\cdot, C) \to \coprod_{i \in I} \mathbf{C}(\cdot, C_i)/\mathrm{r}(\mathbf{C}(\cdot, C_i)$ has the following properties*:

(a) *If S is a semisimple $\mathbf{C}$-module, then*

$$\mathrm{Hom}_{\mathbf{C}}\left(\coprod_{i \in I} \mathbf{C}(\cdot, C_i)/\mathrm{r}\mathbf{C}(\cdot, C_i), S\right) \to \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}(\cdot, C), S)$$

is an isomorphism.

(b) $\mathrm{Ker}(\mathbf{C}(\cdot, C) \to \coprod_{i \in I} \mathbf{C}(\cdot, C_i)/\mathrm{r}\mathbf{C}(\cdot, C_i)) = \coprod_{i \in I} \mathrm{r}\mathbf{C}(\cdot, C_i)$ *is the radical of* $\mathbf{C}(\cdot, C)$.

(c) $\mathrm{r}\mathbf{C}(\cdot, C)$ *is a finitely presented* $\mathbf{C}$-*module.*

Proof. (a) If C is indecomposable, then the fact that $\mathrm{r}\mathbf{C}(\cdot, C)$ is the unique maximal submodule of $\mathrm{r}\mathbf{C}(\cdot, C)$ shows that a) is true in this case. The general situation now follows trivially.

(b) Since $\mathbf{C}(\cdot, C)/\coprod_{i \in I} \mathrm{r}\mathbf{C}(\cdot, C_i)$ is semisimple, it follows that $\coprod_{i \in I} \mathrm{r}\mathbf{C}(\cdot, C_i)$ contains the radical of $\mathbf{C}(\cdot, C)$. On the other hand part (a) shows that every maximal submodule of $\mathbf{C}(\cdot, C)$ contains $\coprod_{i \in I} \mathrm{r}\mathbf{C}(\cdot, C_i)$. Thus $\mathrm{r}\mathbf{C}(\cdot, C)$ contains $\coprod_{i \in I} \mathrm{r}\mathbf{C}(\cdot, C_i)$ and so they are equal.

(c) By (b) we know that

$$0 \to \mathrm{r}\mathbf{C}(\cdot, C) \to \mathbf{C}(\cdot, C) \to \coprod_{i \in I} \mathbf{C}(\cdot, C_i)/\mathrm{r}\mathbf{C}(\cdot, C_i) \to 0$$

is exact. Now $\coprod_{i \in I} \mathbf{C}(\cdot, C_i)/\mathrm{r}\mathbf{C}(\cdot, C_i)$ is finitely presented, since it is a finite sum of simple $\mathbf{C}$-modules each of which is finitely presented (see Proposition 3.2). Hence $\mathrm{r}\mathbf{C}(\cdot, C)$ is finitely presented.

In order to use this result to describe the radical of arbitrary finitely presented $\mathbf{C}$-modules we need the following result.

PROPOSITION 3.4. *Every finitely presented* $\mathbf{C}$-*module has a minimal projective presentation. In particular, every finitely presented C-module has a projective cover.*

Proof. It was shown in [4] that every finitely presented $\mathbf{C}$-module has a minimal projective presentation if $\mathrm{End}_{\mathbf{C}}(\mathbf{C})^{\mathrm{op}}$ is semiperfect for every C in $\mathbf{C}$. Since each $\mathrm{End}_{\mathbf{C}}(C)^{\mathrm{op}}$ is an artin ring it is certainly semiperfect.

We now describe the radical of arbitrary finitely presented $\mathbf{C}$-modules.

PROPOSITION 3.5. *Let* $\mathbf{C}(\cdot, C) \xrightarrow{f} M \to 0$ *be a projective cover for the finitely presented* $\mathbf{C}$*-module* M.

(a) $f^{-1}(\mathrm{r}M) = \mathrm{r}\mathbf{C}(\cdot, C)$ *and so* $f(\mathrm{r}\mathbf{C}(\cdot, C)) = \mathrm{r}M$.

(b) $\mathbf{C}(\cdot, C)/\mathrm{r}\mathbf{C}(\cdot, C)$ *is isomorphic to* $M/\mathrm{r}M$.

(c) $M/\mathrm{r}M$ *is a finitely generated semisimple module with the property that* $\mathrm{Hom}_{\mathbf{C}}(M/\mathrm{r}M, S) \to \mathrm{Hom}_{\mathbf{C}}(M, S)$ *is an isomorphism for each semisimple* $\mathbf{C}$*-module* S.

(d) *A submodule* M' *of* M *contains* $\mathrm{r}M$ *if and only if* M/M' *is semisimple.*

(e) $\mathrm{r}M$ *is finitely presented.*

(f) *If* $g: M \to N$ *is a morphism of finitely presented* $\mathbf{C}$*-modules, then* $g(\mathrm{r}M) \subset \mathrm{r}N$.

(g) *If* $g: M \to N$ *is an epimorphism, then* $g(\mathrm{r}M) = \mathrm{r}N$.

Proof. (a) Let $\{M_i\}_{j\in I}$ be the family of maximal submodules of M. Because f is an epimorphism, it is clear that $f^{-1}(M_j)$ is a maximal submodule of $\mathbf{C}(\cdot, C)$ for each j in J. On the other hand suppose L is a maximal submodule of $\mathbf{C}(\cdot, C)$. The fact that $f: \mathbf{C}(\cdot, C) \to M$ is a projective cover shows that $f(L) \neq M$ (for otherwise $L = \mathbf{C}(\cdot, C)$). Hence $\mathbf{C}(\cdot, C)/L \to M/f(L)$ is a nontrivial epimorphism and therefore an isomorphism since $\mathbf{C}(\cdot, C)/L$ is simple. Thus $f(L)$ is a maximal subobject of M with the property $f^{-1}(f(L)) = L$. Therefore the family $\{f^{-1}(M_i)\}_{i\in I}$ is the family of all maximal submodules of $\mathbf{C}(\cdot, C)$. Because $\mathrm{r}M = \bigcap_{i\in I} M_i$, it follows that

$$f^{-1}(\mathrm{r}M) = f^{-1}\Big(\bigcap_{i\in I} M_i\Big) = \bigcap_{i\in I} f^{-1}(M_i) = \mathrm{r}\mathbf{C}(\cdot, C).$$

Thus $f(\mathrm{r}\mathbf{C}(\cdot, C)) = \mathrm{r}M$ (remember f is an epimorphism).

(b) and (c). These follow readily from (a).

(d) and (e). These follow from (c) and (b) respectively.

(f) Suppose $g: M \to N$ is a morphism in mod($\mathbf{C}$). Then $M/g^{-1}(\mathrm{r}N) \to N/\mathrm{r}N$ is a monomorphism and so $M/g^{-1}(\mathrm{r}N)$ is semisimple. Therefore $g^{-1}(\mathrm{r}N)$ contains $\mathrm{r}M$ and so $g(\mathrm{r}M) \subset \mathrm{r}N$.

(g) Follows from (d) and (f).

Let s.s.mod($\mathbf{C}$) be the full subcategory of semisimple objects in mod($\mathbf{C}$). Then as a consequence of part (f) we see that $M \to M/\mathrm{r}M$ defines a functor mod($\mathbf{C}$) $\to$ s.s.mod($\mathbf{C}$). As a direct consequence of Proposition 3.5 we have the following.

PROPOSITION 3.6. *The functor* mod($\mathbf{C}$) $\to$ s.s.mod($\mathbf{C}$) *given by* $M \to M/\mathrm{r}M$ *has the following properties.*

(a) *The composition* s.s.mod($\mathbf{C}$) $\to^{\mathrm{inc}}$ mod($\mathbf{C}$) $\to$ s.s.mod($\mathbf{C}$) *is the identity.*

(b) mod($\mathbf{C}$) $\to$ s.s.mod($\mathbf{C}$) *is a left adjoint for the inclusion* s.s.mod($\mathbf{C}$) $\to$ mod($\mathbf{C}$) *and so is right exact.*

Another consequence of Proposition 3.5 is the following.

PROPOSITION 3.7. *Let M be in* mod($\mathbf{C}$).

(a) *$M \to M/\mathrm{r}M$ is an essential epimorphism.*

(b) *$M = 0$ if and only if $M/\mathrm{r}M = 0$.*

(c) *An epimorphism $f: P \to M$ is a projective cover if and only if* $\operatorname{Ker} f \subset \mathrm{r}P$.

Proof. (a) Let $\mathbf{C}(\cdot, C) \xrightarrow{f} M$ be a projective cover for M. Since $\mathbf{C}(\cdot, C) \to \mathbf{C}(\cdot, C)/\mathrm{r}\mathbf{C}(\cdot, C)$ is a projective and $M/\mathrm{r}M = \mathbf{C}(\cdot, C)/\mathrm{r}\mathbf{C}(\cdot, C)$ (see Proposition 3.5), we see that the composition $C(\cdot, C) \xrightarrow{f} M \xrightarrow{g} M/\mathrm{r}M$ is an essential epimorphism. Since f is also an essential epimorphism, it follows that g is an essential epimorphism.

(b) Follows from (a).

(c) We have already seen in Proposition 3.5 that if $f: P \to M$ is a projective cover, then $f^{-1}(\mathrm{r}M) = \mathrm{r}P$ which shows that $\operatorname{Ker} f \subset \mathrm{r}P$. Suppose $f: P \to M$ is an epimorphism with P projective and $\operatorname{Ker} f \subset \mathrm{r}P$. Then $M/\mathrm{r}M = P/\mathrm{r}P$ and so the composition $P \to^{f} M \to^{g} M/\mathrm{r}M$ is an essential epimorphism. Hence $P \xrightarrow{f} M$ is an essential epimorphism.

Suppose M is a finitely presented $\mathbf{C}$-module. Then we know that $\mathrm{r}M$ is a finitely presented $\mathbf{C}$-module and $M/\mathrm{r}M$ is a finitely presented semisimple **C**-module. Hence we can define $\mathrm{r}^i M$ for each nonnegative integer i as follows: $\mathrm{r}^0 M = M$, $\mathrm{r}^{i+1} M = \mathrm{r}(\mathrm{r}^i M)$. In this way we obtain a filtration called the *Loewy series* for M

$$M = \mathrm{r}^0 M \supset \mathrm{r}M \supset \mathrm{r}^2 M \supset \cdots$$

with the properties that each $\mathrm{r}^i M$ is a finitely presented **C**-module and $\mathrm{r}^{i+1} M = \mathrm{r}(\mathrm{r}^i M)$ for all i. Hence each $\mathrm{r}^i M/\mathrm{r}^{i+1} M$ is a finitely generated semisimple **C**-module. If $\mathrm{r}^i M \neq 0$ for all i, then we say that the *Loewy length* of M is infinite. Otherwise we define the *Loewy length* of M to be the smallest integer i such that $\mathrm{r}^i M = 0$. It is not difficult to see that the Loewy length of M is finite if and only if M is a finite **C**-module. We denote the Loewy length of M by $\mathrm{LL}(M)$.

We now describe some basic properties of the Loewy length of modules.

PROPOSITION 3.8. *Let M be a finitely presented* **C**-*module.*

(a) *Let $f\colon M \to N$ be a morphism in* mod(**C**).

(i) *If $f\colon M \to N$ is a monomorphism, then* $\mathrm{LL}(M) \leqslant \mathrm{LL}(N)$.

(ii) *If $f\colon M \to N$ is an epimorphism, then* $\mathrm{LL}(M) \geqslant \mathrm{LL}(N)$.

(b) *If $M = \coprod_{i\in I} M_i$ is a finite sum in* mod(**C**), *then* $\mathrm{LL}(M) = \max_{i\in I} \mathrm{LL}(M_i)$.

(c) *Let n be a nonnegative integer. Then $\mathrm{LL}(P) \leqslant n$ for all indecomposable finitely generated projective* **C**-*modules if and only if $\mathrm{LL}(M) \leqslant n$ for all finitely presented* **C**-*modules M.*

(d) *If there is a finite filtration of M in* mod(**C**) $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ *with the M_i/M_{i+1} semisimple for $i = 0,\ldots, n-1$, then* $\mathrm{LL}(M) \leqslant n$.

Proof. (a)(i) Follows from the fact that $f(\mathrm{r}^j M) \subset \mathrm{r}^j N$ for all $j \geqslant 1$ (see Proposition 3.5).

(ii) Follows from the fact that $f(\mathrm{r}^j M) = \mathrm{r}^j N$ for all $j \geqslant 1$ (see Proposition 3.5).

(b) Follows from the fact that $M \to \mathrm{r}^j M$ is a functor from mod(**C**) to mod(**C**), for all $j \geqslant 1$.

(c) Follows from (a)(ii) and (b).

(d) It follows from (a) and Proposition 3.5(d) that $\mathrm{r}^j M \subset M_j$ for all $j \leqslant n$.

We define the Loewy length of mod(**C**), denoted by LL(mod(**C**)) to be the suprenum of the Loewy length of all **C**-modules. By Proposition 3.8 LL(mod(**C**)) is equal to the suprenum of the Loewy lengths of the indecomposable projectives. If $\mathbf{C} = \mathbf{p}(\Lambda)$ for an artin algebra Λ, LL(mod(**C**)) is the usual Loewy length of the artin algebra Λ, and so it will often be denoted by LL(Λ).

We end this section by pointing out the dual concept to the Loewy length.

We recall that the *socle* of M denoted by soc M is the submodule of M generated by all the semisimple submodules of M. Using the fact that the duality D: mod(**C**) $\to$ mod($\mathbf{C}^{\text{op}}$) preserves simple objects, it follows that $\text{soc}(M) = D(D(M)/\text{r}D(M))$ and so $\text{soc}(M)$ is a finitely presented submodule of M. Hence we have that $M/\text{soc}(M)$ is finitely presented so that $\text{soc}(M/\text{soc}(M))$ is also finitely presented. We define $\text{soc}_i(M)$ by induction as follows: $\text{soc}_0(M) = 0$, $\text{soc}_{i+1}(M) =$ preimage of $\text{soc}(M/\text{soc}_i(M))$ under the canonical morphism $M \to M/\text{soc}_i(M)$. We then obtain a filtration

$$0 = \text{soc}_0(M) \subset \text{soc}_1(M) \subset \cdots \subset \text{soc}_{i+1}(M) \subset \cdots$$

of M having the property that each $\text{soc}_i(M)$ is a finitely presented submodule of M such that $\text{soc}_{i+1}(M)/\text{soc}_i(M)$ is the socle of $M/\text{soc}_i(M)$ and so each $\text{soc}_{i+1}(M)/\text{soc}_i(M)$ is a finitely generated semisimple **C**-module. This filtration is called the *socle series* for M. If $\text{soc}_i(M) \neq M$ for all i, we say that the *socle length* of M is infinite. Otherwise, we say that the socle length of M is the smallest integer i such that $\text{soc}_i(M) = M$. We denote the socle length of M by SL(M). Clearly SL(M) is finite if and only if M is a finite **C**-module.

The same type of duality argument as used in this discussion of the socle series also can be used to show that every object in mod(C) has an injective envelope because every object in mod($\mathbf{C}^{\text{op}}$) has a projective cover. In analogy with LL(mod(**C**)), we define the socle length SL(mod(**C**)) to be the suprenum of SL(M) for all M in mod(**C**).

We now point out the following connection between the Loewy length and the socle length.

PROPOSITION 3.9. *Let* **C** *be a dualizing R-variety.*

(a) *If M is a finitely presented* **C**-*module, then*
 (i) $\text{SL}(M) \geqslant \text{LL}(M)$.
 (ii) $\text{SL}(M) = \text{LL}(D(M))$.

(b) $\mathrm{LL}(\mathrm{mod}(\mathbf{C})) = \mathrm{LL}(\mathrm{mod}(\mathbf{C}^{\mathrm{op}}))$.

(c) $\mathrm{SL}(\mathrm{mod}(\mathbf{C})) = \mathrm{LL}(\mathrm{mod}(\mathbf{C}))$.

Proof. (a) Since the socle series for M has the property that $\mathrm{soc}_{i+1}(M)/\mathrm{soc}_i(M)$ is semisimple, (i) follows from Proposition 3.5(d).

(ii) Follows from the fact that $D(\mathrm{r}X) = \mathrm{soc}(D(X))$ for a finitely presented $\mathbf{C}$-module X.

(b) By (a) we have $\mathrm{LL}(\mathrm{mod}(\mathbf{C})) \leqslant \mathrm{SL}(\mathrm{mod}(\mathbf{C})) = \mathrm{LL}(\mathrm{mod}(\mathbf{C}^{\mathrm{op}}))$. Hence (b) follows by symmetry.

(c) Follows from (a) and (b).

4. Adjoints for $\underline{\mathrm{mod}}(\mathbf{D}) \to \mathrm{mod}(\mathbf{D})$

Let $\mathbf{C}$ be a fixed dualizing R-variety. Throughout this section we will denote the dualizing R-variety $\mathrm{mod}(\mathbf{C})$ by $\mathbf{D}$. As we saw in the last section $\mathbf{D}$ is a dualizing R-variety which is an abelian category in which every object has a projective cover and an injective envelope. Our main concern in the rest of this paper is the dualizing R-variety $\mathrm{mod}(\mathbf{D})$ of finitely presented $\mathbf{D}$-modules and a certain full subcategory $\underline{\mathrm{mod}}(\mathbf{D})$ of $\mathrm{mod}(\mathbf{D})$ which we now describe.

We denote by $\underline{\mathrm{mod}}(\mathbf{D})$ the full subcategory of $\mathrm{mod}(\mathbf{D})$ consisting of those $\mathbf{D}$-modules M such that $M(P) = 0$ for all projective objects P in $\mathbf{D}$. Clearly if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $\mathbf{D}$-modules, then M is in $\underline{\mathrm{mod}}(\mathbf{D})$ if and only if M' and M'' are in $\underline{\mathrm{mod}}(\mathbf{D})$. Thus $\underline{\mathrm{mod}}(\mathbf{D})$ is an abelian category and the inclusion $\underline{\mathrm{mod}}(\mathbf{D}) \to \mathrm{mod}(\mathbf{D})$ is exact. We now describe the objects in $\underline{\mathrm{mod}}(\mathbf{D})$ by means of their projective presentations in $\mathrm{mod}(\mathbf{D})$.

PROPOSITION 4.1. *The following statements are equivalent for a* $\mathbf{D}$-*module* M.

(a) *M is in* $\underline{\mathrm{mod}}(\mathbf{D})$.

(b) *If*

$$\mathbf{D}(\cdot, D_1) \xrightarrow{\mathbf{D}(\cdot, f)} \mathbf{D}(\cdot, D_0) \to M \to 0$$

is exact, then $f\colon D_1 \to D_0$ *is an epimorphism in* $\mathbf{D}$.

(c) *There is an exact sequence*

$$\mathbf{D}(\cdot, D_1) \xrightarrow{\mathbf{D}(\cdot,f)} \mathbf{D}(\cdot, D_0) \rightarrow M \rightarrow 0$$

with $f\colon D_1 \rightarrow D_0$ *an epimorphism in* D.

(d) $\mathrm{Hom}_{\mathbf{D}}(M, P) = 0$ *for all projective* **D**-*modules* P.

Proof. (a) implies (b). Suppose $M(P) = 0$ for all projective objects P in **D**. If

$$\mathbf{D}(\cdot, D_1) \xrightarrow{\mathbf{D}(\cdot,f)} \mathbf{D}(\cdot, D_0) \rightarrow M \rightarrow 0$$

is exact, then

$$\mathbf{D}(P, D_1) \xrightarrow{\mathbf{D}(P,f)} \mathbf{D}(P, D_0) \rightarrow 0$$

is exact for all projective objects P in **D**. Since **D** has enough projectives this implies that $f\colon D_1 \rightarrow D_0$ is an epimorphism.

(b) implies (c). Trivial since given any M in mod(**D**) there is an exact sequence

$$\mathbf{D}(\cdot, D_1) \xrightarrow{\mathbf{D}(\cdot,f)} \mathbf{D}(\cdot, D_0) \rightarrow M \rightarrow 0$$

in mod(**D**).

(c) implies (d). Since each projective **D**-module is isomorphic to $\mathbf{D}(\cdot, D)$ for some D in **D**, it suffices to show $\mathrm{Hom}_{\mathbf{D}}(M, \mathbf{D}(\cdot, D)) = 0$ for all D in **D**. Now let $f\colon D_1 \rightarrow D_0$ be an epimorphism such that

$$\mathbf{D}(\cdot, D_1) \xrightarrow{\mathbf{D}(\cdot,f)} \mathbf{D}(\cdot, D_0) \rightarrow M \rightarrow 0$$

is exact. Then we have the exact sequence

$$0 \rightarrow \mathrm{Hom}_{\mathbf{D}}(M, \mathbf{D}(\cdot, D)) \rightarrow \mathrm{Hom}_{\mathbf{D}}(\mathbf{D}(\cdot, D_0), \mathbf{D}(\cdot, D)) \rightarrow \mathrm{Hom}_{\mathbf{D}}(\mathbf{D}(\cdot, D_1), \mathbf{D}(\cdot, D)).$$

Hence $\mathrm{Hom}_{\mathbf{D}}(M, \mathbf{D}(\cdot, D)) = 0$ for all D in **D** if $0 \rightarrow \mathbf{D}(D_0, D) \rightarrow \mathbf{D}(D_1, D)$ is exact for all D in **D**. But this is the case since $f\colon D_1 \rightarrow D_0$ is an epimorphism.

(d) implies (a). Suppose

$$\mathbf{D}(\cdot, D_1) \xrightarrow{\mathbf{D}(\cdot,f)} \mathbf{D}(\cdot, D_0) \rightarrow M \rightarrow 0$$

is exact and $\mathrm{Hom}_{\mathbf{D}}(M, \mathbf{D}(\cdot, D)) = 0$ for all D in **D**. Then the argument used in establishing (c) implies (d) shows that $f\colon D_1 \rightarrow D_0$ is an epimorphism in **D**. Hence if X is a projective object in **D**, then

$\mathbf{D}(X, D_1) \to \mathbf{D}(X, D_0) \to 0$ is exact, which shows that $M(X) = 0$ for all projective objects X in $\mathbf{D}$.

In connection with Proposition 4.1, it is worthwhile pointing out the following.

PROPOSITION 4.2. (a) gl dim mod($\mathbf{D}$) $\leqslant 2$.

(b) *If M is in $\underline{\text{mod}}(\mathbf{D})$, then* $\text{Ext}_{\mathbf{D}}^1(M, \mathbf{D}(\cdot, D)) = 0$ *for all D in* $\mathbf{D}$.

Proof. (a) Let M be a $\mathbf{D}$-module. Let

$$\mathbf{D}(\cdot, D_1) \xrightarrow{\mathbf{D}(\cdot, f)} \mathbf{D}(\cdot, D_0) \to M \to 0$$

be exact. Since $\mathbf{D}$ is abelian, we have an exact sequence $0 \to D_2 \xrightarrow{g} D_1 \xrightarrow{f} D_0$ on $\mathbf{D}$. Hence

$$0 \to \mathbf{D}(\cdot, D_2) \xrightarrow{\mathbf{D}(\cdot, g)} \mathbf{D}(\cdot, D_1) \xrightarrow{\mathbf{D}(\cdot, f)} \mathbf{D}(\cdot, D_0) \to M \to 0$$

is exact and so $pdM \leqslant 2$. Since this is true for every M in mod($\mathbf{D}$), it follows that gl dim mod($\mathbf{D}$) $\leqslant 2$.

(b) Since M is in $\underline{\text{mod}}(\mathbf{D})$, we know by Proposition 4.1 that there is an exact sequence $0 \to D_2 \to D_1 \to D_0 \to 0$ of objects in $\mathbf{D}$ such that $0 \to \mathbf{D}(\cdot, D_2) \to \mathbf{D}(\cdot, D_1) \to \mathbf{D}(\cdot, D_0) \to M \to 0$ is a projective resolution of M. Thus the homology of the complex $0 \to \mathbf{D}(D_0, D) \to \mathbf{D}(D_1, D) \to \mathbf{D}(D_2, D) \to 0$ gives the $\text{Ext}_{\mathbf{D}}^i(M, \mathbf{D}(\cdot, D))$ for each D in $\mathbf{D}$. Hence $\text{Ext}_{\mathbf{D}}^i(M, \mathbf{D}(\cdot, D)) = 0$ for $i = 0, 1$ since the sequence $0 \to D_2 \to D_1 \to D_0 \to 0$ is exact in D.

The fact that the $\mathbf{D}$-modules M in $\underline{\text{mod}}(\mathbf{D})$ are characterized by the property $\text{Hom}_{\mathbf{D}}(M, P) = 0$ for all projective $\mathbf{D}$-modules P, shows that they are very similar to "torsion" modules. The extent of this similarity is shown in the following proposition which we state after giving some definitions.

We say that a $\mathbf{D}$-module M is a *torsion module* if M is in $\underline{\text{mod}}(\mathbf{D})$. We say that M is a *torsionless module* if it is a submodule of a projective $\mathbf{D}$-module.

PROPOSITION 4.3. *For each* $\mathbf{D}$*-module* M, *there is a unique (up to isomorphism) exact sequence* $0 \to M_0 \to M \to \mathbf{D}(\cdot, D_M) \to M_1 \to 0$ *of* $\mathbf{D}$*-modules with the* M_i *in* $\underline{\text{mod}}(\mathbf{D})$ *and* D_M *is an object in* $\mathbf{D}$. *This exact sequence has the following properties.*

(a) *The map* $M \to \mathbf{D}(\cdot, D_M)$ *is completely determined, up to isomorphism, by the property that for each* D *in* $\mathbf{D}$, *the induced map* $\mathrm{Hom}_{\mathbf{D}}(\mathbf{D}(\cdot, D_M), \mathbf{D}(\cdot, D)) \to \mathrm{Hom}_{\mathbf{D}}(M, \mathbf{D}(\cdot, D))$ *is an isomorphism.*

(b) *Given any morphism* $f: M \to N$ *of* $\mathbf{D}$*-modules there is a unique commutative diagram*

$$\begin{array}{ccccccccc} 0 \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & \mathbf{D}(\cdot, D_M) & \longrightarrow & M_1 & \longrightarrow 0 \\ & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f_1} & \\ 0 \longrightarrow & N_0 & \longrightarrow & N & \longrightarrow & D(\cdot, D_N) & \longrightarrow & N_1 & \longrightarrow 0. \end{array}$$

Thus the sequence is functorial in M.

(c) *If* X *is in* $\underline{\mathrm{mod}}(\mathbf{D})$, *then* $\mathrm{Hom}_{\mathbf{D}}(X, M_0) \to \mathrm{Hom}_{\mathbf{D}}(X, M)$ *is an isomorphism.*

(d) *Let* $0 \to M_0 \to M \to M' \to 0$ *be exact. Then* M' *is torsionless and given any torsionless* $\mathbf{D}$*-module* Y, *the map* $\mathrm{Hom}_{\mathbf{D}}(M', Y) \to \mathrm{Hom}_{\mathbf{D}}(M, Y)$ *is an isomorphism.*

Proof. The existence and uniqueness (up to isomorphism) of this exact sequence is shown in [1, Proposition 3.2].

(a) Let $0 \to M_0 \to M \to M' \to 0$ and $0 \to M' \to \mathbf{D}(\cdot, D_M) \to M_2 \to 0$ be exact. If D is in $\mathbf{D}$, then we have the exact sequence

$$0 \to \mathrm{Hom}_{\mathbf{D}}(M_2, \mathbf{D}(\cdot, D)) \to \mathrm{Hom}_{\mathbf{D}}(\mathbf{D}(\cdot, D_M), \mathbf{D}(\cdot, D))$$

$$\to \mathrm{Hom}_{\mathbf{D}}(M', \mathbf{D}(\cdot, D)) \to \mathrm{Ext}_{\mathbf{D}}^1(M_2, \mathbf{D}(\cdot, D)).$$

Since M_2 is in $\underline{\mathrm{mod}}(\mathbf{D})$, it follows that $\mathrm{Hom}_{\mathbf{D}}(M_2, \mathbf{D}(\cdot, D)) = 0 = \mathrm{Ext}_{\mathbf{D}}^1(M_2, \mathbf{D}(\cdot, D))$ (see Proposition 4.2). Hence $\mathrm{Hom}_{\mathbf{D}}(\mathbf{D}(\cdot, D_M), (\cdot, D)) \to \mathrm{Hom}_{\mathbf{D}}(M', \mathbf{D}(\cdot, D))$ is an isomorphism.

On the other hand, the exact sequence $0 \to M_0 \to M \to M' \to 0$ gives the exact sequence

$$0 \to \mathrm{Hom}_{\mathbf{D}}(M', \mathbf{D}(\cdot, D)) \to \mathrm{Hom}_{\mathbf{D}}(M, \mathbf{D}(\cdot, D)) \to \mathrm{Hom}_{\mathbf{D}}(M_0, \mathbf{D}(\cdot, D)).$$

Since M_0 is in $\underline{\mathrm{mod}}(\mathbf{D})$, we have that $\mathrm{Hom}_{\mathbf{D}}(M', \mathbf{D}(\cdot, D)) \to \mathrm{Hom}_{\mathbf{D}}(M, \mathbf{D}(\cdot, D))$ is an isomorphism. Hence $\mathrm{Hom}_{\mathbf{D}}(\mathbf{D}(\cdot, D_M), \mathbf{D}(\cdot, D)) \to \mathrm{Hom}_{\mathbf{D}}(M, \mathbf{D}(\cdot, D))$ is an isomorphism since it is the composition of the two isomorphisms $\text{Hom}_{\mathbf{D}}(\mathbf{D}(\cdot, D_M), \mathbf{D}(\cdot, D)) \to \text{Hom}_{\mathbf{D}}(M', \mathbf{D}(\cdot, D))$ and $\text{Hom}_{\mathbf{D}}(M', \mathbf{D}(\cdot, D)) \to \text{Hom}_{\mathbf{D}}(M, \mathbf{D}(\cdot, D))$.

(b) This is an immediate consequence of (a).

(c) and (d) follow easily from (b).

For each $\mathbf{D}$-module M we define $\text{Im}(M_0 \to M)$ to be the *torsion submodule* of M which we denote by $t(M)$. It follows from Proposition 4.3 that if $f\colon M \to N$ is a morphism of $\mathbf{D}$-modules, then $f(t(M)) \subset t(N)$. Hence we obtain a functor $t\colon \text{mod}(D) \to \underline{\text{mod}}(D)$ by sending M to $t(M)$ and $f\colon M \to N$ to $f \mid t(M)$. It also follows from Proposition 4.3, that $\text{Hom}_{\mathbf{D}}(X, t(M)) \to \text{Hom}_{\mathbf{D}}(X, M)$ is an isomorphism for all X in $\underline{\text{mod}}(\mathbf{D})$, which shows that $t\colon \text{mod}(\mathbf{D}) \to \underline{\text{mod}}(\mathbf{D})$ is a right adjoint to the inclusion $\underline{\text{mod}}(\mathbf{D}) \to \text{mod}(\mathbf{D})$. Thus t is left exact.

Another consequence of Proposition 4.3 is that $M/t(M)$ is a torsionless $\mathbf{D}$-module and the canonical epimorphism $M \to M/t(M) \to 0$ has the property that $\text{Hom}_{\mathbf{D}}(M/t(M), X) \to \text{Hom}_{\mathbf{D}}(M, X)$ is an isomorphism for all torsionless $\mathbf{D}$-modules X. Hence if we denote by $\mathbf{T}(\mathbf{D})$, the full subcategory of $\text{mod}(\mathbf{D})$ consisting of the torsionless $\mathbf{D}$-modules, we have the functor $T\colon \text{mod}(\mathbf{D}) \to \mathbf{T}(\mathbf{D})$ given by $T(M) = M/t(M)$. This is a left adjoint of the inclusion $\mathbf{T}(\mathbf{D}) \to \text{mod}(\mathbf{D})$. It is not difficult to check that T preserves epimorphisms and monomorphisms.

Having shown that the inclusion functor $\underline{\text{mod}}(\mathbf{D}) \to \text{mod}(\mathbf{D})$ has a right adjoint we end this section by showing that it has a left adjoint.

PROPOSITION 4.4. (a) *For each M in* $\text{mod}(\mathbf{D})$, *there is a uniquely determined finitely presented submodule M' of M satisfying*:

(i) *there is an epimorphism* $\mathbf{D}(\cdot, P) \to M'$ *with P a projective object in* $\mathbf{D}$;

(ii) *if* $\mathbf{D}(\cdot, P') \xrightarrow{f} M$ *is any morphism with P' a projective object in* $\mathbf{D}$, *then* $\text{Im} f$ *is contained in* M';

(b) *if* $f\colon M_1 \to M_2$ *is a morphism in* $\text{mod}(\mathbf{D})$, *then* $f(M_1{}') \subset M_2{}'$;

(c) *if for each M in* $\text{mod}(\mathbf{D})$, *we denote the finitely presented* $\mathbf{D}$*-module M/M' by $\underline{M}$, then*:

(i) *$\underline{M}$ is in* $\underline{\text{mod}}(\mathbf{D})$ *for all M in* $\text{mod}(\mathbf{D})$.

(ii) *$M = \underline{M}$ if and only if M is in* $\underline{\text{mod}}(\mathbf{D})$.

(iii) *The epimorphism $M \to \underline{M}$ has the property that the induced map* $\text{Hom}_{\mathbf{D}}(\underline{M}, N) \to \text{Hom}_{\mathbf{D}}(M, N)$ *is an isomorphism for all N in* $\underline{\text{mod}}(\mathbf{D})$.

(iv) *If* $M \xrightarrow{f} N$ *is a morphism in* $\mathbf{D}$, *there is a unique morphism* $\underline{f}\colon \underline{M} \to \underline{N}$ *such that the diagram*

$$\begin{array}{ccc} M & \longrightarrow & \underline{M} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \underline{f}} \\ N & \longrightarrow & \underline{N} \end{array}$$

commutes.

(v) *The functor* $\mathrm{mod}(\mathbf{D}) \to \underline{\mathrm{mod}}(\mathbf{D})$ *given by* $M \mapsto \underline{M}$ *and* $f\colon M \to N$ *goes to* $\underline{f}\colon \underline{M} \to \underline{N}$ *is a left adjoint to the inclusion* $\underline{\mathrm{mod}}(\mathbf{D}) \to \mathrm{mod}(\mathbf{D})$ *and is therefore right exact.*

Proof. (a) Let M be in $\mathrm{mod}(\mathbf{D})$. We know that there is an exact sequence $\mathbf{D}(\cdot, D) \to M \to 0$ with D in $\mathbf{D}$. Let $P \to D \to 0$ be exact with P a projective object in $\mathbf{D}$. We claim that the submodule M' of M which is the image of the composition $\mathbf{D}(\cdot, P) \to \mathbf{D}(\cdot, D) \to M$ has our desired properties.

Clearly M' is a finitely presented $\mathbf{D}$-module since it is a finitely generated submodule of the finitely presented $\mathbf{D}$-module M. It is also obvious from the definition that (i) is true.

(ii) Suppose $\mathbf{D}(\cdot, P') \xrightarrow{f} M$ is a morphism in $\mathrm{mod}(\mathbf{D})$ with P' a projective object in $\mathbf{D}$. Since $\mathbf{D}(\cdot, D) \to M \to 0$ is exact and $\mathbf{D}(\cdot, P')$ is a projective $\mathbf{D}$-module, there is a morphism $g\colon P' \to D$ such that f is the composition

$$\mathbf{D}(\cdot, P') \xrightarrow{\mathbf{D}(\cdot, g)} \mathbf{D}(\cdot, D) \to M.$$

Since $P \to D \to 0$ is exact and P' is projective, there is an $h\colon P' \to P$ such that g is the composition $P' \xrightarrow{h} P \to D$. From this it follows that f is the composition $\mathbf{D}(\cdot, P') \to \mathbf{D}(\cdot, P) \to \mathbf{D}(\cdot, D) \to M$ and so $\mathrm{Im}\, f$ is contained in M'. Clearly there is only one submodule M' of M satisfying (i) and (ii).

(b) This follows trivially from (i) and (ii). Thus the proof of part (a) of the proposition is complete.

(c) This part follows easily from part (a).

5. Minimal Projective Presentations in $\underline{\mathrm{mod}}(\mathbf{D})$

As in the previous section, $\mathbf{D} = \mathrm{mod}(\mathbf{C})$ where $\mathbf{C}$ is a dualizing R-variety. We now use the results of the previous section to investigate the projective objects in $\underline{\mathrm{mod}}(\mathbf{D})$. For ease of notation, we make the convention that for each D in $\mathbf{D}$, the functor $\underline{\mathrm{Hom}_{\mathbf{D}}(\cdot\,, D)}$ will be denoted by $\mathrm{Hom}_{\mathbf{D}}(\cdot, \underline{D})$. A useful preliminary result along these lines is the following.

PROPOSITION 5.1. (a) *Let* $P \xrightarrow{f} D \to 0$ *be an exact sequence of objects in* $\mathbf{D}$ *with* P *a projective object in* $\mathbf{D}$. *If* $g\colon \mathbf{D}(\cdot, D) \to M$ *is an epimorphism, then*

$$\mathbf{D}(\cdot, P) \xrightarrow{g\mathbf{D}(\cdot, f)} M \to \underline{M} \to 0$$

is exact.

(b) *Let* D *be in* $\mathbf{D}$.

(i) $\mathrm{Hom}_{\mathbf{D}}(\mathbf{D}(\cdot, \underline{D}), M) = M(D)$ *for all* M *in* $\underline{\mathrm{mod}}(\mathbf{D})$.

(ii) $\mathbf{D}(\cdot, \underline{D})$ *is projective in* $\underline{\mathrm{mod}}(\mathbf{D})$.

(c) *For each* M *in* $\underline{\mathrm{mod}}(\mathbf{D})$ *there is a projective presentation* $\mathbf{D}(\cdot, \underline{D}_1) \to \mathbf{D}(\cdot, \underline{D}_0) \to M \to 0$.

Proof. (a) This is just the description of the submodule M' of M given in Proposition 4.4.

(b)(i) By (a) if $P \to D \to 0$ is exact with P a projective object in $\mathbf{D}$, then $\mathbf{D}(\cdot, P) \to \mathbf{D}(\cdot, D) \to \mathbf{D}(\cdot, \underline{D}) \to 0$ is exact. Hence $(\mathbf{D}(\cdot, \underline{D}), M) \to (\mathbf{D}(\cdot, D), M)$ is an isomorphism for each $\mathbf{D}$-module M in $\underline{\mathrm{mod}}(\mathbf{D})$. This establishes (i).

(ii) Keeping in mind that a sequence $0 \to M' \to M \to M'' \to 0$ in $\underline{\mathrm{mod}}(\mathbf{D})$ is exact if and only if it is exact as a sequence in $\mathrm{mod}(D)$, it is trivial to show that (i) implies (ii).

(c) Suppose M is in $\underline{\mathrm{mod}}(\mathbf{D})$. Then there is a projective presentation $\mathbf{D}(\cdot, D_1) \to \mathbf{D}(\cdot, D_0) \to M \to 0$ in $\mathrm{mod}(\mathbf{D})$. Applying the right functor $X \mapsto \underline{X}$, we obtain the exact sequence $\mathbf{D}(\cdot, \underline{D}_1) \to \mathbf{D}(\cdot, \underline{D}_0) \to \underline{M} \to 0$ in $\underline{\mathrm{mod}}(\mathbf{D})$. Because M is in $\underline{\mathrm{mod}}(\mathbf{D})$, we know that $\underline{M} = M$ and so we have the desired exact sequence $\mathbf{D}(\cdot, \underline{D}_1) \to \mathbf{D}(\cdot, \underline{D}_0) \to M \to 0$.

This last result suggests that the projectives in $\mathrm{mod}(\mathbf{D})$ and $\underline{\mathrm{mod}}(\mathbf{D})$ should be intimately related. For instance, does the fact that projective covers exist in $\mathrm{mod}(\mathbf{D})$ imply the same for $\underline{\mathrm{mod}}(\mathbf{D})$? We now answer this question in the affirmative.

PROPOSITION 5.2. (a) *If* $M \to M'' \to 0$ *is an essential epimorphism in* $\mathrm{mod}(\mathbf{D})$, *then* $\underline{M} \to \underline{M}'' \to 0$ *is an essential epimorphism in* $\underline{\mathrm{mod}}(\mathbf{D})$.

(b) *If* $\mathbf{D}(\cdot, D) \to M \to 0$ *is a projective cover in* $\mathrm{mod}(\mathbf{D})$, *then* $\mathbf{D}(\cdot, \underline{D}) \to \underline{M} \to 0$ *is a projective cover in* $\underline{\mathrm{mod}}(\mathbf{D})$.

(c) *A* $\mathbf{D}$*-module* M *in* $\underline{\mathrm{mod}}(\mathbf{D})$ *is projective in* $\underline{\mathrm{mod}}(\mathbf{D})$ *if and only if* M *is isomorphic to* $\mathbf{D}(\cdot, \underline{D})$ *for some* D *in* $\mathbf{D}$.

(d) *Every object in* $\underline{\mathrm{mod}}(\mathbf{D})$ *has a projective cover.*

Proof. (a) Let $M \to M'' \to 0$ be an essential epimorphism in mod($\mathbf{D}$). We have already seen that given any X in $\underline{\mathrm{mod}}(\mathbf{D})$, there is an epimorphism $\mathbf{D}(\cdot, \underline{D}) \to X$ for some D in $\mathbf{D}$. Hence, to show that the epimorphism $\underline{M} \to \underline{M}$ is essential, it suffices to show that if a composition $\mathbf{D}(\cdot, \underline{D}) \to \underline{M} \to \underline{M}''$ is an epimorphism, then $\mathbf{D}(\cdot, \underline{D}) \to \underline{M}$ is an epimorphism.

Suppose $\mathbf{D}(\cdot, \underline{D}) \to \underline{M}$ is a morphism such that the composition $\mathbf{D}(\cdot, \underline{D}) \to \underline{M} \to \underline{M}''$ is an epimorphism. Since $\mathbf{D}(\cdot, D)$ is projective in mod($\mathbf{D}$) and $M \to \underline{M}$ is an epimorphism, there is a commutative diagram

$$\begin{array}{ccccc} \mathbf{D}(\cdot, D) & \longrightarrow & M & \longrightarrow & M'' \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{D}(\cdot, \underline{D}) & \longrightarrow & \underline{M} & \longrightarrow & \underline{M}'' \end{array}$$

with the composition of the bottom row an epimorphism. By Proposition 5.1, we know there is an exact sequence $\mathbf{D}(\cdot, P) \to M'' \to \underline{M}'' \to 0$ of $\mathbf{D}$-modules with P projective in $\mathbf{D}$. Since $\mathbf{D}(\cdot, P)$ is a projective in mod($\mathbf{D}$) and $M \to M'' \to 0$ is exact, it follows that the morphism $\mathbf{D}(\cdot, P) \to M''$ can be factored through $M \to M''$. Thus we get a morphism $\mathbf{D}(\cdot, D) \coprod \mathbf{D}(\cdot, P) \to M$ which is easily seen to have the property that the composition $\mathbf{D}(\cdot, D) \coprod \mathbf{D}(\cdot, P) \to M \to M''$ is an epimorphism. Since $M \to M''$ is an essential epimorphism, it follows that $\mathbf{D}(\cdot, D) \coprod \mathbf{D}(\cdot, P) \to M$ is an epimorphism. But $\mathbf{D}(\cdot, P) = 0$ since P is a projective object in D. Consequently $\mathbf{D}(\cdot, \underline{D}) \to \underline{M}$ is an epimorphism which finishes the proof of (a).

(b) Follows easily from (a) and the fact that $\mathbf{D}(\cdot, \underline{D})$ is projective in $\underline{\mathrm{mod}}(\mathbf{D})$ (see Proposition 5.1).

(c) Let M be a projective object in $\underline{\mathrm{mod}}(\mathbf{D})$ and let $\mathbf{D}(\cdot, D) \to M$ be its projective cover in mod($\mathbf{D}$). Then $\mathbf{D}(\cdot, \underline{D}) \to \underline{M}$ is a projective cover for $\underline{M}$ in $\underline{\mathrm{mod}}(\mathbf{D})$ by part (b). But $\underline{M} = M$ since M is in $\underline{\mathrm{mod}}(\mathbf{D})$. Therefore $\mathbf{D}(\cdot, \underline{D}) \to M$ is a projective cover in $\underline{\mathrm{mod}}(\mathbf{D})$ of the projective object M and so is an isomorphism.

(d) Easy consequence of previous results and the fact that every object in mod($\mathbf{D}$) has a projective cover.

Since the functor $\operatorname{mod}(\mathbf{D}) \to \underline{\operatorname{mod}}(\mathbf{D})$ given by $X \to \underline{X}$ is right exact, this last result suggests that if $\mathbf{D}(\cdot, D_1) \to \mathbf{D}(\cdot, D_0) \to M \to 0$ is a minimal projective presentation for M in $\operatorname{mod}(D)$, then $\mathbf{D}(\cdot, \underline{D}_1) \to \mathbf{D}(\cdot, \underline{D}_0) \to \underline{M} \to 0$ is a minimal projective presentation for $\underline{M}$ in $\underline{\operatorname{mod}}(\mathbf{D})$. While this is not true generally, it is true if M is in $\underline{\operatorname{mod}}(\mathbf{D})$ as we will now show. To do this, we need the following preliminary result.

Proposition 5.3. *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be an exact sequence in* $\operatorname{mod}(\mathbf{D})$. *If* M_3 *is in* $\underline{\operatorname{mod}}(\mathbf{D})$, *then the sequence* $0 \to \underline{M}_1 \to \underline{M}_2 \to \underline{M}_3 \to 0$ *is exact.*

Proof. Let M_1' be the kernel of the epimorphism $M_1 \to \underline{M}_1$. Then we have the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & M_1' & = & M_1' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \underline{M}_1 & \longrightarrow & M_2/M_1' & \longrightarrow & M_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $\underline{M}_1$ and M_3 are in $\underline{\operatorname{mod}}(\mathbf{D})$, it follows that M_2/M_1' is in $\underline{\operatorname{mod}}(\mathbf{D})$ because $\underline{\operatorname{mod}}(\mathbf{D})$ is closed under taking extensions.

Since there is an exact sequence $\mathbf{D}(\cdot, P) \to M_1 \to \underline{M}_1 \to 0$ with P a projective object in $\mathbf{D}$, there is an exact sequence $\mathbf{D}(\cdot, P) \to M_2 \to M_2/M_1' \to 0$. Therefore M_1' is contained in $M_2 = \operatorname{Ker}(M_2 \to \underline{M}_2)$. Also, because $\mathbf{D}(\cdot, \underline{P}) \to \underline{M}_2 \to \underline{M}_2/M_1' \to 0$ is exact and $\mathbf{D}(\cdot, \underline{P}) = 0$, it follows that $\underline{M}_2 \to \underline{M}_2/M_1'$ is an isomorphism. But $\underline{M}_2/M_1' = M_2/M_1'$ since M_2/M_1' is in $\underline{\operatorname{mod}}(\mathbf{D})$. Therefore $\underline{M}_2 \to M_2/M_1'$ is an isomorphism which shows that $M_2' = M_1'$ and hence the sequence $0 \to \underline{M}_1 \to \underline{M}_2 \to \underline{M}_3 \to 0$ is exact.

Corollary 5.4. *Let* $\mathbf{D}(\cdot, D_1) \to \mathbf{D}(\cdot, D_0) \to M \to 0$ *be a minimal projective presentation of* M *in* $\operatorname{mod}(\mathbf{D})$. *If* M *is in* $\underline{\operatorname{mod}}(\mathbf{D})$, *then*

$\mathbf{D}(\cdot, D_1) \to \mathbf{D}(\cdot, D_0) \to M \to 0$ *is a minimal projective presentation of* M *in* $\underline{\text{mod}}(\mathbf{D})$.

Proof. We already know by Proposition 5.2 that if we let $L = \text{Ker}(\mathbf{D}(\cdot, D_0) \to M)$, then $\mathbf{D}(\cdot, D_1) \to \underline{L}$ and $\mathbf{D}(\cdot, D_0) \to \underline{M}$ are projective covers in $\underline{\text{mod}}(\mathbf{D})$. Since M is in $\underline{\text{mod}}(\mathbf{D})$, we know that $\underline{M} = M$ and $0 \to \underline{L} \to \mathbf{D}(\cdot, \underline{D}_0) \to M \to 0$ is exact. Hence the exact sequence $\mathbf{D}(\cdot, \underline{D}_1) \to \mathbf{D}(\cdot, \underline{D}_0) \to M \to 0$ is a minimal projective presentation of M in $\underline{\text{mod}}(\mathbf{D})$.

6. The Category $\mathbf{D}/P$

As in the previous section $\mathbf{D} = \text{mod}(\mathbf{C})$ where $\mathbf{C}$ is a dualizing R-variety. We now use the description of the projective objects in $\underline{\text{mod}}(\mathbf{D})$ given in the previous section to show that $\underline{\text{mod}}(\mathbf{D})$ is the category of finitely presented modules over a dualizing R-variety.

We have already seen in the last section that $\mathbf{D}(\cdot, \underline{D})$ is a projective object in $\underline{\text{mod}}(\mathbf{D})$ for each D in $\mathbf{D}$. Since

$$\coprod_{i=1}^{n} \mathbf{D}(\cdot, \underline{D}_i) = \mathbf{D}\left(\cdot, \underline{\coprod_{i=1}^{n} D_i}\right)$$

for all finite families $D_1, \ldots, D_n$ of objects in $\mathbf{D}$, we see that the full subcategory of $\underline{\text{mod}}(\mathbf{D})$ consisting of the $\mathbf{D}$-modules $\mathbf{D}(\cdot, \underline{D})$ for all D in $\mathbf{D}$ is a skeletally small additive R-subcategory of $\underline{\text{mod}}(\mathbf{D})$. From the fact that if M is a projective object in $\underline{\text{mod}}(\mathbf{D})$, then M is isomorphic to $\mathbf{D}(\cdot, \underline{D})$ for some D in $\mathbf{D}$ (see Proposition 5.2), it follows that the full subcategory of $\underline{\text{mod}}(\mathbf{D})$ consisting of all $\mathbf{D}$-modules of the form $\mathbf{D}(\cdot, \underline{D})$ is an R-variety which is equivalent to $\mathbf{p}(\underline{\text{mod}}(\mathbf{D}))$, the full subcategory of all projective objects in $\underline{\text{mod}}(\mathbf{D})$. This observation enables us to give another description of $\mathbf{p}(\underline{\text{mod}}(\mathbf{D}))$.

Suppose D is in $\mathbf{D}$ and $P \to D \to 0$ is exact in $\mathbf{D}$ with P a projective object in $\mathbf{D}$. Then we know that $\mathbf{D}(\cdot, P) \to \mathbf{D}(\cdot, D) \to \mathbf{D}(\cdot, \underline{D}) \to 0$ is exact. Hence for each X in $\mathbf{D}$ we have that

$$\text{Hom}_{\mathbf{D}}(\mathbf{D}(\cdot, X), \mathbf{D}(\cdot, P)) \to \text{Hom}_{\mathbf{D}}(\mathbf{D}(\cdot, X), \mathbf{D}(\cdot, D))$$
$$\to \text{Hom}_{\mathbf{D}}(\mathbf{D}(\cdot, X), \mathbf{D}(\cdot, \underline{D})) \to 0$$

is exact. Since $\text{Hom}_{\mathbf{D}}(\mathbf{D}(\cdot, X), \text{Hom}_{\mathbf{D}}(\cdot, \underline{D})) = \text{Hom}_{\mathbf{D}}(\mathbf{D}(\cdot, \underline{X}), \text{Hom}_{\mathbf{D}}(\cdot, \underline{D}))$, it follows that $\mathbf{D}(X, P) \to \mathbf{D}(X, D) \to \text{Hom}_{\mathbf{D}}(\mathbf{D}(\cdot, \underline{X}), \text{Hom}_{\mathbf{D}}(\cdot, \underline{D})) \to 0$

is exact. Hence $\mathrm{Hom}_{\mathbf{D}}(\mathbf{D}(\cdot, X), \mathrm{Hom}_{\mathbf{D}}(\cdot, \underline{D}))$ is the same as the R-module $\mathbf{D}(X, D)$ modulo the R-submodule $P(X, D)$ consisting of all morphisms from X to D which factor through the epimorphism $P \to D \to 0$. It is not difficult to see that $P(X, D)$ actually consists of all morphisms from X to D which factor through any projective object in $\mathbf{D}$. Hence we see that the functor $F: \mathbf{D} \to \underline{\mathrm{mod}}(\mathbf{D})$ given by $F(D) = \mathbf{D}(\cdot, \underline{D})$ is a full functor with the property that $P(D_1, D_2) = \mathrm{Ker}(F: \mathbf{D}(D_1, D_2) \to \mathrm{Hom}_{\mathbf{D}}(F(D_1), F(D_2)))$. It follows from this observation that the collection $P(D_1, D_2)$ of R-submodules of $\mathbf{D}(D_1, D_2)$ defines a relation in $\mathbf{D}$ which we denote by P. The category $\mathbf{D}/P$ is called the category $\mathbf{D}$ *modulo projectives* or the *projective stabilization* of $\mathbf{D}$. From our previous discussion, it is obvious that the functor $\mathbf{D}/P \to \underline{\mathrm{mod}}(\mathbf{D})$ given by $D \mapsto \mathbf{D}(\cdot, \underline{D})$ is a fully faithful functor which induces an equivalence of categories $\mathbf{D}/P \to \mathbf{p}(\underline{\mathrm{mod}}(\mathbf{D}))$. Since $\mathbf{p}(\underline{\mathrm{mod}}(\mathbf{D}))$ is an R-variety, it follows that $\mathbf{D}/P$ is an R-variety. To simplify notation we make the convention that we will write $\underline{D}$ instead of D when we are viewing an object D in $\mathbf{D}$ as an object in $\mathbf{D}/P$. Also if $f: D_1 \to D_2$ is a morphism in $\mathbf{D}$, we denote its image in $\mathbf{D}(D_1, D_2)/P(D_1, D_2)$ by $\underline{f}: \underline{D}_1 \to \underline{D}_2$. In this notation, the canonical morphism $\mathbf{D} \to \mathbf{D}/P$ takes the form $D \mapsto \underline{D}$ for all D in $\mathbf{D}$ and $f \mapsto \underline{f}$ for all morphisms $f: D_1 \to D_2$ in $\mathbf{D}$. Finally it is clear that the diagram

$$\begin{array}{ccc} \mathbf{D} & \longrightarrow & \mathrm{mod}(D) \\ \downarrow & & \downarrow \\ \mathbf{D}/P & \longrightarrow & \underline{\mathrm{mod}}(\mathbf{D}) \end{array}$$

commutes where $\mathbf{D} \to \mathrm{mod}(\mathbf{D})$ is given by $D \mapsto \mathbf{D}(\cdot, D)$, $\mathbf{D}/P \mapsto \underline{\mathrm{mod}}(\mathbf{D})$ is given by $D \mapsto \mathbf{D}(\cdot, \underline{D})$ and the other functors are the usual ones.

Let M be a $\mathbf{D}^{\mathrm{op}}$-module. Then it is not difficult to see that $M(P) = 0$ for all projective objects in $\mathbf{D}$ if and only if for each pair of objects D_1 and D_2 in $\mathbf{D}$ and $f: D_1 \to D_2$ in $P(D_1, D_2)$ we have $M(f) = 0$. Thus by Proposition 1.1, we see that the functor $((\mathbf{D}/P), Ab) \to (\mathbf{D}, Ab)$, induced by $\mathbf{D} \to \mathbf{D}/P$, gives an isomorphism of $((\mathbf{D}/P), Ab)$ and the full subcategory of $(\mathbf{D}, Ab)$ consisting of those $\mathbf{D}^{\mathrm{op}}$-modules M such that $M(P) = 0$ for all projective objects P in $\mathbf{D}$. This isomorphism shows that the functor $\mathrm{mod}((\mathbf{D}/P)^{\mathrm{op}}) \to \underline{\mathrm{mod}}(\mathbf{D}^{\mathrm{op}})$ given by $M \mapsto N$, where $N(D) = M(\underline{D})$ for all D in $\mathbf{D}$ is an isomorphism of categories, which we often view as an identification. Combining this observation with the fact that $\mathbf{D}^{\mathrm{op}}$ is a dualizing R-variety since $\mathbf{C}$ is a dualizing R-variety, it is not difficult to establish the following proposition.

PROPOSITION 6.1. *The R-variety $\mathbf{D}/P$ has the following properties.*

(a) $(\mathbf{D}/P)^{\mathrm{op}}$ *has pseudokernels.*

(b) *Given any M in* $\mathrm{mod}(\mathbf{D}/P^{\mathrm{op}})$, *there is a $\underline{D}$ in $\mathbf{D}/P$ such that the morphism*

$$\mathrm{Hom}_{(\mathbf{D}/P)^{\mathrm{op}}}(N, M) \to \mathrm{Hom}_{\mathrm{End}_{(\mathbf{D}/P)^{\mathrm{op}}(\underline{D})^{\mathrm{op}}}}(N(\underline{D}), M(\underline{D}))$$

is an isomorphism for all N in $\mathrm{mod}(\mathbf{D}/P^{\mathrm{op}})$.

Proof. (a) As we have already seen, $\mathbf{D}/P$ has pseudokernels if and only if $\mathrm{mod}(\mathbf{D}/P^{\mathrm{op}})$ is abelian. But $\mathrm{mod}(\mathbf{D}/P)$ is abelian since it is isomorphic to the abelian category $\underline{\mathrm{mod}}(\mathbf{D})$.

(b) Using the identification of $\mathrm{mod}(\mathbf{D}/P^{\mathrm{op}})$ with $\underline{\mathrm{mod}}(\mathbf{D}^{\mathrm{op}})$ we can view our given $\mathbf{D}/P^{\mathrm{op}}$-module M as being in $\underline{\mathrm{mod}}(\mathbf{D}^{\mathrm{op}})$. Since $\mathbf{D}^{\mathrm{op}}$ is a dualizing R-variety, we know there is a D in $\mathbf{D}^{\mathrm{op}}$ such that

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{op}}}(N, M) \to \mathrm{Hom}_{\mathrm{End}_{\mathbf{D}^{\mathrm{op}}(D)}}(N(D), M(D))$$

is an isomorphism for all N in $\mathrm{mod}(\mathbf{D}^{\mathrm{op}})$. But if N is in $\underline{\mathrm{mod}}(\mathbf{D}^{\mathrm{op}})$, then $N(D) = N(\underline{D})$.

Similarly $M(D) = M(\underline{D})$. Hence $\underline{D}$ in $\mathbf{D}/P$ has the property that

$$\mathrm{Hom}_{(\mathbf{D}/P)^{\mathrm{op}}}(N, M) \to \mathrm{Hom}_{\mathrm{End}_{(\mathbf{D}/P)^{\mathrm{op}}(\underline{D})}}(N(\underline{D}), M(\underline{D}))$$

is an isomorphism for all N in $\mathrm{mod}((\mathbf{D}/P)^{\mathrm{op}})$.

In view of Theorem 2.4, to show that $\mathbf{D}/P$ is a dualizing R-variety, it suffices to show that $(\mathbf{D}/P)^{\mathrm{op}}$ has the same properties as those just established for $\mathbf{D}/P$ in Proposition 6.1. This will follow from the fact which we now establish that $(\mathbf{D}/P)^{\mathrm{op}}$ is equivalent to $\mathbf{D}^{\mathrm{op}}/P$. In order to show this we need to introduce a contravariant functor $\mathrm{Mod}(\mathbf{C}) \to \mathrm{Mod}(\mathbf{C}^{\mathrm{op}})$ which is a generalization of the functor $\mathrm{Mod}(\Lambda) \to \mathrm{Mod}(\Lambda^{\mathrm{op}})$ given by $M \mapsto \mathrm{Hom}_\Lambda(M, \Lambda)$ for all Λ-modules M.

For each $\mathbf{C}$-module M define $M^*\colon \mathbf{C} \to Ab$ by $M^*(C) = \mathrm{Hom}_{\mathbf{C}}(M, \mathbf{C}(\cdot, C))$. Clearly M^* is a $\mathbf{C}^{\mathrm{op}}$-module. In this way we obtain the contravariant functor $\mathrm{Mod}(\mathbf{C}) \to \mathrm{Mod}(\mathbf{C}^{\mathrm{op}})$ given by $M \mapsto M^*$. Also for each M in $\mathrm{Mod}(\mathbf{C})$ we can define the morphism $f\colon M \to M^{**}$ as follows. For each C in $\mathbf{C}$, we have to define $f_C\colon M(C) \to M^{**}(C)$. Now let x be in $M(C)$. We have to associate with x a morphism $f_C(x)\colon M^* \to \mathbf{C}(C, \cdot)$, i.e., for every X in $\mathbf{C}$ we have to define $(f_C(x))_X\colon M^*(X) \to \mathbf{C}(C, X)$. But $M^*(X) = \mathrm{Hom}_{\mathbf{C}}(M, \mathbf{C}(\cdot, X))$. Hence we obtain a morphism $(f_C(x))_X\colon M^*(X) \to \mathbf{C}(C, X))$ for each X in $\mathbf{C}$ by setting $(f_C(x))_X(g) = g(x)$

for each g in $\mathrm{Hom}_{\mathbf{C}}(M, \mathbf{C}(\cdot, X)) = M^*(X)$. It is not difficult to check that the morphisms $f_C: M(C) \to M^{**}(C)$ defined in this way give a morphism of $\mathbf{C}$-modules $f: M \to M^{**}$. Straightforward calculations also show that given any morphism $g: M \to N$ of $\mathbf{C}$-modules the diagram

$$\begin{array}{ccc} M & \longrightarrow & M^{**} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g^{**}} \\ N & \longrightarrow & N^{**} \end{array}$$

commutes.

Suppose C is in $\mathbf{C}$. Then $\mathbf{C}(\cdot, C)^*(X) = \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}(\cdot, C), \mathbf{C}(\cdot, X)) = \mathbf{C}(C, X))$ for all X in $\mathbf{C}$. Hence $\mathbf{C}(\cdot, C)^* = \mathbf{C}(C, \cdot)$. A similar calculation shows that $\mathbf{C}(C, \cdot)^* = \mathbf{C}(\cdot, C)$. From this it is easy to see that for each finitely generated projective $\mathbf{C}$-module P we have that P^* is a finitely generated projective $\mathbf{C}^{\mathrm{op}}$-module and $P \to P^{**}$ is an isomorphism.

Choose for each finitely presented $\mathbf{C}$-module M a fixed minimal projective presentation $\mathbf{C}(\cdot, C_1(M)) \to \mathbf{C}(\cdot, C_0(M)) \to M \to 0$. Then define the $\mathbf{C}^{\mathrm{op}}$-module $\mathrm{Tr}(M)$ to be $\mathrm{Coker}(\mathbf{C}(\cdot, C_0(M))^* \to \mathbf{C}(\cdot, C_1(M)^*)$ which we call the *transpose* of M. Obviously $\mathrm{Tr}(M)$ is a finitely presented $\mathbf{C}^{\mathrm{op}}$-module. We can obviously do the same thing for finitely presented $\mathbf{C}^{\mathrm{op}}$-modules. Because $\mathbf{C}(\cdot, C_1(M))^{**} \to \mathbf{C}(\cdot, C_0(M))^{**}$ is isomorphic to the morphism $\mathbf{C}(\cdot, C_1(M)) \to \mathbf{C}(\cdot, C_0(M))$ it follows that:

(a) $\mathbf{C}(\cdot, C_0(M))^* \to \mathbf{C}(\cdot, C_1(M))^* \to \mathrm{Tr}(M) \to 0$ is a minimal projective presentation of $\mathrm{Tr}(M)$;

(b) $\mathrm{Tr}(M) = 0$ if and only if M is projective;

(c) for each finitely presented $\mathbf{C}$-module M we have

(i) $\mathrm{Tr}(\mathrm{Tr}(M))$ has no nontrivial projective summands,

(ii) there is a uniquely determined (up to isomorphism) finitely generated projective $\mathbf{C}$-module P such that $\mathrm{Tr}(\mathrm{Tr}(M)) \coprod P$ is isomorphic to M;

(d) suppose M is a $\mathbf{C}$-module with no nonzero projective summands:

(i) $\mathrm{Tr}(\mathrm{Tr}(M))$ is isomorphic to M,

(ii) M is indecomposable if and only if $\mathrm{Tr}(M)$ is indecomposable;

(e) if M_1 and M_2 are $\mathbf{C}$-modules with no nonzero projective summands, then M_1 and M_2 are isomorphic $\mathbf{C}$-modules if and only if $\mathrm{Tr}(M_1)$ and $\mathrm{Tr}(M_2)$ are isomorphic $\mathbf{C}^{\mathrm{op}}$-modules.

Suppose we are given a morphism $f\colon M \to N$ in mod(**C**). Then we can find a commutative diagram

$$\begin{array}{ccccccc} \mathbf{C}(\cdot, C_1(M)) & \longrightarrow & \mathbf{C}(\cdot, C_0(M)) & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\ \mathbf{C}(\cdot, C_1(N)) & \longrightarrow & \mathbf{C}(\cdot, C_0(N)) & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

which gives rise to a unique morphism $\mathrm{Tr}_{f_0,f_1}(f)\colon \mathrm{Tr}(N) \to \mathrm{Tr}(M)$ in mod($\mathbf{C}^{\mathrm{op}}$) such that the diagram

$$\begin{array}{ccccccc} \mathbf{C}(\cdot, C_0(N))^* & \longrightarrow & \mathbf{C}(\cdot, C_1(N))^* & \longrightarrow & \mathrm{Tr}(N) & \longrightarrow & 0 \\ \downarrow{\scriptstyle f_0^*} & & \downarrow{\scriptstyle f_1^*} & & \downarrow{\scriptstyle \mathrm{Tr}_{f_0,f_1}(f)} & & \\ \mathbf{C}(\cdot, C_0(M))^* & \longrightarrow & \mathbf{C}(\cdot, C_1(M))^* & \longrightarrow & \mathrm{Tr}(M) & \longrightarrow & 0 \end{array}$$

is commutative. Although the morphism $\mathrm{Tr}_{f_0,f_1}(f)$ depends on the particular choice of f_0 and f_1, it is not difficult to see that if we choose a different commutative diagram

$$\begin{array}{ccccccc} \mathbf{C}(\cdot, C_1(M)) & \longrightarrow & \mathbf{C}(\cdot, C_0(M)) & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle f_1'} & & \downarrow{\scriptstyle f_0'} & & \downarrow{\scriptstyle f} & & \\ \mathbf{C}(\cdot, C_1(N)) & \longrightarrow & \mathbf{C}(\cdot, C_0(N)) & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

then $\mathrm{Tr}_{f_0,f_1}(f) - \mathrm{Tr}_{f_0',f_1'}(f)\colon \mathrm{Tr}(N) \to \mathrm{Tr}(M)$ factors through a projective $\mathbf{C}^{\mathrm{op}}$-module. In this way we obtain the contravariant functor $\mathrm{mod}(\mathbf{C}) \to \mathrm{mod}(\mathbf{C}^{\mathrm{op}})/P$ given by $M \mapsto \mathrm{Tr}(M)$ and $f \mapsto \underline{\mathrm{Tr}(f)}$. Since this functor takes projectives in mod(**C**) to zero in $\mathrm{mod}(\mathbf{C}^{\mathrm{op}})/P$, it follows that there is a unique contravariant functor, which we call the *transpose* and denote by $\mathrm{Tr}\colon \mathrm{mod}(\mathbf{C})/P \to \mathrm{mod}(\mathbf{C}^{\mathrm{op}})/P$, such that the composition $\mathrm{mod}(\mathbf{C}) \to \mathrm{mod}(\mathbf{C})/P \xrightarrow{\mathrm{Tr}} \mathrm{mod}(\mathbf{C}^{\mathrm{op}})/P$ is the contravariant functor $\mathrm{mod}(\mathbf{C}) \to \mathrm{mod}(\mathbf{C}^{\mathrm{op}})/P$. Obviously we also have the contravariant functor $\mathrm{Tr}\colon \mathrm{mod}(\mathbf{C}^{\mathrm{op}})/P \to \mathrm{mod}(\mathbf{C})/P$ defined in a similar manner. It is not difficult to see that the transpose functors are dualities which are inverses of each other. This substantiates our earlier claim that $(\mathrm{mod}(\mathbf{C})/P)^{\mathrm{op}}$ and $\mathrm{mod}(\mathbf{C}^{\mathrm{op}})/P$ are equivalent categories. Hence $(\mathrm{mod}(\mathbf{C})/P)^{\mathrm{op}}$ also has the properties proved in Proposition 6.1 for $\mathrm{mod}(\mathbf{C})/P$. Therefore, applying Theorem 2.4 we obtain the following proposition.

PROPOSITION 6.2. *Let* $\mathbf{C}$ *be a dualizing R-variety.*

(a) $\text{mod}(\mathbf{C})/P$ *is a dualizing R-variety.*

(b) $\text{Tr}: \text{mod}(\mathbf{C})/P \to \text{mod}(\mathbf{C}^{\text{op}})/P$ *is a duality.*

(c) *The functor* $\underline{D}: \text{mod}(\text{mod}(\mathbf{C})/P) \to \text{mod}(\text{mod}(\mathbf{C}^{\text{op}})/P)$ *given by* $\underline{D}(M)(X) = D(M(\text{Tr}(X))$ *for all* X *in* $\text{mod}(\mathbf{C}^{\text{op}})/P$ *is a duality.*

(d) *The functor* $\underline{D}: \underline{\text{mod}}(\text{mod}(\mathbf{C})) \to \underline{\text{mod}}(\text{mod}(\mathbf{C}^{\text{op}}))$ *given by* $\underline{D}(M)(X) = D(M(\text{Tr}(X)))$ *for all* X *in* $\text{mod}(\mathbf{C}^{\text{op}})$ *is a duality.*

Proof. (a) and (b). These were proven in the discussion preceding the statement of the proposition.

(c) Since $\text{mod}(\mathbf{C})/P$ is a dualizing R-variety, the functor

$$\underline{D}: \text{mod}(\text{mod}(\mathbf{C})/P) \to \text{mod}((\text{mod}(\mathbf{C})/P)^{\text{op}}),$$

given by $\underline{D}(M)(X) = D(M(X))$ for all X in $\text{mod}(\mathbf{C})/P$, is a duality. Combining this with the fact that the functor $\text{Tr}: \text{mod}(\mathbf{C}^{\text{op}})/P \to (\text{mod}(\mathbf{C})/P)$ is a duality gives the desired result.

(d) Follows from (c) by means of the identifications of $\text{mod}(\text{mod}(\mathbf{C})/P)$ with $\underline{\text{mod}}(\text{mod}(\mathbf{C}))$ and of $\text{mod}(\text{mod}(\mathbf{C}^{\text{op}})/P)$ with $\underline{\text{mod}}(\text{mod}(\mathbf{C}^{\text{op}}))$.

7. THE DUALITY $\underline{D}$

We now use the results of the previous sections to develop a basic computational tool that will be used throughout the rest of this work. As usual $\mathbf{C}$ is a fixed dualizing R-variety. We begin by investigating the duality $\underline{D}: \underline{\text{mod}}(\text{mod}(\mathbf{C})) \to \underline{\text{mod}}(\text{mod}(\mathbf{C}^{\text{op}}))$ defined in the previous section which is given by $\underline{D}(M)(X) = D(M(\text{Tr}(X)))$ for all X in $\text{mod}(\mathbf{C}^{\text{op}})$. Our immediate aim is to show that for each $\mathbf{C}$-module A we have $\underline{D}((\cdot, A)) = \text{Ext}^1_{\mathbf{C}^{\text{op}}}(\cdot, D(A))$, where $D(A)$ is the $\mathbf{C}^{\text{op}}$-module given by the duality $D: \text{mod}(\mathbf{C}) \to \text{mod}(\mathbf{C}^{\text{op}})$. To prove this we need some preliminaries about tensor products of $\mathbf{C}$-modules.

We recall that for each variety $\mathbf{C}$, there is a unique functor $\otimes: \text{Mod}(\mathbf{C}^{\text{op}}) \times \text{Mod}(\mathbf{C}) \to Ab$, called the tensor product having the following properties:

(a) For each M in $\text{Mod}(\mathbf{C}^{\text{op}})$, the functor $M\otimes: \text{Mod}(\mathbf{C}) \to Ab$ given by $(M\otimes)(N) = M \otimes N$ for all N in $\text{Mod}(\mathbf{C})$ is a right exact functor which commutes with sums and has the property $M \otimes (\cdot, C) = M(C)$ for all C in $\mathbf{C}$.

(b) For each N in $\mathrm{Mod}(\mathbf{C})$, the functor $\otimes N\colon \mathrm{Mod}(\mathbf{C}^{\mathrm{op}}) \to Ab$ given by $(\otimes N)(M) = M \otimes N$ for all M in $\mathrm{Mod}(\mathbf{C}^{\mathrm{op}})$ is a right exact functor which commutes with sums and has the property that $(C, \cdot) \otimes N = N(C)$ for all C in $\mathbf{C}$.

Let M be an object in $\mathrm{mod}(\mathbf{C})$. Associated with M is the functor $M^*\otimes\colon \mathrm{mod}(\mathbf{C}) \to Ab$ given by $X \mapsto M^* \otimes X$ for all X in $\mathrm{mod}(\mathbf{C})$ and the functor $\mathrm{Hom}_{\mathbf{C}}(M, \cdot)\colon \mathrm{mod}(\mathbf{C}) \to Ab$ given by $X \mapsto \mathrm{Hom}_{\mathbf{C}}(M, X)$ for all X in $\mathrm{mod}(\mathbf{C})$. On the basis of the properties of the tensor product listed above it is not difficult to see that there is a unique (up to isomorphism) morphism $M^*\otimes \to \mathrm{Hom}_{\mathbf{C}}(M, \cdot)$ such that $M^* \otimes (\cdot, C) \to \mathrm{Hom}_{\mathbf{C}}(M, (\cdot, C))$ is the identity for all C in $\mathbf{C}$ (remember $M^* \otimes (\cdot, C) = M^*(C) = \mathrm{Hom}_{\mathbf{C}}(M, (\cdot, C))$.

LEMMA 7.1. *Let M be in* $\mathrm{mod}(\mathbf{C})$. *Then*

$$\mathrm{Hom}_{\mathbf{C}}(\underline{M}, \underline{X}) = \mathrm{Coker}(M^* \otimes X \to \mathrm{Hom}_{\mathbf{C}}(M, X))$$

for all X in $\mathrm{mod}(\mathbf{C})$.

Proof. Because X is in $\mathrm{mod}(\mathbf{C})$, we know there is an exact sequence $(\cdot, C) \to X \to 0$ for some C in $\mathbf{C}$. From this it follows that we have the exact commutative diagram

$$\begin{array}{ccccc}
M^* \otimes (\cdot, C) & \longrightarrow & M^* \otimes X & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\mathrm{Hom}_{\mathbf{C}}(M, X) & = & \mathrm{Hom}_{\mathbf{C}}(M, X) & & \\
\downarrow & & & & \\
\mathrm{Hom}_{\mathbf{C}}(\underline{M}, \underline{X}) & & & & \\
\downarrow & & & & \\
0 & & & &
\end{array}$$

which gives our desired result.

PROPOSITION 7.2. *Let M be in* $\mathrm{mod}(\mathbf{C})$. *Then the functors* $\mathrm{Tor}_1(\mathrm{Tr}(M), \cdot)\colon \mathrm{mod}(\mathbf{C}) \to Ab$ *and* $\mathrm{Hom}_{\mathbf{C}}(\underline{M}, \cdot)\colon \mathrm{mod}(\mathbf{C}) \to Ab$ *are isomorphic where*

$$\mathrm{Tor}_1(\mathrm{Tr}(M), \cdot)(X) = \mathrm{Tor}_1(\mathrm{Tr}(M), X)$$

and $\mathrm{Hom}_{\mathbf{C}}(\underline{M}, \cdot)(X) = \mathrm{Hom}_{\mathbf{C}}(\underline{M}, \underline{X})$ *for all X in* $\mathrm{mod}\,\mathbf{C}$.

Proof. Let $\mathbf{C}(\cdot, C_1) \to \mathbf{C}(\cdot, C_0) \to M \to 0$ be a minimal projective presentation of M. Then we obtain the exact sequence of finitely presented $\mathbf{C}^{\mathrm{op}}$-modules

$$0 \to M^* \to \mathbf{C}(\cdot, C_0)^* \to \mathbf{C}(\cdot, C_1)^* \to \mathrm{Tr}(M) \to 0.$$

Now $\mathbf{C}(\cdot, C_i)^* = \mathbf{C}(C_i, \cdot)$ for $i = 0, 1$ and so tensoring with a finitely presented $\mathbf{C}$-module X, we obtain the sequence

$$M^* \otimes X \xrightarrow{f} \mathbf{C}(C_0, \cdot) \otimes X \xrightarrow{g} \mathbf{C}(C_1, \cdot) \otimes X.$$

Clearly $\mathrm{Tor}_1(\mathrm{Tr}(M), X) = \mathrm{Ker}\, g/\mathrm{Im}\, f$. Since $C(C_i, \cdot) \otimes X = X(C_i)$ for $i = 0, 1$, we have that

$$\begin{aligned} \mathrm{Ker}(\mathbf{C}(C_0, \cdot) \otimes X \to \mathbf{C}(C_1, \cdot) \otimes X) &= \mathrm{Ker}(X(C_0) \to X(C_1)) \\ = \mathrm{Ker}(\mathrm{Hom}_{\mathbf{C}}(\mathbf{C}(\cdot, C_0), X) &\to \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}(\cdot, C_1), X)) = \mathrm{Hom}_{\mathbf{C}}(M, X). \end{aligned}$$

Hence we have the exact sequence $M^* \otimes X \to \mathrm{Hom}_{\mathbf{C}}(M, X) \to \mathrm{Tor}_1(\mathrm{Tr}(M), X) \to 0$ which is easily seen to be functorial in X. Applying Lemma 7.1, we have that the functors $\mathrm{Hom}_{\mathbf{C}}(\underline{M}, \cdot)$ and $\mathrm{Tor}_1(\mathrm{Tr}(M), \cdot)$ are isomorphic.

In order to proceed further we need the fact that since $\mathbf{C}$ is an R-variety, there is a unique R-module structure on the abelian group $M \otimes N$ for each M in $\mathrm{Mod}(\mathbf{C}^{\mathrm{op}})$ and each N in $\mathrm{Mod}(\mathbf{C}^{\mathrm{op}})$ satisfying:

(a) for each morphism of $\mathbf{C}$-modules $N \xrightarrow{f} N'$, the morphism $M \otimes f\colon M \otimes N \to M \otimes N'$ is a morphism of R-modules;

(b) Similar condition as (a) on the other side;

(c) $M \otimes \mathbf{C}(\cdot, C) = M(C)$, $\mathbf{C}(C, \cdot) \otimes N = N(C)$ where $M(C)$ and $N(C)$ have their usual R-module structure.

Moreover, for each R-module L, there is an isomorphism $\mathrm{Hom}_R(M \otimes N, L) \to \mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(M, \mathrm{Hom}_R(N, L))$ which is functorial in M, N and L where $\mathrm{Hom}_R(N, L)$ is the $\mathbf{C}^{\mathrm{op}}$-module $\mathrm{Hom}_R(N, L)(X) = \mathrm{Hom}_R(N(X), L)$. We now apply these remarks to prove the following.

PROPOSITION 7.3. *Let N be in* $\mathrm{mod}(\mathbf{C})$. *Then the functors* $D(\mathrm{Tor}_i(\cdot, N))$ *and* $\mathrm{Ext}^i_{\mathbf{C}^{\mathrm{op}}}(\cdot, D(N))$ *from* $\mathrm{mod}(\mathbf{C}^{\mathrm{op}})$ *to Ab are isomorphic.*

Proof. Let $\cdots \to P_1 \to P_0 \to M \to 0$ be a projective resolution in $\mathrm{mod}(\mathbf{C}^{\mathrm{op}})$ for the $\mathbf{C}^{\mathrm{op}}$-module M. Then letting P be this projective resolution we have by our previous remarks that

$$\mathrm{Hom}_R(P \otimes N, E) = \mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(P, \mathrm{Hom}_R(N, E))$$

where E is an injective envelop for the R-module $R/\mathfrak{r}$. Taking homology of both sides and using the fact that E is injective, we have that

$$\mathrm{Hom}_R(H_i(P \otimes N), E) = H_i(\mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(P, \mathrm{Hom}_R(N, E)).$$

This gives the desired result that $D(\mathrm{Tor}_i(\cdot, N)) = \mathrm{Ext}^i_{\mathbf{C}^{\mathrm{op}}}(\cdot, D(N))$ for all i.

We are now in a position to establish the result cited in the beginning of this section.

PROPOSITION 7.4. *The duality functor*

$$\underline{D}\colon \underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C})) \to \underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}^{\mathrm{op}}))$$

has the property that $\underline{D}(\mathrm{Hom}_{\mathbf{C}}(\cdot, M)) = \mathrm{Ext}^1_{\mathbf{C}^{\mathrm{op}}}(\cdot, D(M))$ *for all* M *in* mod(**C**). *Hence an object* L *in* $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}^{\mathrm{op}}))$ *is injective if and only if* $L \approx \mathrm{Ext}^1_{\mathbf{C}^{\mathrm{op}}}(\cdot, X)$ *for some* X *in* mod($\mathbf{C}^{\mathrm{op}}$).

Proof. For each X in mod($\mathbf{C}^{\mathrm{op}}$) we know that $\underline{D}(\mathrm{Hom}_{\mathbf{C}}(\cdot, M))(X) = D(\mathrm{Hom}_{\mathbf{C}}(\mathrm{Tr}(\underline{X}), \underline{M}))$. But by Proposition 7.2 $\mathrm{Hom}_{\mathbf{C}}(\mathrm{Tr}(\underline{X}), \underline{M}) = \mathrm{Tor}_1(\mathrm{Tr}\mathrm{Tr}(\underline{X}), \underline{M})$. Since $\mathrm{Tr}\mathrm{Tr}(\underline{X}) = \underline{X}$, we have $\mathrm{Tor}_1(\mathrm{Tr}\mathrm{Tr}(\underline{X}), \underline{M}) = \mathrm{Tor}_1(\underline{X}, \underline{M}) = \mathrm{Tor}_1(X, M)$. Thus $\underline{D}(\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{M})) = D(\mathrm{Tor}_1(\cdot, M))$. But $D(\mathrm{Tor}_1(\cdot, M)) = \mathrm{Ext}^1_{\mathbf{C}^{\mathrm{op}}}(\cdot, C(M))$ by Proposition 7.3. Thus $D(\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{M}) = \mathrm{Ext}^1_{\mathbf{C}^{\mathrm{op}}}(\cdot, D(M)$ for all M in mod(**C**).

The second half of the proposition follows from what we have just proved since we know that an object in $\underline{\mathrm{mod}}$(mod(**C**)) is projective if and only if it is isomorphic to $\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{M})$ for some M in mod(**C**) and the fact that $\underline{D}\colon \underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C})) \to \underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}^{\mathrm{op}})$ is a duality.

Suppose M is in $\underline{\mathrm{mod}}$(mod(**C**)). Then we know by Proposition 4.1 that there is an epimorphism $A_1 \to A_0$ in mod(**C**) such that the sequence $\mathrm{Hom}_{\mathbf{C}}(\cdot, A) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_0) \to M \to 0$ is a minimal projective presentation for M in mod(mod(**C**)). We showed in Corollary 5.4 that the exact sequence $\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A}_1) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A}_0) \to M \to 0$ is a minimal projective presentation of M in mod′(mod(**C**)). Since mod(**C**) is abelian we have an exact sequence $0 \to A_2 \to A_1 \to A_0 \to 0$ which gives rise to the exact sequence

$$\mathrm{Hom}_{\mathbf{C}}(\cdot, A_1) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_0) \to \mathrm{Ext}_{\mathbf{C}}^1(\cdot, A_2) \to \mathrm{Ext}_{\mathbf{C}}^1(\cdot, A_1)$$

and hence to the exact sequence

$$0 \to M \to \mathrm{Ext}_{\mathbf{C}}^1(\cdot, A_2) \to \mathrm{Ext}_{\mathbf{C}}^1(\cdot, A_1)$$

which, by Proposition 7.4, we know is an injective copresentation of M. Our aim is to show that this is a minimal injective copresentation of M.

We begin by defining an exact sequence $0 \to A_2 \to A_1 \to A_0 \to 0$ in mod(**C**) to be a *minimal* exact sequence if whenever the exact sequence can be written as a sum of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_2' & \longrightarrow & A_1' & \longrightarrow & A_0' & \longrightarrow & 0 \\ & & \amalg & & \amalg & & \amalg & & \\ 0 & \longrightarrow & A_2'' & \longrightarrow & A_1'' & \longrightarrow & A_0'' & \longrightarrow & 0 \end{array}$$

with $0 \to A_2'' \to A_1'' \to A_0'' \to 0$ a split exact sequence, then A_1'' is zero and consequently so are A_2'' and A_0''. We saw in [4] that an exact sequence $0 \to A_2 \to A_1 \to A_0 \to 0$ in mod(**C**) is minimal if and only if $\mathrm{Hom}_{\mathbf{C}}(\cdot, A_1) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_0) \to M \to 0$ is a minimal projective presentation for M in mod(mod(**C**)). From this it follows easily that an exact sequence $0 \to A_2 \to A_1 \to A_0 \to 0$ in mod(**C**) is minimal if and only if $0 \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_2) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_1) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_0) \to M \to 0$ is a minimal projective resolution of M in mod(mod(**C**)). Hence given any M in $\underline{\mathrm{mod}}$(mod(**C**)), there is a unique (up to isomorphism) minimal exact sequence $0 \to A_2 \to A_1 \to A_0 \to 0$ in mod(**C**) such that

$$0 \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_2) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_1) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_0) \to M \to 0$$

is a minimal projective resolution of M. Such a minimal exact sequence is call a *minimal exact sequence associated* with M.

We now prove the main result of this section.

THEOREM 7.5. *Let* D: mod(**C**) $\to$ mod($\mathbf{C}^{\mathrm{op}}$) *and* $\underline{D}$: $\underline{\mathrm{mod}}$(mod(**C**)) $\to$ $\underline{\mathrm{mod}}$(mod($\mathbf{C}^{\mathrm{op}}$)) *be the usual dualities. Suppose* M *is in* $\underline{\mathrm{mod}}$(mod(**C**)) *and* $0 \to A_2 \to A_1 \to A_0 \to 0$ *is an exact sequence in* mod(**C**) *such that*

$$0 \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_2) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_1) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_0) \to M \to 0$$

is exact. Let N *be in* $\underline{\mathrm{mod}}$(mod($\mathbf{C}^{\mathrm{op}}$)) *such that the sequence*

$$0 \to \mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(\cdot, D(A_0) \to \mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(\cdot, D(A_1) \to \mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(\cdot, D(A_2)) \to N \to 0$$

is exact.

(a) *N is isomorphic to* $\underline{D}(M)$.

(b) $0 \to A_2 \to A_1 \to A_0 \to 0$ *is a minimal exact sequence in* mod(**C**) *if and only if* $0 \to D(A_0) \to D(A_1) \to D(A_2) \to 0$ *is a minimal exact sequence in* mod($\mathbf{C}^{\mathrm{op}}$).

(c) *If* $0 \to A_2 \to A_1 \to A_0 \to 0$ *is a minimal exact sequence in* $\text{mod}(\mathbf{C})$, *then*:

(i) $\text{Hom}_{\mathbf{C}}(\cdot, \underline{A}_1) \to \text{Hom}_{\mathbf{C}}(\cdot, \underline{A}_0) \to M \to 0$ *is a minimal projective presentation for* M.

(ii) $0 \to M \to \text{Ext}^1(\cdot, A_2) \to \text{Ext}^1(\cdot, A_1)$ *is a minimal injective copresentation for* M.

Proof. (a) We know that $\text{Hom}_{\mathbf{C}}(\cdot, \underline{A}_1) \to \text{Hom}_{\mathbf{C}}(\cdot, \underline{A}_0) \to M \to 0$ is exact and so

$$0 \to \underline{D}(M) \to \underline{D}(\text{Hom}_{\mathbf{C}}(\cdot, A_0)) \to \underline{D}(\text{Hom}_{\mathbf{C}}(\cdot, A_1))$$

is exact. But by Proposition 7.4 $\underline{D}(\text{Hom}_{\mathbf{C}}(\cdot, \bar{A}_i) = \text{Ext}^1_{\mathbf{C}^{\text{op}}}(\cdot, D(A_i))$ for $i = 0, 1$. Hence we have the exact sequence

$$0 \to \underline{D}(M) \to \text{Ext}^1_{\mathbf{C}^{\text{op}}}(\cdot, D(A_0)) \to \text{Ext}^1_{\mathbf{C}^{\text{op}}}(\cdot, D(A_1)).$$

On the other hand, since

$$0 \to \text{Hom}_{\mathbf{C}^{\text{op}}}(\cdot, D(A_0)) \to \text{Hom}_{\mathbf{C}^{\text{op}}}(\cdot, D(A_1)) \to \text{Hom}_{\mathbf{C}^{\text{op}}}(\cdot, D(A_2)) \to N \to 0$$

is exact, it follows that

$$0 \to N \to \text{Ext}^1_{\mathbf{C}^{\text{op}}}(\cdot, D(A_0)) \to \text{Ext}^1_{\mathbf{C}^{\text{op}}}(\cdot, D(A_1))$$

is exact. Hence $\underline{D}(M) = N$.

(b) Easy consequence of the definition involved.

(c) Suppose $0 \to A_2 \to A_1 \to A_0 \to 0$ is a minimal exact sequence in $\text{mod}(\mathbf{C})$. Then $0 \to D(A_0) \to D(A_1) \to D(A_2) \to 0$ is a minimal exact sequence in $\text{mod}(\mathbf{C}^{\text{op}})$ by (b). Therefore

$$\text{Hom}_{\mathbf{C}^{\text{op}}}(\cdot, \underline{D(A_1)}) \to \text{Hom}_{\mathbf{C}^{\text{op}}}(\cdot, \underline{D(A_2)}) \to N \to 0$$

is a minimal projective presentation for N and so

$$0 \to \underline{D}(N) \to \underline{D}(\text{Hom}_{\mathbf{C}^{\text{op}}}(\cdot, \underline{D(A_2)})) \to \underline{D}(\text{Hom}_{\mathbf{C}^{\text{op}}}(\cdot, \underline{D(A_1)}))$$

is a minimal injective copresentation for $\underline{D}(N)$ which by (a) is isomorphic to M. But by Proposition 7.4, we have that $\underline{D}(\text{Hom}_{\mathbf{C}^{\text{op}}}(D(A_i)) = \text{Ext}_{\mathbf{C}}^1(\cdot, A_i)$ for $i = 1, 2$. So we have that

$$0 \to M \to \text{Ext}_{\mathbf{C}}^1(\cdot, A_2) \to \text{Ext}_{\mathbf{C}}^1(\cdot, A_1)$$

is a minimal injective copresentation for M. This proves part (ii). Part (i) was already established earlier.

So far we have been concentrating on how to describe minimal projective presentations and minimal injective copresentations for objects in $\underline{\text{mod}}(\text{mod}(\mathbf{C}))$. We now show how to construct complete projective and injective resolutions for objects in $\underline{\text{mod}}(\text{mod}(\mathbf{C}))$. Unfortunately these are not minimal in general.

Suppose we are given an M in $\underline{\text{mod}}(\text{mod}(\mathbf{C}))$ and an exact sequence $0 \to A_2 \to A_1 \to A_0 \to 0$ in $\text{mod}(\mathbf{C})$ such that

$$\text{Hom}_{\mathbf{C}}(\cdot, A_1) \to \text{Hom}_{\mathbf{C}}(\cdot, A_0) \to M \to 0$$

is exact. The long exact sequence

$$\text{Hom}_{\mathbf{C}}(\cdot, A_1) \to \text{Hom}_{\mathbf{C}}(\cdot, A_0) \to \text{Ext}_{\mathbf{C}}^1(\cdot, A_2) \to \text{Ext}_{\mathbf{C}}^1(\cdot, A_1) \to \text{Ext}_{\mathbf{C}}^1(\cdot, A_0) \to \text{Ext}_{\mathbf{C}}^2(\cdot, A_2) \to \cdots$$

gives an exact sequence

$$0 \to M \to \text{Ext}_{\mathbf{C}}^1(\cdot, A_2) \to \text{Ext}_{\mathbf{C}}^1(\cdot, A_1) \to \text{Ext}_{\mathbf{C}}^1(\cdot, A_2) \to \text{Ext}_{\mathbf{C}}^2(\cdot, A_0) \to \text{Ext}_{\mathbf{C}}^2(\cdot, A_1) \to \cdots$$

which is obviously an injective resolution for M since $\text{Ext}_{\mathbf{C}}^i(\cdot, A)$ is injective in $\underline{\text{mod}}(\text{mod}(\mathbf{C}))$ for all $i > 0$. Obviously if $0 \to A_2 \to A_1 \to A_0 \to 0$ is a minimal exact sequence in $\text{mod}(\mathbf{C})$, then $0 \to M \to \text{Ext}_{\mathbf{C}}^1(\cdot, A_2) \to \text{Ext}_{\mathbf{C}}^1(\cdot, A_1)$ is a minimal injective copresentation for M.

On the other hand, suppose that

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1^2 & \longrightarrow & P_1^1 & \longrightarrow & P_1^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0^2 & \longrightarrow & P_0^1 & \longrightarrow & P_0^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_2 & \longrightarrow & A_1 & \longrightarrow & A_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

is an exact commutative diagram with P_j^i projective objects in $\text{mod}(\mathbf{C})$

for all j and $i = 0, 1, 2$. Denote by $\pi_i(\mathrm{Hom}_{\mathbf{C}}(\cdot, A_j))$ the ith homology group of the complex

$$\cdots \to \mathrm{Hom}_{\mathbf{C}}(\cdot, P_1^j) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, P_0^j) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_j) \to 0.$$

Then it is well known that the $\pi_i(\mathrm{Hom}_{\mathbf{C}}(\cdot, A_j))$ are independent of the projective resolutions used and that there is a long exact sequence

$$\cdots \to \pi_1(\mathrm{Hom}_{\mathbf{C}}(\cdot, A_2)) \to \pi_1(\mathrm{Hom}_{\mathbf{C}}(\cdot, A_1)) \to \pi_1(\mathrm{Hom}_{\mathbf{C}}(\cdot, A_0))$$
$$\to \pi_0(\mathrm{Hom}_{\mathbf{C}}(\cdot, A_2)) \to \pi_0(\mathrm{Hom}_{\mathbf{C}}(\cdot, A_1)) \to \pi_0(\mathrm{Hom}_{\mathbf{C}}(\cdot, A_2)). \quad [8]$$

If we denote $\mathrm{Ker}(P_i^j \to P_{i-1}^j)$ by $\Omega^{i+1}A_j$ for $i = 1, 2,\ldots$, and we denote $\mathrm{Ker}(P_0 \to A)$ by $\Omega^1 A_j$ and let $\Omega^0 A_j = A_j$, then it is not difficult to see that $\pi_i(\mathrm{Hom}_{\mathbf{C}}(\cdot, A_j)) = \mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{\Omega^i A_j})$ for $i = 0, 1,\ldots$. Hence this gives a projective resolution

$$\cdots \to \mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{\Omega^1 A_1}) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{\Omega^1 A_0}) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A}_2) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A}_1)$$
$$\to \mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A}_0) \to M \to 0.$$

If $0 \to A_2 \to A_1 \to A_0 \to 0$ is minimal, we have shown that

$$\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A}_1) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A}_0) \to M \to 0$$

is a minimal projective presentation for M.

We have already seen that the functor $\mathrm{mod}(\mathbf{C}) \to \underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$ given by $M \mapsto \mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{M})$ induces an equivalence between $\mathrm{mod}(\mathbf{C})/P$ and the full subcategory of projective objects in $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$. We now derive a similar description of the full subcategory of injective objects in $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$.

We have already seen that M in $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$ is injective if and only if M is isomorphic to $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, A)$ for some A in $\mathrm{mod}(\mathbf{C})$. Consider the functor $\mathrm{mod}(\mathbf{C}) \to \underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$ given by $A \mapsto \mathrm{Ext}_{\mathbf{C}}^1(\cdot, A)$ for each A in $\mathrm{mod}(\mathbf{C})$. It is well known (see [9]) that this functor is full and that a morphism $f: A \to A'$ in $\mathrm{mod}(\mathbf{C})$ has the property that $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, f)$: $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, A) \to \mathrm{Ext}_{\mathbf{C}}^1(\cdot, A')$ is zero if and only if f factors through an injective $\mathbf{C}$-module.

The following facts are easily checked.

(a) Let $0 \to A \xrightarrow{g} E(A)$ be an injective envelope for A. Then a morphism $f: A \to A'$ factors through an injective $\mathbf{C}$-module if and only if there is an $h: E(A) \to A'$ such that $f = hg$.

(b) If for each pair (A_1, A_2) of objects in mod($\mathbf{C}$) we denote by $E(A_1, A_2)$ the R-submodule of $\text{Hom}_{\mathbf{C}}(A_1, A_2)$ consisting of all $f: A_1 \to A_2$ which factor through injective objects, then the family of R-submodules $E(A_1, A_2)$ is a relation on mod($\mathbf{C}$) which we denote by E.

(c) The functor $\text{mod}(\mathbf{C}) \to \underline{\text{mod}}(\text{mod}(C))$ given by $A \mapsto \text{Ext}_{\mathbf{C}}^1(\cdot, A)$ induces an equivalence between $\text{mod}(\mathbf{C})/E$ and the full subcategory of injective objects in $\underline{\text{mod}}(\text{mod}(\mathbf{C}))$.

(d) The duality $D: \text{mod}(\mathbf{C}) \to \text{mod}(\mathbf{C}^{\text{op}})$ induces a duality

$$D: \text{mod}(\mathbf{C})/P \to \text{mod}(\mathbf{C}^{\text{op}})/E.$$

(e) The composition $\text{mod}(\mathbf{C})/P \xrightarrow{\text{Tr}} \text{mod}(\mathbf{C}^{\text{op}})/P \xrightarrow{D} \text{mod}(\mathbf{C})/E$ is an equivalence of categories $\text{mod}(\mathbf{C})/P \to \text{mod}(\mathbf{C})/E$ given by $\underline{A} \mapsto D(\text{Tr}(\underline{A}))$.

When we are considering an object A in mod($\mathbf{C}$) as an object in $\text{mod}(\mathbf{C})/I$, we will usually denote that fact by writing $\bar{A}$ for A. Also we will often denote by $\bar{f}$ the image in $\text{Hom}_{\mathbf{C}}(A_1, A_2)/E(A_1, A_2)$ of an element f in $\text{Hom}_{\mathbf{C}}(A_1, A_2)$. In this notation, the canonical morphism $\text{mod}(\mathbf{C}) \to \text{mod}(\mathbf{C})/E$ takes the form $A \mapsto \underline{A}$ and $f \mapsto \bar{f}$.

8. Stable Equivalence

Let $\mathbf{C}$ and $\mathbf{C}'$ be two dualizing R-varieties. A *projective equivalence* $G: \mathbf{C} \to \mathbf{C}'$ is an equivalence of categories $G: \text{mod}(\mathbf{C})/P \to \text{mod}(\mathbf{C}')/P$. We say that $\mathbf{C}$ and $\mathbf{C}'$ are *projectively equivalent* if and only if there is a projective equivalence $G: \mathbf{C} \to \mathbf{C}'$. Being projectively equivalent is obviously an equivalence relation on dualizing R-varieties. An *injective equivalence* $H: \mathbf{C} \to \mathbf{C}'$ is an equivalence of categories $H: \text{mod}(\mathbf{C})/E \to \text{mod}(\mathbf{C}')/E$. We say that $\mathbf{C}$ and $\mathbf{C}'$ are *injectively equivalent* if and only if there is an injective equivalence $H: \mathbf{C} \to \mathbf{C}'$. Being injectively equivalent is obviously an equivalence relation on dualizing R-varieties.

If $G: \mathbf{C} \to \mathbf{C}'$ is a projective equivalence then the equivalences of categories $\text{mod}(\mathbf{C})/E \to \text{mod}(\mathbf{C})/P$ and $\text{mod}(\mathbf{C}')/P \to \text{mod}(\mathbf{C}')/E$ described in the last section give rise to the injective equivalence $\bar{G}: \mathbf{C} \to \mathbf{C}'$ which is the composition

$$\text{mod}(\mathbf{C})/E \longrightarrow \text{mod}(\mathbf{C})/P \xrightarrow{G} \text{mod}(\mathbf{C}')/P \longrightarrow \text{mod}(\mathbf{C}')/E.$$

Similarly, associated with an injective equivalence $H\colon \mathbf{C} \to \mathbf{C}'$, there is the projective equivalence $\underline{H}\colon \mathbf{C} \to \mathbf{C}'$ given by the composition

$$\mathrm{mod}(\mathbf{C})/P \longrightarrow \mathrm{mod}(\mathbf{C})/E \xrightarrow{H} \mathrm{mod}(\mathbf{C}')/E \longrightarrow \mathrm{mod}(\mathbf{C}')/P.$$

Thus we see that $\mathbf{C}$ and $\mathbf{C}'$ are projectively equivalent if and only if they are injectively equivalent. We say that $\mathbf{C}$ and $\mathbf{C}'$ are *stably equivalent* if they are projectively, or equivalently, injectively equivalent. It is the aim of this series of papers to study stably equivalent dualizing R-varieties. We would like to be able to determine when two dualizing R-varieties are stably equivalent as well as determine what properties stably equivalent dualizing R-varieties have in common.

Suppose $G\colon \mathbf{C} \to \mathbf{C}'$ is a projective equivalence of categories. Then the equivalence of categories $G\colon \mathrm{mod}(\mathbf{C})/P \to \mathrm{mod}(\mathbf{C}')/P$ obviously induces an equivalence of categories $\mathrm{mod}(\mathrm{mod}(\mathbf{C})/P \to \mathrm{mod}(\mathrm{mod}(\mathbf{C}')/P)$. Using the identifications

$$\mathrm{mod}(\mathrm{mod}(\mathbf{C})/P) = \underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C})) \quad \text{and} \quad \mathrm{mod}(\mathrm{mod}(\mathbf{C}')/P) = \underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}')),$$

we have that the projective equivalence $G\colon \mathbf{C} \to \mathbf{C}'$ induces the equivalence of categories $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C})) \to \underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}'))$.

On the other hand suppose we are given an equivalence of categories $T\colon \underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C})) \to \underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}'))$. The T induces an equivalence of categories between the categories of projective objects

$$T\colon \mathbf{p}(\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C})) \to \mathbf{p}(\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}'))).$$

Hence the composition

$$\mathrm{mod}(\mathbf{C})/P \longrightarrow \mathbf{p}(\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))) \xrightarrow{T} \mathbf{p}(\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}'))) \longrightarrow \mathrm{mod}(\mathbf{C}')/P$$

where $\mathrm{mod}(\mathbf{C})/P \to \mathbf{p}(\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$ and $\mathbf{p}(\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}')) \to \mathrm{mod}(\mathbf{C}')/P$ are the usual equivalences of categories (see Section 6), is an equivalence of categories $\mathrm{mod}(\mathbf{C})/P \to \mathrm{mod}(\mathbf{C}')/P$. Thus associated with an equivalence of categories $T\colon \underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C})) \to \underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}'))$ is a projective equivalence $\mathbf{C} \to \mathbf{C}'$.

Thus we see that there is a systematic way of going back and forth between projective equivalences $\mathbf{C} \to \mathbf{C}'$ and equivalences of categories $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C})) \to \underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}'))$. In particular $\mathbf{C}$ and $\mathbf{C}'$ are stably equivalent if and only if $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$ and $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}'))$ are equivalent categories. Hence anything about $\mathbf{C}$ which can be deduced from the category $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$ will be shared by any other $\mathbf{C}'$ stably equivalent to $\mathbf{C}$. Since the category $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$ has more structure than the categories $\mathbf{C}$, $\mathrm{mod}(\mathbf{C})/P$ or $\mathrm{mod}(\mathbf{C})/E$, the category $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$ plays a major role in studying stably equivalent dualizing R-varieties.

Obviously, a similar discussion to that given for projective equivalence can also be carried out for injective equivalence. The details are left to the reader to supply. We now end this preliminary discussion of stable equivalence by pointing out the following useful technical device for interpreting results concerning $\mathrm{mod}(\mathbf{C})/P$ as results about $\mathrm{mod}(\mathbf{C})$.

We denote by $\mathrm{mod}_P(\mathbf{C})$, the full subcategory of $\mathrm{mod}(\mathbf{C})$ consisting of all $\mathbf{C}$-modules with no nontrivial projective summands. Clearly $\mathrm{mod}_P(\mathbf{C})$ is an R-variety of $\mathrm{mod}(\mathbf{C})$. The functor $\mathrm{mod}_P(\mathbf{C}) \to \mathrm{mod}(\mathbf{C})/P$ is obviously full and dense (i.e., given any M in $\mathrm{mod}(\mathbf{C})/P$, there is an N in $\mathrm{mod}_P(\mathbf{C})$ such that $\underline{N}$ is isomorphic to $\underline{M}$). Since N in $\mathrm{mod}_P(\mathbf{C})$ has no nontrivial projective summands, it follows that $\mathrm{Ker}(\mathrm{End}_{\mathbf{C}}(N) \to \mathrm{End}(\underline{N}))$ is contained in $\mathrm{rad}(\mathrm{End}_{\mathbf{C}}(N))$ (see [2]). From this it follows that a morphism $f\,AN \to N'$ in $\mathrm{mod}_P(\mathbf{C})$ is an isomorphism if and only if $\underline{f}\colon \underline{N} \to \underline{N}'$ is an isomorphism in $\mathrm{mod}(\mathbf{C})/P$. In other words the functor $\mathrm{mod}_P(\mathbf{C}) \to \mathrm{mod}(\mathbf{C})/P$ is a representation equivalence (see [3]). As a consequence we have the following useful lemma.

LEMMA 8.1. *If $f\colon A \to B$ is a morphism and A has no projective summands, then f splits if $\underline{f}\colon \underline{A} \to \underline{B}$ splits.*

Finally, suppose that $G\colon \mathbf{C} \to \mathbf{C}'$ is a projective equivalence. Then the equivalence $G\colon \mathrm{mod}(\mathbf{C})/P \to \mathrm{mod}(\mathbf{C}')/P$ obviously has the property that $G(M)$ is zero if and only if M in $\mathrm{mod}(\mathbf{C})$ is projective. Thus G gives a map $G\colon Ob(\mathrm{mod}_P(\mathbf{C})) \to Ob(\mathrm{mod}_P(\mathbf{C}'))$ which has the following properties.

(a) M_1 and M_2 in $\mathrm{mod}_P(\mathbf{C})$ are isomorphic if and only if $G(M_1)$ and $G(M_2)$ in $\mathrm{mod}_P(\mathbf{C}')$ are isomorphic.

(b) If M is isomorphic to a finite sum $\coprod_{i=1}^{n} M_i$, then $G(M)$ is isomorphic to $\coprod_{i=1}^{n} G(M_i)$.

(c) M is indecomposable if and only if $G(M)$ is indecomposable.

Similarly, we define $\mathrm{mod}_E(\mathbf{C})$ to be the full subcategory of $\mathrm{mod}(\mathbf{C})$ which has no nontrivial injective summands. We leave it to the reader to state and prove the obvious analogs for $\mathrm{mod}_E(\mathbf{C})$ that we just discussed for $\mathrm{mod}_P(\mathbf{C})$.

9. Properties of $\underline{\text{mod}}$(mod $\mathbf{C}$)

Throughout this section we assume that $\mathbf{C}$ is a dualizing R-variety. Our object is to show what information about mod($\mathbf{C}$) can be derived from the knowledge of the projective and injective dimensions of objects in $\underline{\text{mod}}$(mod($\mathbf{C}$)). To facilitate interpreting what this information tells about the relationship between stably equivalent dualizing R-varieties, we introduce the following notations which will remain fixed throughout this discussion.

Suppose $\mathbf{C}$ and $\mathbf{C}'$ are stably equivalent dualizing R-varieties. Let G: mod($\mathbf{C}$)$/P \to$ mod($\mathbf{C}'$)$/P$ be an equivalence of categories. Because the functors mod($\mathbf{C}$)$/P \to \underline{\text{mod}}$(mod($\mathbf{C}$)) and mod($C'$)$/P \to \underline{\text{mod}}$(mod($C'$)), given respectively by $\underline{X} \to \text{Hom}_{\mathbf{C}}(\cdot, \underline{X})$ and $\underline{X}' \to \text{Hom}_{\mathbf{C}'}(\cdot, \underline{X}')$ for all X in mod($\mathbf{C}$) and X' in mod($\mathbf{C}'$), induce an equivalence mod($\mathbf{C}$)$/P \to$ $\mathbf{p}$(mod(mod($\mathbf{C}'$))), it follows that there is an equivalence of categories $\underline{G}$: $\underline{\text{mod}}$(mod($\mathbf{C}$)) $\to \underline{\text{mod}}$(mod($\mathbf{C}'$)) which makes the diagram

$$\begin{array}{ccc} \text{mod}(\mathbf{C})/P & \xrightarrow{\;G\;} & \text{mod}(\mathbf{C}')/P \\ \downarrow & & \downarrow \\ \underline{\text{mod}}(\text{mod}(\mathbf{C})) & \xrightarrow{\;\underline{G}\;} & \underline{\text{mod}}(\text{mod}(\mathbf{C}')) \end{array}$$

commute. We remind the reader that associated with the equivalence G: mod($\mathbf{C}$)$/P \to$ mod($\mathbf{C}'$)$/P$ is the map G: $Ob(\text{mod}_P(\mathbf{C})) \to Ob(\text{mod}_P(\mathbf{C}'))$ with the properties described at the end of Section 8.

Since $\mathbf{C}$ and $\mathbf{C}'$ are stably equivalent, we also know that there is an equivalence of categories H: mod($\mathbf{C}$)$/E \to$ mod($\mathbf{C}'$)$/E$. In analogy with the above, associated with H is an equivalence of categories $\underline{H}$:($\underline{\text{mod}}$(mod($\mathbf{C}$)) $\to \underline{\text{mod}}$(mod($\mathbf{C}'$)) having the following property.

Denote the canonical functor mod($\mathbf{C}$) $\to$ mod($\mathbf{C}$)$/E$ by $X \to \bar{X}$ for all X in mod($\mathbf{C}$) and let mod($\mathbf{C}$)$/E \to \underline{\text{mod}}$(mod($\mathbf{C}$)) be the fully faithful functor given by $\bar{X} \to \text{Ext}_{\mathbf{C}}^1(X)$ for all X in mod($\mathbf{C}$). Then the diagram

$$\begin{array}{ccc} \text{mod}(\mathbf{C})/E & \xrightarrow{\;H\;} & \text{mod}(\mathbf{C}')/E \\ \downarrow & & \downarrow \\ \underline{\text{mod}}(\text{mod}(\mathbf{C})) & \xrightarrow{\;\underline{H}\;} & \underline{\text{mod}}(\text{mod}(\mathbf{C}')) \end{array}$$

commutes.

We remind the reader that associated with the functor H: mod($\mathbf{C}$)$/E \to$ mod($\mathbf{C}'$)$/E$ is the map H: $Ob(\text{mod}_E(\mathbf{C})) \to Ob(\text{mod}_E(\mathbf{C}'))$ having the properties given at the end of Section 8.

Proposition 9.1. *Let M be a nonzero object in* $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$ *and* $0 \to A_2 \to A_1 \to A_0 \to 0$ *a minimal exact sequence in* $\mathrm{mod}(\mathbf{C})$ *such that* $0 \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_2) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_1) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_0) \to M \to 0$ *is a minimal projective resolution of M in* $\mathrm{mod}(\mathrm{mod}(\mathbf{C}))$.

(a) *M is projective in* $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$ *if and only if A_1 is projective in* $\mathrm{mod}(\mathbf{C})$. *If M is projective in* $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$, *then M is isomorphic to* $\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A}_0)$.

(b) *M is injective in* $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$ *if and only if A_1 is injective in* $\mathrm{mod}(\mathbf{C})$. *If M is injective in* $\mathrm{mod}(\mathrm{mod}(\mathbf{C}))$, *then M is isomorphic to* $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, A_2)$.

Proof. (a) Since $0 \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_2) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_1) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_0) \to M \to 0$ is a minimal projective resolution in $\mathrm{mod}(\mathbf{C})$, it follows from Corollary 5.4 that $\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A}_0) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A}_1) \to M \to 0$ is a minimal projective presentation for M in $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$. Hence M is projective in $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$ if and only if $\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A}_2) = 0$ or, equivalently, if and only if $\underline{A}_2 = 0$. Therefore M is projective in $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$ if and only if A_2 is projective in $\mathrm{mod}(\mathbf{C})$. The rest of part (a) is obvious.

(b) Follows in a manner similar to part (a) using the fact that $0 \to M \to \mathrm{Ext}_{\mathbf{C}}^1(\cdot, A_2) \to \mathrm{Ext}_{\mathbf{C}}^1(\cdot, A_1)$ is a minimal injective copresentation of M since $0 \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_2) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_1) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A_0) \to M \to 0$ is a minimal projective resolution of M in $\mathrm{mod}(\mathbf{C})$ (see Theorem 7.5).

We are particularly interested in knowing when $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, A)$ is projective and $\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A})$ is injective in $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$.

Corollary 9.2. *Let A be in* $\mathrm{mod}(\mathbf{C})$.

(a) *If A has no nonzero projective summands, then* $(\cdot, \underline{A})$ *is injective in* $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$ *if and only if the projective cover P of A in* $\mathrm{mod}(\mathbf{C})$ *is injective in* $\mathrm{mod}(\mathbf{C})$.

(b) *If A has no nonzero injective summands, then* $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, A)$ *is projective in* $\underline{\mathrm{mod}}(\mathrm{mod}(\mathbf{C}))$ *if and only if the injective envelope $E(A)$ of A in* $\mathrm{mod}(\mathbf{C})$ *is projective in* $\mathrm{mod}(\mathbf{C})$.

Proof. Let $P \to A \to 0$ be a projective cover for A in $\mathrm{mod}(\mathbf{C})$. Since A has no nonzero projective summands, the exact sequence $0 \to K \to P \to A \to 0$ is a minimal exact sequence in $\mathrm{mod}(\mathbf{C})$. Hence $0 \to \mathrm{Hom}_{\mathbf{C}}(\cdot, K) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, P) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A}) \to 0$ is a minimal projective resolution of $\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A})$ in mod(mod(**C**)). Applying Proposition 9.1 we have that $\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A})$ is injective in $\underline{\mathrm{mod}}$(mod(**C**)) if and only if P is injective in mod(**C**).

(b) Let $0 \to A \to E(A)$ be an injective envelope for A in mod(**C**). Since A has no nonzero injective summands, the exact sequence of **C**-modules $0 \to A \to E(A) \to L \to 0$ is minimal. Hence the exact sequence

$$0 \to \mathrm{Hom}_{\mathbf{C}}(\cdot, A) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, E(A)) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, L) \to \mathrm{Ext}_{\mathbf{C}}^1(\cdot, A) \to 0$$

is a minimal projective resolution of $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, A) \to 0$ in mod(mod(**C**)). Applying Proposition 9.1 we have that $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, A)$ is projective in $\underline{\mathrm{mod}}$(mod(**C**)) if and only if $E(A)$ is projective in mod(**C**).

We now show how this result can be interpreted to give information about stably equivalent dualizing R-varieties.

Proposition 9.3. *Let H: $Ob(\mathrm{mod}_E(\mathbf{C})) \to Ob(\mathrm{mod}_E(\mathbf{C}'))$ be the map given by the equivalence of categories H:* $\mathrm{mod}(\mathbf{C})/E \to \mathrm{mod}(\mathbf{C}')/E$.

(a) *If M is in* $\mathrm{mod}_E(\mathbf{C})$, *then $E(H(M))$, the injective envelope of $H(M)$ in* mod(**C**′) *is projective if and only if $E(M)$, the injective envelope of M in* mod(**C**), *is projective.*

(b) *H gives a bijection between the isomorphism classes of* **C**-*modules in* $\mathrm{mod}_E(\mathbf{C})$ *whose injective envelopes are projective and the isomorphism classes of* **C**-*modules in* $\mathrm{mod}_E(\mathbf{C}')$ *whose injective envelopes are projective.*

(c) *Suppose neither* mod(**C**) *nor* mod(**C**′) *have any simple objects which are both projective and injective. Then* mod(**C**) *has no nonzero objects which are both projective and injective if and only if* mod(**C**′) *has no nonzero objects which are both projective and injective.*

Proof. (a) Since $H(M) = 0$ if and only if $M = 0$, we can assume that $M \neq 0$. Let H: $\underline{\mathrm{mod}}$(mod(**C**)) $\to$ $\underline{\mathrm{mod}}$(mod(**C**′)) be an equivalence of categories such that $\underline{H}(\mathrm{Ext}_{\mathbf{C}}^1(\cdot, A)) = \mathrm{Ext}_{\mathbf{C}'}^1(\cdot, H(A))$ for all A in mod(**C**). Since M and $H(M)$ are nonzero objects of $\mathrm{mod}_E(\mathbf{C})$ and $\mathrm{mod}_E(\mathbf{C}')$ respectively, $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, M) \neq 0$ and $\mathrm{Ext}_{\mathbf{C}'}^1(\cdot, H(M)) \neq 0$. Then clearly $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, M)$ is projective in $\underline{\mathrm{mod}}$(mod(**C**)) if and only if $\underline{H}(\mathrm{Ext}_{\mathbf{C}}^1(\cdot, M)) = \mathrm{Ext}_{\mathbf{C}'}^1(\cdot, H(M))$ is projective in $\underline{\mathrm{mod}}$(mod(**C**)). Thus $E(M)$ is projective in mod(**C**) if and only if $E(H(M))$ is projective in mod(**C**′).

(b) Follows from (a) and formal properties of H.

(c) We show that if mod(**C**) has a nonzero object which is projective and injective, then mod($\mathbf{C}'$) also has a nonzero object which is both projective and injective. Let E be an indecomposable object in mod($\mathbf{C}$) which is both projective and injective. Since no simple object in mod($\mathbf{C}$) is both projective and injective, E is not equal to its socle which is a simple $\mathbf{C}$-module S. Since S is not injective, it follows that $\text{Ext}_{\mathbf{C}}^1(\cdot, S)$ is projective in $\underline{\text{mod}}$(mod($\mathbf{C}$)). Hence $\text{Ext}_{\mathbf{C}'}^1(\cdot, H(S))$ is a nonzero projective object in $\underline{\text{mod}}$(mod($\mathbf{C}'$)). Therefore, by part (a) the injective envelope of $H(S)$ is a nonzero projective and injective object in mod($\mathbf{C}$). The result now follows by symmetry.

A dual argument establishes the following proposition.

Proposition 9.4. *Let G: $Ob(\text{mod}_P(\mathbf{C})) \to Ob(\text{mod}_P(\mathbf{C}'))$ be the map given by the equivalence of categories G:* mod($\mathbf{C}$)/$P \to$ mod($\mathbf{C}'$)/P.

(a) *If M is in* $\text{mod}_P(\mathbf{C})$, *then a projective cover for $G(M)$ in* mod($\mathbf{C}'$) *is injective if and only if a projective cover for M in* mod($\mathbf{C}$) *is injective.*

(b) *G gives a bijection between the isomorphism classes of* $\mathbf{C}$*-modules in* $\text{mod}_P(\mathbf{C})$ *whose projective covers in* mod($\mathbf{C}$) *are injective and the isomorphism classes of* $\mathbf{C}'$*-modules in* $\text{mod}_P(\mathbf{C}')$ *whose projective covers in* mod($\mathbf{C}'$) *are injective.*

Our next result concerning the projective and injective dimensions of the objects in $\underline{\text{mod}}$(mod($\mathbf{C}$)) requires the following technical fact.

Lemma 9.5. *Let S be a simple projective object in* mod($\mathbf{C}$). *Then S is injective if and only if S is not contained in any nonsimple indecomposable projective in* mod($\mathbf{C}$).

Proof. Obviously if S is injective, then S is the only indecomposable $\mathbf{C}$-module containing S.

Suppose S is not injective and let $0 \to S \to E(S)$ be an injective envelope for S. Let $\coprod_{i=1}^n P_i \xrightarrow{f} (S) \to 0$ be a projective cover for $E(S)$ in mod($\mathbf{C}$) where each P_i is indecomposable. Then no P_i is simple, for otherwise $\text{Im}(P_i \to E(S))$ would be contained in soc($E(S)$) which is contained in r($E(S)$). But this contradicts the fact that $\coprod_{i=1}^n P_i \to E(S)$ has the property that $\coprod P_i/\text{r}P_i \to E(S)/\text{r}E(S)$ is an isomorphism since it is a projective cover for $E(S)$. Since S is projective and $\coprod_{i=1}^n P_i \to E(S)$ is an epimorphism, it follows that there is a morphism h: $S \to \coprod_{i=1}^n P_i$ such that $g = fh$. Because $g \neq 0$, $h \neq 0$ and so for some j the composition $S \to \coprod_{i=1}^n P_i \to P_j$ is not zero. Thus S is contained in the indecomposable nonsimple projective P_j in mod($\mathbf{C}$).

Proposition 9.6. *The injective envelope $E(P)$ is projective in* mod(**C**) *for each projective P in* mod(**C**) *if and only if the injective envelope in* $\underline{\text{mod}}$(mod(**C**)) *is projective in* mod(mod(**C**)) *for each projective in* $\underline{\text{mod}}$(mod(**C**)).

Proof. Suppose that $E(P)$ is projective for each projective P in mod(**C**). Then $E(K)$ is projective for each submodule K of a projective P, since $E(K)$ is contained in $E(P)$ and is hence a summand of $E(P)$. Since each nonzero projective object in $\underline{\text{mod}}$(mod(**C**)) is isomorphic to $\text{Hom}_{\mathbf{C}}(\cdot, \underline{M})$ with M in $\text{mod}_P(\mathbf{C})$, we only have to show that the injective envelope $E(\text{Hom}_{\mathbf{C}}(\cdot, \underline{M}))$ in $\underline{\text{mod}}$(mod(**C**)) is projective in $\underline{\text{mod}}$(mod(**C**)). Let $0 \to K \to P \to M \to 0$ be exact in mod(**C**) with $P \to M \to 0$ a projective cover for M. Then $0 \to M \to \text{Ext}_{\mathbf{C}}^1(\cdot, K) \to \text{Ext}^1(\cdot, P)$ is a minimal injective copresentation of M in $\underline{\text{mod}}$(mod(**C**)). But since $E(K)$ is projective in mod(**C**), it follows from Corollary 9.2 that $\text{Ext}_{\mathbf{C}}^1(\cdot, K)$ is projective in $\underline{\text{mod}}$(mod(**C**)). Hence the injective envelope of $\text{Hom}_{\mathbf{C}}(\cdot, M)$ in $\underline{\text{mod}}$(mod(**C**)) is also projective in $\underline{\text{mod}}$(mod(**C**)).

Suppose now that for each M in $\text{mod}_P(\mathbf{C})$, we have that the injective envelope of $\text{Hom}_{\mathbf{C}}(\cdot, \underline{M})$ in $\underline{\text{mod}}$(mod(**C**)) is projective in $\underline{\text{mod}}$(mod(**C**)). By Corollary 9.2 this is equivalent to assuming that if M is in $\text{mod}_P(\mathbf{C})$ and $0 \to K \to P \to M \to 0$ is exact in mod(**C**) with $P \to M \to 0$ a projective cover, then the injective envelope of K in mod(**C**) is projective in mod(**C**).

We now want to show that if P is an indecomposable projective object in mod(**C**) which is not injective then its injective envelope is projective in mod(**C**).

Suppose P is a nonsimple projective object in mod(**C**). Then the exact sequence $0 \to \text{r}P \to P \to P/\text{r}P \to 0$ has the following properties

(a) $P/\text{r}P$ is in $\text{mod}_P(\mathbf{C})$;

(b) $P \to P/\text{r}P \to 0$ is a projective cover and

(c) $0 \to \text{r}P \to P$ is an essential monomorphism.

Hence $E(P) = E(\text{r}P)$ which is a projective object in mod(**C**) by hypothesis.

Suppose P is a simple projective in mod(**C**). If P is injective, there is nothing to prove. If P is not injective, then by Lemma 9.5, there is a monomorphism $0 \to P \to P'$ with P' an indecomposable nonsimple projective object in mod(**C**). Since $E(P')$ is projective, it follows that $E(P)$, which is a summand of $E(P')$, is also projective.

Before stating the next result, we give the following definition: Let $\mathbf{C}$ be an arbitrary dualizing R-variety. We say that the dominant dimension of $\mathbf{C}$ is at least n if for each P projective in mod($\mathbf{C}$) a minimal injective resolution in mod($\mathbf{C}$)

$$0 \to P \to E_0(P) \to E_1(P) \to \cdots$$

has the property that $E_j(P)$ is projective for $j < n$. The dominant dimension of $\mathbf{C}$ is denoted by dom dim $\mathbf{C}$.

COROLLARY 9.7. *For a dualizing R-variety* $\mathbf{C}$ *the following statements are equivalent.*

(a) dom dim $\mathbf{C} \geqslant 1$.

(b) dom dim mod($\mathbf{C}$)$/P \geqslant 1$.

(c) dom dim mod($\mathbf{C}$)$/P \geqslant 2$.

Proof. The equivalence of (a) and (b) was established in Proposition 9.6. Since (c) obviously implies (b) it is only necessary to show that (a) implies (c).

(a) implies (c). Let M be a nonprojective object in mod(C) and let $0 \to K \to P \to M \to 0$ be exact with $P \to M \to 0$ a projective cover. Since $0 \to K \to P \to M \to 0$ is minimal, it follows that

$$0 \to \operatorname{Hom}_{\mathbf{C}}(\cdot, M) \to \operatorname{Ext}_{\mathbf{C}}^1(\cdot, K) \to \operatorname{Ext}_{\mathbf{C}}^1(\cdot, P)$$

is a minimal injective copresentation of $\operatorname{Hom}_{\mathbf{C}}(\cdot, M)$ in $\underline{\text{mod}}$(mod($\mathbf{C}$)). Since dom dim $\mathbf{C} \geqslant 1$, we know that the injective envelope of P, and hence that of K, is projective. But this implies, by Corollary 9.2, that $\operatorname{Ext}_{\mathbf{C}}^1(\cdot, K)$ and $\operatorname{Ext}_{\mathbf{C}}^1(\cdot, P)$ are projective in $\underline{\text{mod}}$(mod($\mathbf{C}$)). Since this is true for all projectives $\operatorname{Hom}_{\mathbf{C}}(\cdot, \underline{M})$ in $\underline{\text{mod}}$(mod($\mathbf{C}$)), we have that dom dim mod($\mathbf{C}$)$/P \geqslant 2$.

As an immediate consequence of Corollary 9.7 we have

COROLLARY 9.8. *If* $\mathbf{C}$ *and* $\mathbf{C}'$ *are stably equivalent, then* dom dim $\mathbf{C} \geqslant 1$ *if and only if* dom dim $\mathbf{C}' \geqslant 1$.

Having examined when an M in $\underline{\text{mod}}$(mod($\mathbf{C}$)) is either projective or injective, we now turn our attention to determining when pd $M \leqslant 1$ and id $M \leqslant 1$ where pd M is the projective dimension of M and id M is the injective dimension of M.

PROPOSITION 9.9. *Let M be in* $\underline{\text{mod}}(\text{mod}(\mathbf{C}))$ *and let* $0 \to A_2 \xrightarrow{f} A_1 \xrightarrow{g} A_0 \to 0$ *be a minimal exact sequence in* $\text{mod}(\mathbf{C})$ *such that*

$$0 \to \text{Hom}_{\mathbf{C}}(\cdot, A_2) \to \text{Hom}_{\mathbf{C}}(\cdot, A_1) \to \text{Hom}_{\mathbf{C}}(\cdot, A_0) \to M \to 0$$

is exact.

(a) $\text{pd}\, M \leqslant 1$ *in* $\underline{\text{mod}}(\text{mod}(\mathbf{C}))$ *if and only if* $f\colon A_2 \to A_1$ *factors through a projective in* $\text{mod}(\mathbf{C})$.

(b) $\text{id}\, M \leqslant 1$ *in* $\underline{\text{mod}}(\text{mod}(\mathbf{C}))$ *if and only if* $g\colon A_1 \to A_0$ *factors through an injective in* $\text{mod}(\mathbf{C})$.

Proof. (a) Recall that

$$\text{Hom}_{\mathbf{C}}(\cdot, \underline{A}_2) \to \text{Hom}_{\mathbf{C}}(\cdot, \underline{A}_1) \to \text{Hom}_{\mathbf{C}}(\cdot, \underline{A}_0) \to M \to 0$$

is exact (see Section 7). Since the $\text{Hom}_{\mathbf{C}}(\cdot, A_i)$ are projective in $\underline{\text{mod}}(\text{mod}(\mathbf{C}))$ and $\text{Hom}_{\mathbf{C}}(\cdot, A_1) \to \text{Hom}_{\mathbf{C}}(\cdot, A_0) \to M \to 0$ is a minimal projective presentation in $\underline{\text{mod}}(\text{mod}(\mathbf{C}))$, it follows that $\text{pd}\, M \leqslant 1$ in $\underline{\text{mod}}(\text{mod}(\mathbf{C}))$ if and only if the morphism $\text{Hom}_{\mathbf{C}}(\cdot, \underline{A}_2) \to \text{Hom}_{\mathbf{C}}(\cdot, \underline{A}_1)$ is zero. This is equivalent to saying that the morphism $f\colon A_2 \to A_1$ factors through a projective in $\text{mod}(\mathbf{C})$.

(b) We know that $0 \to M \to \text{Ext}_{\mathbf{C}}^1(\cdot, A_2) \to \text{Ext}_{\mathbf{C}}^1(\cdot, A_1) \to \text{Ext}_{\mathbf{C}}^1(\cdot, A_0)$ is the beginning of an injective resolution for M in $\underline{\text{mod}}(\text{mod}(\mathbf{C}))$ with the property that $0 \to M \to \text{Ext}_{\mathbf{C}}^1(\cdot, A_2) \to \text{Ext}_{\mathbf{C}}^1(\cdot, A_1)$ is a minimal injective copresentation for M. Hence $\text{id}\, M \leqslant 1$ in $\underline{\text{mod}}(\text{mod}(\mathbf{C}))$ if and only if the morphism $\text{Ext}_{\mathbf{C}}^1(\cdot, A_1) \to \text{Ext}_{\mathbf{C}}^1(\cdot, A_0)$ is zero. But this is the case if and only if $g\colon A_1 \to A_2$ factors through an injective object in $\text{mod}(\mathbf{C})$.

Before specializing these results to the cases $M = \text{Hom}_{\mathbf{C}}(\cdot, \underline{A})$ and $M = \text{Ext}_{\mathbf{C}}^1(\cdot, \underline{A})$, it is convenient to introduce some definitions.

Let $\mathbf{C}$ be an arbitrary dualizing R-variety and M in $\text{mod}(\mathbf{C})$. We say that M is *torsionless* if there is a monomorphism $0 \to M \to P$ with P a projective object in $\text{mod}(\mathbf{C})$. We say that M is *cotorsionless* if there is an epimorphism $Q \to M \to 0$ with Q injective in $\text{mod}(\mathbf{C})$.

PROPOSITION 9.10. *Let A be an indecomposable object in* $\text{mod}(\mathbf{C})$.

(a) *If A is not injective, then* $\text{pd}\, \text{Ext}_{\mathbf{C}}^1(\cdot, A) \leqslant 1$ *in* $\underline{\text{mod}}(\text{mod}(\mathbf{C}))$ *if and only if A is a torsionless object in* $\text{mod}(\mathbf{C})$.

(b) *If A is not projective, then* $\text{id}\, \text{Hom}_{\mathbf{C}}(\cdot, \underline{A}) \leqslant 1$ *in* $\underline{\text{mod}}(\text{mod}(\mathbf{C}))$ *if and only if A is a cotorsionless object in* $\text{mod}(\mathbf{C})$.

Proof. (a) Since A is indecomposable and not injective, we know that if $0 \to A \to E(A)$ is an injective envelope for A, then $0 \to A \xrightarrow{f} E(A) \to E(A)/A \to 0$ is minimal in mod($\mathbf{C}$). Hence by Proposition 9.9 we know that pd $\text{Ext}_{\mathbf{C}}^1(\cdot, A) \leqslant 1$ in $\underline{\text{mod}}$(mod($\mathbf{C}$)) if and only if $0 \to A \xrightarrow{f} E(A)$ factors through a projective object P in mod($\mathbf{C}$). Now, if $f: A \to E(A)$ factors as $A \xrightarrow{g} P \xrightarrow{h} E(A)$, then g must be a monomorphism since f is a monomorphism and so A is torsionless. On the other hand, if we are given a monomorphism $0 \to A \to^g P$, then there is an $h: P \to E(A)$ such that $f = hg$ since $E(A)$ is injective. This completes the proof of (a).

(b) Let $0 \to K \to P \xrightarrow{g} A \to 0$ be exact in mod($\mathbf{C}$) with P a projective cover for A. Since A is indecomposable and not projective, $0 \to K \to P \to A \to 0$ is minimal in mod($\mathbf{C}$). Since $\text{Hom}_{\mathbf{C}}(\cdot, P) \to \text{Hom}_{\mathbf{C}}(\cdot, A) \to \text{Hom}_{\mathbf{C}}(\cdot, A) \to 0$ is exact, it follows by Proposition 9.9 that id $\text{Hom}_{\mathbf{C}}(\cdot, \underline{A}) \leqslant 1$ in $\underline{\text{mod}}$(mod($\mathbf{C}$)) if and only if $g: P \to A$ factors through an injective object in mod($\mathbf{C}$). It is not difficult to show that $g: P \to A$ factors through an injective if and only if A is a factor module of an injective $\mathbf{C}$-module in mod(C).

As an immediate consequence of Proposition 9.10, we have the following.

COROLLARY 9.11. *Let* $\mathbf{C}$ *and* $\mathbf{C}'$ *be two stably equivalent dualizing* R*-varieties.*

(a) *Suppose* $H: \mathbf{C} \to \mathbf{C}'$ *is an injective equivalence. Then*

$$H: Ob(\text{mod}_E(\mathbf{C})) \to Ob(\text{mod}_E(\mathbf{C}'))$$

induces a bijection between the isomorphism classes of indecomposable non-injective torsionless objects in mod($\mathbf{C}$) *and those in* mod($\mathbf{C}'$).

(b) *Suppose* $G: \mathbf{C} \to \mathbf{C}'$ *is a projective equivalence. Then*

$$G: Ob(\text{mod}_P(\mathbf{C})) \to Ob(\text{mod}_P(\mathbf{C}'))$$

induces a bijection between the isomorphism classes of indecomposable non-projective cotorsionless modules in mod($\mathbf{C}$) *and those in* mod($\mathbf{C}'$).

For each A in mod($\mathbf{C}$), we denote by

$$0 \to \text{Hom}_{\mathbf{C}}(\cdot, \underline{A}) \to E_0(\text{Hom}_{\mathbf{C}}(\cdot, \underline{A})) \to E_1(\text{Hom}_{\mathbf{C}}(\cdot, \underline{A}))$$

a minimal injective copresentation of $\text{Hom}_{\mathbf{C}}(\cdot, \underline{A})$ in $\underline{\text{mod}}$(mod($\mathbf{C}$)). We now investigate which **C**-modules M have the property that $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, M)$ is a summand of $E_i(\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A}))$ for some A in mod(**C**) and $i = 0$ or 1.

PROPOSITION 9.12. *Let M be an indecomposable noninjective object in* mod(**C**).

(a) *There is some A in* mod(**C**) *such that $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, M)$ is a summand of $E_1(\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A}))$ if and only if M is a nonsimple projective in* mod(**C**).

(b) *There is an A in* mod(**C**) *such that $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, M)$ is a summand of $E_0(\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A}))$ if and only if $M \subset \mathrm{r}P$ for some projective module P in* mod(**C**).

Proof. (a) Let $M = P$ be a nonsimple indecomposable projective **C**-module. Then $0 \to \mathrm{r}P \to P \to P/\mathrm{r}P$ is a minimal exact sequence in mod(**C**). Then $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, P) = E_1((\cdot, P/\mathrm{r}P))$ and so we are done.

Suppose now that $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, M)$ is a summand of $E_1(\mathrm{Hom}(\cdot, \bar{A}))$ for some A in mod(**C**). We can assume, without loss of generality, that A is a nonprojective indecomposable object in **C**. Since A is not projective, we know that $0 \to K \to P \to A \to 0$ with $P \to A \to 0$ a projective cover is minimal in mod(**C**). Hence $E_1(\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A})) = \mathrm{Ext}_{\mathbf{C}}^1(\cdot, P)$ and so $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, M)$ is a summand of $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, P)$. Thus there are morphisms $M \xrightarrow{f} P \xrightarrow{g} M$ such that $\overline{M} \xrightarrow{\bar{f}} \overline{P} \xrightarrow{\bar{g}} \overline{M}$ is the identity on $\overline{M}$. But since M is an indecomposable, noninjective object in mod(**C**), it follows that $gf\colon M \to M$ is an isomorphism (see Lemma 8.1). Therefore M is projective because it is a summand of P. But M can not be simple. For if $P \to A \to 0$ is a projective cover, then any simple summand of P is also a summand of A. This is a contradiction since we have assumed that A is an indecomposable, nonprojective object in mod(**C**). Therefore M is a nonsimple, indecomposable projective object in mod(**C**).

(b) Assume that $M \subset \mathrm{r}P$ for some projective object P in mod(**C**). Then there is an exact sequence $0 \to M \to P \to A \to 0$ with $\mathrm{Im}(M \to P)$ contained in $\mathrm{r}P$. Then by Proposition 3.7 $P \to A \to 0$ is a projective cover. Without loss of generality we may assume that no summand of A is projective and so $0 \to M \to P \to A \to 0$ is a minimal exact sequence in mod(**C**). Then $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, M) = E_0(\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A}))$.

Suppose that $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, M)$ is a summand of $E_0(\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A}))$. Again without loss of generality, we may assume that A is indecomposable and not projective. Hence if $0 \to K \to P \to A \to 0$ is exact with $P \to A \to 0$ a projective cover, then $E_0(\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{A})) = \mathrm{Ext}^1(\cdot, K)$.

Therefore M is a summand of K which is contained in rP. This completes the proof of the proposition.

For each A in mod(**C**), we denote by

$$P_1(\mathrm{Ext}_{\mathbf{C}}^1(\cdot, A)) \to P_0(\mathrm{Ext}_{\mathbf{C}}^1(\cdot, A)) \to \mathrm{Ext}_{\mathbf{C}}^1(\cdot, A) \to 0$$

a minimal projective presentation of $\mathrm{Ext}_{\mathbf{C}}^1(\cdot, A)$ in $\underline{\mathrm{mod}}$(mod(**C**)). The following result, which is nothing more than the dual of Proposition 9.12, can be proven using arguments dual to those for Proposition 9.12.

PROPOSITION 9.13. *Let M be an indecomposable nonprojective object in* mod(**C**).

(a) *There is some A in* mod(**C**) *such that* $\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{M})$ *is a summand of* $P_1(\mathrm{Ext}_{\mathbf{C}}^1(\cdot, A))$ *if and only if M is a nonsimple injective object in* mod(**C**).

(b) *There is an A in* mod(**C**) *such that* $\mathrm{Hom}_{\mathbf{C}}(\cdot, \underline{M})$ *is a summand of* $P_0(\mathrm{Ext}_{\mathbf{C}}^1(\cdot, A))$ *if and only if* $M = E/E'$ *where E is injective in* mod(**C**) *and* $E' \supset S$ *or E.*

As an immediate consequence of Propositions 9.12 and 9.13 we have the following.

COROLLARY 9.14. *Suppose* **C** *and* **C**′ *are stably equivalent dualizing R-varieties.*

(a) *If* $H: \mathbf{C} \to \mathbf{C}'$ *is an injective equivalence, then the map* $H: Ob(\mathrm{mod}_E(\mathbf{C})) \to Ob(\mathrm{mod}_E(\mathbf{C}'))$ *induces a bijection between the isomorphism classes of indecomposable nonsimple noninjective projective modules in* mod(**C**) *and* mod(**C**′).

(b) *If* $G: \mathbf{C} \to \mathbf{C}'$ *is a projective equivalence, then the map* $G: Ob(\mathrm{mod}_P(\mathbf{C})) \to Ob(\mathrm{mod}_P(\mathbf{C}))$ *induces a bijection between the isomorphism classes of indecomposable nonsimple nonprojective injective modules in* mod(**C**) *and* mod(**C**′).

10. gl dim $\underline{\mathrm{mod}}$(mod **C**) $\leqslant 2$

Let **C** be a dualizing R-variety and **D** = mod(**C**). We want to apply our previous results to give a description of what it means for **D** that gl dim $\underline{\mathrm{mod}}$(**D**) $= i$, for $i = 0, 1, 2$. It is most convenient to first investigate the case gl dim $\underline{\mathrm{mod}}$(**D**) $\leqslant 2$.

THEOREM 10.1. *The following are equivalent for a dualizing R-variety* $\mathbf{D} = \text{mod}(\mathbf{C})$.

(a) gl dim $\underline{\text{mod}}(\mathbf{D}) \leqslant 2$.

(b) *If A is an indecomposable torsionless nonprojective object in* $\mathbf{D}$, *then a monomorphism* $0 \to A \to B$ *either splits or factors through a projective object in* $\mathbf{D}$.

(c) (i) *Eeach indecomposable torsionless object in* $\mathbf{D}$ *is simple or projective.*

(ii) *If* $\text{soc}\, P \subset A \subset \text{r}P$, *for a projective object* P *in* $\mathbf{D}$, *then* $\text{Hom}_{\mathbf{D}}(S, \text{r}P/A) = 0$ *for each torsionless simple object* S *in* $\mathbf{D}$.

Proof. (a) implies (b). Let A be an indecomposable torsionless nonprojective object in $\mathbf{D}$ and $f: A \to B$ a monomorphism which does not factor through any projective object. We then want to show that f splits. We can clearly assume that A is not injective. Consider the exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, where $C = \text{Coker}\, f$. Let F be the object determined by the exact sequence

$$\to \text{Hom}_{\mathbf{C}}(\cdot, \underline{A}) \xrightarrow{(\cdot,f)} \text{Hom}_{\mathbf{C}}(\cdot, \underline{B}) \xrightarrow{(\cdot,g)} \text{Hom}_{\mathbf{C}}(\cdot, \underline{C}) \to F \to 0.$$

Since by assumption gl dim $\underline{\text{mod}}(D) \leqslant 2$, $\text{Im}(\cdot, \underline{f})$ is a projective subobject of $\text{Hom}_{\mathbf{C}}(\cdot, \underline{B})$, and because we have assumed that $f: A \to B$ does not factor through a projective object, $\text{Im}(\cdot, \underline{f})$ is not zero. Since A is indecomposable, $\text{Hom}_{\mathbf{C}}(\cdot, \underline{A})$ is also indecomposable by Lemma 8.1. Hence $(\cdot, \underline{f})$ must be a monomorphism. Consider now the exact sequences

$$0 \to \text{Hom}_{\mathbf{C}}(\cdot, \underline{A}) \xrightarrow{(\cdot,f)} \text{Hom}_{\mathbf{C}}(\cdot, \underline{B}) \to K \to 0$$
$$0 \to K \to \text{Hom}_{\mathbf{C}}(\cdot, \underline{C}) \to F \to 0$$
$$0 \to F \to \text{Ext}_{\mathbf{C}}^1(\cdot, A),$$

where $K = \text{Im}(\cdot, g)$.

Because A is torsionless, we know by Proposition 9.10 that $\text{pd}\, \text{Ext}_{\mathbf{C}}^1(\cdot, A)$ is at most 1. Since gl dim $\underline{\text{mod}}(\mathbf{D}) \leqslant 2$, it is then easy to see that $\text{pd}\, F \leqslant 1$. It follows that K is projective, hence $(\cdot, \underline{f})$ splits. Hence $\underline{f}: \underline{A} \to \underline{B}$ splits, and because A has no projective summand, we conclude by Lemma 8.1 that $f: A \to B$ splits.

(b) implies (c)(i). Let A be an indecomposable torsionless nonprojective object in $\mathbf{D}$. We want to show that A is simple. Assume to the contrary that A is not simple. Hence $A \neq A/\text{r}A$. Let B be a simple summand of the semisimple module $A/\mathrm{r}A$, and $g: A \to B$ an epimorphism. Clearly g is not a monomorphism. Since A is torsionless, there is a monomorphism $f: A \to P$ with P projective. Consider the monomorphism $h = f \coprod g: A \to P \coprod B$. Since g is an epimorphism and A has no nonzero projective summands, g does not factor through any projective object. It follows that h does not factor through any projective object. By assumption we then know that $h: A \to P \coprod B$ splits. Since A is indecomposable and nonprojective, A must be a summand of B by the Krull–Schmidt theorem. This is a contradiction, which shows that A must be simple.

(ii) Let P be a projective object in $\mathbf{D}$. Let A be an object in $\mathbf{D}$, where $\operatorname{soc} P \subset A \subset \mathrm{r}P$. Let S be a torsionless simple object in $\mathbf{D}$, and assume that there is a nonzero map $f: S \to \mathrm{r}P/A \subset P/A$. Then $f: S \to P/A$ does not split, since $f(S) \subset \mathrm{r}(P/A) = \mathrm{r}P/A$. By assumption, f then factors through a projective object, hence through P. This implies that f is zero, since $\operatorname{soc} P \subset A$. This contradiction shows that $\operatorname{Hom}(S, P/A)$ is zero.

(c) implies (a). Let F be an object in $\underline{\mathrm{mod}}(\mathbf{D})$. From Section 7 we know that F has a projective resolution

$$\to \operatorname{Hom}_{\mathbf{C}}(\cdot, \underline{\Omega^1 C}) \xrightarrow{(\cdot,f)} \operatorname{Hom}_{\mathbf{C}}(\cdot, \underline{A}) \to \operatorname{Hom}_{\mathbf{C}}(\cdot, \underline{B}) \to \operatorname{Hom}_{\mathbf{C}}(\cdot, \underline{C}) \to F \to 0$$

where $0 \to A \to B \to C \to 0$ is a minimal exact sequence in $\mathbf{D}$. Consider the morphism $\underline{f}: \underline{\Omega^1 C} \to \underline{A}$. We want to show that $\underline{f}: \underline{\Omega^1 C} \to \underline{A}$ decomposes into the direct sum of an isomorphism and a zero map. For this implies that $\operatorname{Coim}(\cdot, \underline{f})$ is projective, and hence that $\operatorname{pd} F$ is at most 2.

Since $\Omega^1 C$ is torsionless, $\Omega^1 C = S_1 \coprod \cdots \coprod S_n \coprod P$, where the S_i are simple nonprojective and P is projective. Write $A = B \coprod C \coprod P$, where B is semisimple with no projective summand, P is projective, and C has no projective or simple summand. Let $f': S_1 \coprod \cdots \coprod S_n \to B \coprod C$ be such that $\underline{f'} = \underline{f}$. Let $p: B \coprod C \to C$ denote the natural projection, and for each S_i, consider the induced map $f_i: S_i \to B \coprod C$ and $f_i' = pf_i: S_i \to C$. Write $C = P/K$, where P is the projective cover of C, and consider the following exact commutative diagram.

$$\begin{array}{ccccccccc} & & & & S_i & & & & \\ & & & & \downarrow{\scriptstyle f_i} & & & & \\ 0 \longrightarrow & \operatorname{soc} P \cap K & \xrightarrow{i} & \operatorname{soc} P & \xrightarrow{g} & P/K & \xrightarrow{h} & P/(K, \operatorname{soc} P) & \longrightarrow 0 \end{array}$$

Since $C = P/K$ has no simple summand, $f_i'(S_i) \subset \mathrm{r}(P/K)$, hence $hf_i'(S') \subset \mathrm{r}(P/(K, \operatorname{soc} P)) = \mathrm{r}P/(K, \operatorname{soc} P)$. By our assumption, $hf_i'(S)$ is zero, hence $f_i'(S_i) \subset g(\operatorname{soc} P)$. Since $\operatorname{soc} P$ is semisimple, $g\colon \operatorname{soc} P \to g(\operatorname{soc} P)$ splits. We can now conclude that $f_i'\colon S_i \to P/K$ factors through P, so that f_i' is zero. We can then assume that f_i' is zero. It now follows that $f'(S_1 \amalg \cdots \amalg S_n) \subset B$. Since B is semisimple, f' is the direct sum of an isomorphism and a zero map, hence so is $f' = f$. This finishes the proof of (c) $\mapsto$ (a), and the proof of Theorem 10.1.

We now show that if gl dim(**D**) $\leqslant 1$, then **D** belongs to the above class. More generally, we have the following.

PROPOSITION 10.2. *If* gl dim(**D**) $\leqslant n$, *then* gl dim $\underline{\mathrm{mod}}$(**D**) $\leqslant 3n - 1$. *Hence if* **D** *is stably equivalent to a dualizing R-variety* **D**$'$ *with* gl dim(**D**$'$) $\leqslant n$, *then* gl dim $\underline{\mathrm{mod}}$(**D**) $\leqslant 3n - 1$.

Proof. Let F be an indecomposable object in $\underline{\mathrm{mod}}$(**D**) and $0 \to A \to B \to C \to 0$ the minimal exact sequence associated with F, as described in Section 7. Then we have seen that

$$0 \to F \to \operatorname{Ext}_{\mathbf{C}}^1(\cdot, A) \to \operatorname{Ext}_{\mathbf{C}}^1(\cdot, B) \to \operatorname{Ext}_{\mathbf{C}}^1(\cdot, C) \to \operatorname{Ext}_{\mathbf{C}}^2(\cdot, A) \to \operatorname{Ext}_{\mathbf{C}}^2(\cdot, B)$$
$$\to \operatorname{Ext}_{\mathbf{C}}^2(\cdot, C) \to \cdots \to \operatorname{Ext}_{\mathbf{C}}^i(\cdot, A) \to \operatorname{Ext}_{\mathbf{C}}^i(\cdot, B) \to \operatorname{Ext}_{\mathbf{C}}^i(\cdot, C) \to \cdots$$

gives an injective resolution for F. Hence gl dim $\underline{\mathrm{mod}}(D) \leqslant 3n - 1$. The rest of the proposition follows from this.

COROLLARY 10.3. *If* gl dim mod(**D**) $\leqslant 1$, *then* gl dim $\underline{\mathrm{mod}}$(**D**) $\leqslant 2$. *Hence if* **D** *is stably equivalent to an hereditary dualizing R-variety, then* gl dim $\underline{\mathrm{mod}}$(**D**) $\leqslant 2$.

This result naturally raises the question of whether **D** is stably equivalent to an hereditary dualizing R-variety if gl dim $\underline{\mathrm{mod}}$(**D**) $\leqslant 2$. While this is not true for arbitrary dualizing R-varieties **D** it is true for a class of dualizing R-varieties which includes **D** $=$ mod(Λ) where Λ is an artin R-algebra. This and related questions will be discussed in the next paper in this series [7].

As for when gl dim $\underline{\mathrm{mod}}$(**D**) is exactly 2, we will have the answer once we have characterized what it means for gl dim $\underline{\mathrm{mod}}$(**D**) to be at most 1, which we proceed to do next.

THEOREM 10.4. *The following are equivalent for a dualizing R-variety* **D** $=$ mod(**C**).

(a) gl dim $\underline{\text{mod}}(\mathbf{D}) \leqslant 1$.

(b) *Each indecomposable object in* $\mathbf{D}$ *is injective, projective or torsionless simple.*

(c) $\mathbf{D}$ *has Loewy length at most* 2, *and each indecomposable object in D is simple, projective or injective.*

Proof. (a) implies (b). Assume gl dim $\underline{\text{mod}}(\mathbf{D}) \leqslant 1$. Let A be an indecomposable noninjective object. Since gl dim $\underline{\text{mod}}(\mathbf{D}) \leqslant 1$, pd $\text{Ext}_{\mathbf{C}}^1(\cdot, A) \leqslant 1$.

Since $0 \to A \xrightarrow{f} E(A) \to E(A)/A \to 0$, where $E(A)$ denotes the injective envelope of A, is the minimal exact sequence associated with $\text{Ext}_{\mathbf{C}}^1(\cdot, A)$, we know by Proposition 9.9 that $f: A \to E(A)$ factors through a projective object P. Since f is a monomorphism, we have a monomorphism from A to P, so that A is torsionless. By Theorem 10.1 we conclude that A is projective or (torsionless) simple.

(b) implies (c). Assume that every indecomposable object in $\mathbf{D}$ is injective, projective or torsionless simple. We want to show $\text{LL}(\mathbf{D}) \leqslant 2$, that is $\text{r}^2P = 0$ for all indecomposable projectives (see Section 3). Assume to the contrary that there is an indecomposable projective object P with $\text{LL}(P) > 2$. Then $P' = P/\text{r}^2P$ is neither projective nor simple, hence by assumption P/r^2P is injective. Since P/r^2P is indecomposable injective, $\text{soc}(P/\text{r}^2P) = \text{r}P/\text{r}^2P = S$ is simple. Since S is clearly neither injective nor projective, S is torsionless simple by assumption. Because P/r^2P is injective, and S is torsionless, the inclusion map $f: S \to P/\text{r}^2P$ factors through a projective object. Hence f factors through P, so that we have the following commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{g} & P \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle h} & \\ P/\text{r}^2P & & \end{array}$$

where $g: P \to P/\text{r}^2P$ is the natural map. Now $g(S) \subset \text{r}P$. But $\text{r}P$ is indecomposable since $\text{r}P/\text{r}^2P$ is simple, and $\text{r}P$ is not simple. Hence $g(S) \subset \text{r}^2P$. This implies that hg is zero, so that f is zero, a contradiction.

(c) implies (a). Assume $\text{LL}(\mathbf{D}) \leqslant 2$, and that each indecomposable object in $\mathbf{D}$ is simple, projective or injective. We want to show that gl dim $\underline{\text{mod}}(\mathbf{D})$ is at most 1. Let F be an object in $\underline{\text{mod}}(\mathbf{D})$, and $0 \to A \xrightarrow{f} B \to C \to 0$ the associated minimal exact sequence. Then

$$\cdots \to \text{Hom}_{\mathbf{C}}(\cdot, \underline{A}) \xrightarrow{(\cdot, f)} \text{Hom}_{\mathbf{C}}(\cdot, \underline{B}) \to \text{Hom}_{\mathbf{C}}(\cdot, \underline{C}) \to F \to 0$$

is a projective resolution for F, minimal in the first 2 terms, by Corollary 5.4. We want to show that $(\cdot, \underline{f})$ is zero. Since $0 \to A \xrightarrow{f} B \to C \to 0$ is minimal, A can not have any injective summands. Hence we can write $A = S \coprod P$, where S is semisimple with no projective summands and P is projective. Write $B = T \coprod Q_1 \coprod Q_2 \coprod E$, where T is semisimple with no projective summand, Q_1 is semisimple projective, Q_2 projective with no simple summands and E is injective. Consider the induced map $f'\colon S \to B$. Assume first that $pf'\colon S \to T$ is not zero, where $p\colon B \to T$ is the natural projection. Then the map $f'\colon S \to B$ would have an isomorphism as a summand, hence so would $\underline{f}'\colon \underline{S} \to \underline{B}$. But this contradicts the fact that the projective resolution for F is minimal in the first 2 terms. Hence we conclude that pf' is zero. This implies that $f'(S) \subset Q_2 \coprod E$. Let now $g\colon Q' \to Q_2 \coprod E$ be the projective cover of $Q \coprod E$. Since $\mathrm{LL}(\mathbf{D}) \leqslant 2$, $\mathrm{r}Q'$ is semi-simple, and so is $g(\mathrm{r}Q') = \mathrm{r}(Q_2 \coprod E) = \mathrm{soc}(Q_2 \coprod E)$. Hence $g\colon \mathrm{r}Q' \to \mathrm{r}(Q_2 \coprod E)$ splits, and since $f'(S) \subset \mathrm{r}(Q_2 \coprod E)$, we conclude that $f'\colon S \to Q_2 \coprod E$ factors through Q'. This shows that $\underline{f}' = \underline{f}$ is zero, and we are done.

Examples of $\mathbf{D} = \mathrm{mod}\,\Lambda$, where Λ is an artin algebra such that the conditions of Theorem 10.2 are satisfied are

$$\Lambda = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} k & 0 & 0 \\ k & k & 0 \\ k & 0 & k \end{pmatrix} \qquad \text{and} \qquad \Lambda = k[X]/(X^2)$$

where k is a field.

In connection with Theorem 10.2 it is interesting to observe the following.

EXAMPLE 10.5. Let

$$\Lambda = \begin{pmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{pmatrix},$$

k a field. It is well known that Λ is Nakayama (generalized uniserial) and hereditary. The indecomposable projective modules are given by the columns,

$$P_1 = \begin{pmatrix} k \\ k \\ k \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 \\ k \\ k \end{pmatrix} \qquad \text{and} \qquad P_3 = \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix},$$

and each indecomposable module is a factor module of one of the P_i.

We then observe that each indecomposable Λ-module is injective, projective or simple. Now P_3 is simple, and $L(P_2) = 2$, so any proper factor is simple. Further P_1 is injective, hence so are all factor modules, since Λ is hereditary.

Since $\mathrm{LL}(\Lambda) = 3$, this example shows that $\mathrm{LL}(D) \leqslant 2$ is not a consequence of each indecomposable object being simple, projective or injective. And it also shows that each indecomposable object being projective, injective or torsionless simple is not a consequence of each indecomposable object being projective, injective or simple.

Before we go on to the case gl dim $\underline{\mathrm{mod}}(\mathbf{D}) = 0$, we make the following definition. A dualizing R-variety $\mathbf{D} = \mathrm{mod}(\mathbf{C})$ is said to be generalized uniserial, or *Nakayama*, if for each indecomposable object M which is either projective or injective, the submodules in $\mathbf{D}$ are totally ordered with respect to inclusion. For a Nakayama artin algebra Λ it is well known that for each indecomposable object M in $\mathrm{mod}(\Lambda)$, the subobjects of M are totally ordered by inclusion. It is also well known that each indecomposable Λ-module is the factor of an indecomposable projective Λ-module. We further have

PROPOSITION 10.6. $\mathbf{D} = \mathrm{mod}(\mathbf{C})$ *is Nakayama if and only if for each* C *in* $\mathbf{C}$, $\mathrm{mod}\,\mathbf{V}(C)$ *is Nakayama.*

Proof. Since we have a duality between $\mathrm{mod}(\mathbf{C})$ and $\mathrm{mod}(\mathbf{C}^{\mathrm{op}})$, and between $\mathrm{mod}\,\mathbf{V}(C)$ and $\mathrm{mod}\,\mathbf{V}(C^{\mathrm{op}})$, it is sufficient to consider only indecomposable projective objects. Let C' be an indecomposable object in $\mathbf{C}$. $(\cdot, C')$ does not have a unique composition series as an object of $\mathbf{D} = \mathrm{mod}\,\mathbf{C}$ if and only if there are maps $f\colon (\cdot, C_1) \to (\cdot, C')$ and $g\colon (\cdot, C_2) \to (\cdot, C')$ such that $\mathrm{Im}\, f$ and $\mathrm{Im}\, g$ are noncomparable subobjects of $(\cdot, C')$. If $(\cdot, C')$ has 2 subobjects, neither of which is contained in the other, $(\cdot, C')$ also does in $\mathrm{mod}\,\mathbf{V}(C)$, where $C = C' \coprod C_1 \coprod C_2$.

If conversely there is a C, where for some C' in $\mathbf{V}(C)$, $(\cdot, C')$ has 2 noncomparable subobjects, we have f and g as above with C_1 and C_2 in $\mathbf{V}(C)$, and $\mathrm{Im}\, f$ and $\mathrm{Im}\, g$ noncomparable subobjects of $(\cdot, C')$. Then clearly the subobjects of $(\cdot, C')$ in $\mathrm{mod}(\mathbf{C})$ are not totally ordered by inclusion. This completes the proof of the proposition.

We are now ready to prove

THEOREM 10.7. gl dim $\underline{\mathrm{mod}}(\mathbf{D}) = 0$ *if and only if* $\mathbf{D}$ *is Nakayama and* $\mathrm{LL}(\mathbf{D}) \leqslant 2$.

Proof. Assume first that $\mathbf{D} = \mathrm{mod}(\mathbf{C})$ is Nakayama and $\mathrm{LL}(\mathbf{D}) \leqslant 2$, and let M be an indecomposable object in **D**. Let $(\cdot, C_1) \xrightarrow{f} (\cdot, C_2) \to M \to 0$ be a minimal projective presentation for M. f is then an indecomposable map. Let $C = C_1 \coprod C_2$, and consider mod $\mathbf{V}(C)$. Consider the exact sequence $(\cdot, C_1) \to^f (\cdot, C_2) \to N \to 0$ in mod $\mathbf{V}(C)$. Here N is indecomposable since f is. By Proposition 10.6, End$(\mathbf{C})^{\mathrm{op}}$ is a Nakayama artin algebra. Hence the indecomposable objects in mod$(\mathbf{V}(C))$ are factors of indecomposable projectives. Therefore the projective cover of N, $(\cdot, C_2)$ is indecomposable in mod$(\mathbf{V}(C))$, hence in **D**. Therefore we have established that any indecomposable object M in **D** is a factor of an indecomposable projective object. Since LL(**D**) $\leqslant 2$ it follows that each indecomposable nonprojective object in **D** is simple. This finishes the proof that gl dim $\underline{\mathrm{mod}}$(**D**) is zero.

Assume now gl dim $\underline{\mathrm{mod}}$(**D**) $= 0$. Then LL(**D**) $\leqslant 2$ by Theorem 10.4. Let $0 \to A \to B \xrightarrow{g} C \to 0$ be a minimal exact sequence in **D**. Because Coker$(\cdot, g)$ is projective and injective in $\underline{\mathrm{mod}}$(**D**), we conclude by Proposition 9.1 that B is both projective and injective. If P is an indecomposable projective object which is not simple, it follows that P is injective, since $0 \to \mathrm{r}P \to P \to P/\mathrm{r}P \to 0$ is then a minimal exact sequence. Hence $\mathrm{r}P$ is simple. Similarly, if $E = E(S)$ is an indecomposable nonsimple injective object, $0 \to S \to E(S) \to E(S)/S \to 0$ is a minimal exact sequence. Hence $E(S)$ is projective, so that $E(S)/S$ is simple, and we conclude that **D** is Nakayama.

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