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# New and old results on flocks of circle planes ☆

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## 1. Flocks and packings of circle planes

The term ‘flock’ probably originated with Dembowski [20] (see pp. 254, 255) and was defined as a set of mutually disjoint circles which partition all but two points of an inversive plane.

More generally, we may define a flock of a circle plane.

**Definition 1.1.** Let $C$ be a circle plane.

(1) A flock of $C$ is a set of mutually disjoint circles which partition the points of $C-S$ where $S$ is a set of points consisting of at most two points, one point or no points respectively as the circle plane is a Möbius (inversive), Laguerre, or Minkowski plane.

(2) A packing (resolution, parallelism) of a circle plane is a set of mutually disjoint flocks such that every circle is in exactly one flock of the set.

Note that packings cannot exist for finite Möbius planes due simply to the parameters (when there are $n^2+1$ points, there are $n(n^2+1)$ circles whereas a flock consists of $n-1$ circles). However, packings are possible for finite Minkowski planes of order $n$ and consist of $n(n-1)$ flocks of $n+1$ circles each. Similarly, a packing for a finite Laguerre plane of order $n$ would consist of $n^2$ flocks of $n$ circles each.

Furthermore, when the circle geometry is a finite inversive plane then the set $S$ must have cardinality two. However, there are infinite flocks of inversive planes where $S$ could be 0 or 1.

**Definition 1.2.** A flock of an inversive plane is said to be a $i$-point flock if and only if the partition excludes $i$ points for $i=0,1,2$.

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### 1.1. The classical models

*Elliptic*: Let $E$ be an elliptic quadric in $\mathrm{PG}(3,K)$ where $K$ is a field. The plane sections of $E$ containing more than a point are conics in the corresponding planes. Define 'points' as points of $E$ and 'circles' as non-singular conics of intersection. This structure is the classical Möbius (inversive) plane.

*Quadratic Cones*: Let $C_o$ be a conic of a plane in $\mathrm{PG}(3,K)$ where $K$ is a field. Let $v$ be a point of $\mathrm{PG}(3,K) - C_o$ and form the cone $C$ of points $v\alpha$ for $\alpha$ in $C_o$. The nontrivial plane sections of $C$ not on $v$ are conics in the corresponding planes. Define 'points' as points of $C - \{v\}$, 'circles' as non-singular conics of intersection, where two points are defined to be 'parallel' if and only if they lie on the same line of the cone. This provides the classical Laguerre plane.

*Hyperbolic*: Let $H$ be a hyperbolic quadric in $\mathrm{PG}(3,K)$ where $K$ is a field. Let $R_+$ and $R_-$ denote the two sets of ruling lines. Defines 'points' as points of $H$ and 'circles' as non-singular conics of intersection of planes, where two points are $+$ parallel or $-$ parallel if and only if they lie on a line of $R_+$ or $R_-$ respectively. We then obtain the classical Minkowski plane.

### 1.2. Flocks of affine planes

The local or derived structure relative to a point $P$ in a circle plane is an affine plane. Notice that each parallelism of the circle plane produces an infinite point or parallel class of the derived affine plane (e.g. [61, pp. 14, 15]). That is, each circle of the circle geometry not on $P$ becomes an oval of the projective extension of the affine plane which intersects the line at infinity of the projective extension in 0, 1, or 2 points respectively as the circle plane is Möbius, Laguerre, or Minkowski.

For example, let $p$ be any point of an elliptic quadric $Q$. Let $\pi_p^+$ be any plane not on $p$. Let $\ell_\infty$ denote the intersection line with the tangent plane to $p$ and $\pi_p^+$. Let $\pi_p$ denote the affine plane $\pi_p^+ - \ell_\infty$. Let $F$ be any $i$-point flock of $Q$. Then, there are the following situations: If $F$ is a 0-point flock then $F$ induces a partition of the points of $\pi_p - M$ by projective conics (conics of the projective plane which lie in the associated affine plane) where $M$ is a line of $\pi_p$.

If $F$ is a 1-point flock excluding the point $t$ and $p = t$ then $F$ induces a partition of the points of $\pi_p$ by projective conics. We call such a covering a projective cover. If $t \neq p$ then $F$ induces a partition of $\pi_p - \{M \cup \{s\}\}$ by projective conics where $M$ is a line of $\pi_p$ and $s$ is a point of $\pi_p$ not incident with $M$. Here we call the cover a non-projective cover.

Similarly, if $F$ is a 2-point flock excluding the points $t$ and $s$ and $p$ is either $t$ or $s$ then $F$ induces a partition of the points of $\pi_p - \{w\}$ by projective conics for $w$ a point of $\pi_p$. In this case, we call the partition a projective partition. If $p$ is neither $t$ or $s$ then $F$ induces a partition of $\pi_p - \{M \cup \{w, r\}\}$ by projective conics where $M$ is a line of $\pi_p$ and $w$ and $r$ are points of $\pi_p$ which are not incident with $M$. We call such a partition a non-projective partition.

Hence, we define 'elliptic' flocks of affine planes as follows:

**Definition 1.3.** Let $\pi$ be any affine plane. A oval of the projective extension of $\pi$ is called an affine oval, tangent oval, or secant oval if and only if the oval shares 0, 1, or 2 points with the line at infinity. Two ovals shall be said to be affinely disjoint if and only if they are disjoint on the affine plane.

**Definition 1.4.** Let $\pi$ be any affine plane.

- A 0-point elliptic flock of $\pi$ is a covering of the points of $\pi - M$ by a set of mutually disjoint affine ovals where $M$ is a line of $\pi$.
- A 1-point projective elliptic flock of $\pi$ is a covering of the points of $\pi$ by a set of mutually disjoint affine ovals.
- A 1-point non-projective elliptic flock of $\pi$ is a covering of the points of $\pi - \{M \cup \{s\}\}$ by a set of mutually disjoint affine ovals for $(s, M)$ a non-incident point-line pair of $\pi$.
- A 2-point projective elliptic flock of $\pi$ is a covering of the points of $\pi - \{w\}$ by a set of mutually disjoint affine ovals.
- A 2-point non-projective elliptic flock of $\pi$ is a covering of the points of $\pi - \{M \cup \{w, r\}\}$ by a set of mutually disjoint affine ovals where $M$ is a line of $\pi$ and $w, r$ are points of $\pi$ which are not incident with $M$.

Now let $H$ be a hyperbolic quadic and $p$ any point of $H$ and $\pi_p^+$ any plane which does not contain $p$ and let $F$ be a flock of $H$. Then, by projection, the conic containing $p$ maps to a line $M$, the lines of the quadric not on $p$ map to the affine lines incident with an infinite point $(\infty)$ or $(0)$ and $F$ induces a partition of $\pi_p - M$ by a set of secant conics sharing the same infinite points $(\infty)$ and $(0)$ where $M$ is not incident with $(\infty)$ or $(0)$.

If $C$ be a quadratic cone and $p$ any point of $C$ and $\pi_p^+$ any plane which does not contain $p$ and $F$ a flock of $C$. Then, by projection, $F$ induces a partition of the points of $\pi_p - M$ by a set of tangent conics sharing the same infinite point $(\infty)$ where $(\infty)$ is not incident with the line $M$ in the projective extension.

**Definition 1.5.** Let $\pi$ be an affine plane.

- A hyperbolic flock of $\pi$ is a partition of the points of $\pi - M$ by a set of mutually affinely disjoint secant ovals sharing the same infinite points $(\infty)$ and $(0)$ where $M$ is a line of $\pi$ not incident with $(\infty)$ or $(0)$.
- A conical flock of $\pi$ is a partition of the points of $\pi - M$ by a set of mutually affinely disjoint tangent ovals sharing the same infinite point $(\infty)$ where $M$ is not incident with $(\infty)$.

### *1.3. Almost classical models*

**Definition 1.6** (*Egg-like or ovoidal*). Let $\Sigma$ be a 3-dimensional projective space over a skewfield $K$. An ovoid of $\Sigma$ is a set of points no three collinear and such that there exists a unique tangent plane at each point $P$. Any Möbius plane obtained via sections of an ovoid is said to be egg-like or ovoidal.

Similarly, we may consider oval cones in $\mathrm{PG}(3,K)$.

**Definition 1.7** (*Oval-cones*). Let $\pi$ be any projective plane. An oval of $\pi$ is a set of points no three collinear and such that there exists a unique tangent line at each point. Let $O$ be an oval in a Desarguesian projective plane $\pi$ over a skewfield $K$ and embed $\pi$ into a $\mathrm{PG}(3,K)$. Let $v$ be a point $\mathrm{PG}(3,K)-\{v\}$ and form the set of lines $v\alpha$ for all $\alpha$ in $O$. This set of points is called an oval cone. Corresponding to the points not equal to $v$ and sections of the oval cone is a Laguerre plane which we call an oval cone Laguerre plane.

**Remark 1.** Flocks of oval-cones, quadratic cones, elliptic quadrics, ovoids, or hyperbolic quadrics are easily obtained by taking the set of non-trivial intersecting planes that share a line. Such flocks of these structures (of the corresponding Möbius, Laguerre, and Minkowski planes) are said to be linear.

As we have seen,

**Remark 2.** (1) A flock of the classical Möbius plane produces a 0-point, 1-point, or 2-point elliptic-flock of any derived affine plane depending on whether the point used in the local structure is covered or not by the flock.

(2) A flock of the classical Laguerre plane provides a conical flock of any associated derived affine plane.

(3) A flock of the classical Minkowski plane provides a hyperbolic flock of any associated local affine plane.

**Remark 3.** (1) A finite Minkowski plane is equivalent to a sharply 3-transitive set of permutations. A slightly weaker geometric structure may be defined for infinite sharply 3-transitive sets of permutations. In either case, in this context a flock is simply a sharply 1-transitive subset of the set of permutations.

(2) There are similar geometric structures based on sharply $k$-transitive sets of permutations. It is possible to consider flocks of these objects via sharply 1-transitive sets.

We end this section with a few questions:

*If an affine plane* $\pi$ *admits an elliptic* (0)-*point*, 1-*point* (*projective or non-projective*) *or* 2-*point* (*projective or non-projective*), *hyperbolic or elliptic flock*, *when* *is there an extension to a circle geometry so that* $\pi$ *is isomorphic to a local structure*?

The reader might be interested in the article of Thas [69] which shows that the Desarguesian planes of odd order have unique one point extensions.

*Are there non-linear flocks of the any of the known Möbius, Laguerre, or Minkowski planes*? *In particular, are there non-linear flocks of the classical or almost classical models*?

## 2. Existence and non-existence results

### 2.1. The classical cases

We first consider the classical cases. This is equivalent to asking if there are non-linear flocks of elliptic or hyperbolic quadrics or of quadratic cones.

#### 2.1.1. Finite order — elliptic

Thas [63] for $q$ even and Orr [55] for $q$ odd have shown that there are no non-linear flocks of an elliptic quadric. Furthermore, Thas pointed out to the author that his argument is valid for any ovoid in $\mathrm{PG}(3,K)$ of even order. Furthermore, Barlotti [8] proved that egg-like inversive planes of odd order are always classical. Hence, there can be no non-linear flock of a finite egg-like inversive plane.

#### 2.1.2. Infinite-order — elliptic.

Actually, in a footnote of [20, p. 255], Dembowski mentions that there is a flock of the elliptic quadric in $\mathrm{PG}(3,R)$ where $R$ is the field of real numbers such that the planes of the flock share one of two lines. That is, the flock is 'bilinear' but not linear.

More generally, Biliotti and myself [11] have used this idea to construct infinitely many flocks of elliptic quadrics over certain ordered fields $K$ such that the planes of the flock share one of $n$ -lines where $n$ is a positive integer and is called an $n$-linear flock. A flock of an elliptic quadric covers all points but 0,1, or 2 points and is called a $i$-flock for $i=0,1,2$ respectively. By varying the ordered fields, it is possible to construct $i$-flocks for any such $i$. Furthermore, it is possible to show that for $K$ the field of real numbers, the number of mutually nonisomorphic 2-linear flocks is infinite.

Also, Riesinger [59] has provided some new examples of non-linear flocks of infinite elliptic quadrics over the field of real numbers.

#### 2.1.3. Finite order-hyperbolic

All hyperbolic flocks of finite order have been determined by the work of Thas [64], Bader and Lunardon [2], Bonisoli [4], and Kallaher [49,50]. Thas has previously given an example of some non-linear flocks which we shall call the Thas hyperbolic flocks. There are exactly three other non-linear flocks of order 11, 23, and 59 due independently to Bader [1], Johnson [33] and Baker and Ebert [5] for 11 and 23. We shall call these the BBEJ flocks.

Hence,

**Theorem 2.1** (Thas [64]; Bader and Lunardon [2]). *A flock of a finite hyperbolic quadric is either linear, a Thas hyperbolic flock or one of the BBEJ flocks.*

It might be noted that the main ingredient of the proof of this classification theorem is the result of Thas [67] that given a flock of a finite hyperbolic quadric of odd order then, for each conic of the flock, there is always an involution which fixes the conic pointwise and leaves the flock invariant. We shall call such flocks **Bol flocks**. So, every finite hyperbolic flock of odd order is a Bol flock which, in turn, is either linear, a Thas flock or a BBEJ flock.

Dembowski [20] proved that corresponding to any finite inversive plane of even order is an ovoid in a $PG(3, 2^a)$ such that points of the ovoid and sections of the ovoid produce the points and circles of the circle plane.

It turns out that the derived affine planes coming from egg-like inversive planes are always Desarguesian. Similarly, this is true for oval type Laguerre planes.

Using the classification of flocks of a finite hyperbolic quadric, Thas [66,69] recently proved.

**Theorem 2.2** (Thas [66,69]). *Any inversive plane of odd order must be classical if there exists merely one derived affine plane which is Desarguesian.*

It might be noted that the idea of the proof comes from the observation of Thas that a hyperbolic flock of an affine Desarguesian plane of odd finite order may be realized as originating from a flock of a hyperbolic quadric and since all of these are either linear or Thas when the order is not $11, 23$, or 59 this provides a framework to study the ovals of the local affine plane inherited from the circles of the inversive plane. Furthermore, the use of the set of involutions leaving the flock invariant also may be used when the order is 11, 23, or 59.

*2.1.4. Infinite-order — hyperbolic*

I recently found that it is possible to construct infinitely many mutually non-isomorphic hyperbolic flocks over certain infinite fields by using suitable André planes. All of the flocks are Bol flocks (see [41]).

However, it is not necessarily the case that all hyperbolic flocks are Bol as Riesinger [60,59] provides some examples of infinite hyperbolic flocks which are not Bol (also see [41] for a proof that the Riesinger flocks are not Bol).

*2.1.5. Finite-order — quadratic cones*

There are many finite examples and there is an entire theory being developed on flocks of quadratic cones. It will become clearer later that there are many connections to these flocks and other geometric objects of interest.

#### 2.1.6. *Infinite-order — quadratic cones*

De Clerck and Van Maldeghem [19] have given several examples of infinite flocks of quadratic cones most of which have finite analogues. Furthermore, in [24], various other new examples are given. In [12], there are some constructions of infinite flocks of quadratic cones such that planes of the flock share one of two lines but not all planes share one line. Thas notes in [65] that any finite flock of a quadratic cone with the property that more than half of the planes of the flock share a line must be linear. Hence, the new infinite examples of Biliotti and Johnson [12] have no finite analogue.

The two-line flocks of Biliotti and Johnson [12] are generalized in [11] to what might be called $n$-linear flocks. Furthermore, there are a wealth of infinite examples in [43]. There are also examples which have finite analogues which may be obtained from generalized quadrangles in [4].

### 2.2. *Almost classical cases*

#### 2.2.1. *Finite-order — ovoidal*

We have noted that there can be no non-linear flocks for finite ovoidal planes.

#### 2.2.2. *Infinite-order — ovoidal*

There have been essentially no studies on flocks of infinite ovoidal Möbius planes.

Open question: *Are there non-linear flocks of any infinite ovoidal nonclassical Möbius plane*?

#### 2.2.3. *Finite-order — oval cones — ( oval cone Laguerre planes)* .

In [21], there is a non-linear flock of a cone built from a translation oval which is due to Thas. We shall call the class the set of *translation oval flocks of Thas.*

Recently, Cherowitzo [16] has constructed several new families of flocks of translation oval cones which are related to the translation oval flocks of Thas.

#### 2.2.4. *Infinite-order-oval cones*

Based on the theory of translation oval cones of Cherowitzo, it is possible to consider infinite flocks of translation oval cones over infinite fields of characteristic two. In particular, it is possible to define infinite analogues of the translation oval flocks of Thas. For example, when $\sigma$ is any field automorphism of a perfect field of characteristic 2 such that the mappings $x \to x^{\sigma-1}$ and $x \to x^{\sigma+1}$ are bijective, then there is an infinite flock which admits a doubly transitive automorphism group acting on the planes of the flock (see [17]).

In both the finite and infinite cases, the only known examples over $\mathrm{PG}(3,K)$, for $K$ a field of characteristic 2, are constructed on translation oval cones based on an automorphism of the field. That is, there is an automorphism $\sigma$ such that the cone is represented by the equation $x^{\sigma-1}z = y^{\sigma}$ while representing the points in $\mathrm{PG}(3,K)$ in the form $(x, y, z, w)$ and the field $K$ has characteristic two.

There are infinitely many flocks of semi-elliptic cones which correspond to flocks of quadratic cones in the sense that the same planes can be used to cover both cones. A semi-elliptic oval in $\mathrm{PG}(3,R)$ where $R$ is the field of real numbers is an oval obtained by glueing together two semi-ellipses with the same major axis. (see [43]). For example, let $O_a = \{(1,t^2,0); t \geqslant 0, t \in R\} \cup \{(1,at^2,0); t < 0, t \in R\}$ for $a > 0$ be an oval in $\mathrm{PG}(3,R)$ which we shall call a $\{a\}$-semi-elliptic oval. The cone obtained by projecting from $(0,0,0,1)$ to $O_a$ is called the $\{a\}$-semi-elliptic oval cone. We present $\mathrm{PG}(3,R)$ by homogeneous coordinates $(x_o,x_1,x_2,x_3)$ for $x_i \in R$ and $i = 1,2,3,4$.

**Theorem 2.3** (Johnson and Liu [43]). *Let $x_o x_1 = x_2^2$ define a quadratic cone with vertex $(0,0,0,1)$ in $\mathrm{PG}(3,R)$ where $R$ is the field of read numbers.*

*If $F = \{\{\pi_t; tx_o + f(t)x_1 + x_3 = 0\}; t \text{ in } R\}$ defines a flock of the quadratic cone in $\mathrm{PG}(3,R)$ then the same planes define a flock of any $\{a\}$-semi-elliptic oval cone for $a > 0$.*

Undoubtedly, there are many non-linear flocks over fields of odd or zero characteristic over oval cones which are not quadratic, translation, or semi-elliptic.

Also, it might be possible to consider flocks of various objects in $\mathrm{PG}(3,K)$ where $K$ is a skewfield but not a field. Again, there have been essentially no studies of such objects.

### 2.3. The remaining types and packings

There have been no studies of flocks of non-oval type Laguerre planes, however, there is a class of Minkowski planes for which flocks do occur.

As mentioned above, finite Minkowski planes are equivalent to sharply 3-transitive sets of permutations. Acting on $\mathrm{GF}(q)$, $q$ odd, and for a fixed automorphism $\sigma$, the following mappings $x \to (x^\tau a+b)/(x^\tau c+d)$ such that $\tau=\sigma$ or 1 if and only if $ad-bc$ is nonsquare or nonzero square respectively, determine a sharply 3-transitive set of permutations. Denote the corresponding Minkowski plane by $M(q)^\sigma$.

In [14], Bonisoli constructs flocks of $M(q)^\sigma$ as follows. Let $E$ denote a cyclic subgroup of order $(q+1)/2$ and $g$ an element of $\mathrm{P\Gamma L}(2,q)$ that normalizes $E$. Then Bonisoli determines $g$'s such that $E \cup Eg$ is a sharply transitive subset of the above set of mappings. This provides a flock of the corresponding Minkowski plane. Furthermore, by taking the images under $\mathrm{PG}(2,q)$, there are packings or resolutions constructed.

Since we have noted that it is possible to construct packings of finite Minkowski planes, we end this section with some questions.

*Are there packings of any finite Laguerre plane*?

We also may ask a somewhat ambiguous but related question.

*If $\pi$ is an affine plane with sufficiently many flocks*, *is it almost classical*?

For example, considering what would be the case of oval type Laguerre planes, we could ask:

*If $\pi$ is an affine plane such that, for every line M, there is a conical flock relative to M, is $\pi$ Desarguesian and is it a derived affine plane of an oval type Laguerre plane?*

Similarly,

*if $\pi$ is an affine plane such that, for every line M, there is a hyperbolic flock relative to M, is $\pi$ Desarguesian and is it a derived affine plane of a classical Minkowski plane?*

## 3. Connections to combinatorial structures

Undoubtedly, the current interest in flocks in general arises from the fact that corresponding to flocks are certain translation planes and esoteric projective planes, various generalized quadrangles, and types of families of ovals.

In particular, a flock of an elliptic or hyperbolic quadric or quadratic cone in $\mathrm{PG}(3,K)$ for $K$ an arbitrary field produces a translation plane with spread in $\mathrm{PG}(3,K)$ as follows:

Embed the quadric or quadratic cone in the Klein quadric $Q_5$ of $\mathrm{PG}(5,K)$, take the polar planes of the planes containing the conics of the flock and consider the conics of intersection with $Q_5$ and the polar planes. It turns out that each set of intersections defines an ovoid in $\mathrm{PG}(5,K)$ which then determines a spread of lines in $\mathrm{PG}(3,K)$ using Plücker coordinates. This construction when $K$ is finite is due independently to Thas and Walker [70] and is called the Thas–Walker construction.

### *3.1. Translation planes and flocks of classical circle planes*

The following result turns out to be valid for arbitrary fields.

**Theorem 3.1** (Biliotti and Johnson [11]; Gevaert and Johnson [22]; Gevaert et al. [23]; Jha and Johnson [24]; Johnson [33,41]).

(1) *A translation plane with spread in* $\mathrm{PG}(3,K)$ *corresponds to a flock of a quadratic cone if and only if*

(1.1) *the spread is a union of reguli that mutually share a line or equivalently*

(1.2) *there is an elation group E in the translation complement such that some orbit of components union the axis is a regulus in* $\mathrm{PG}(3,K)$.

(2) *A translation plane with spread in* $\mathrm{PG}(3,K)$ *corresponds to a flock of a hyperbolic quadric if and only if*

(2.1) *the spread is a union of reguli that mutually share two lines or equivalently*

(2.2) *there is a homology group H in the translation complement such that some orbit of components union the axis and coaxis is a regulus in* $\mathrm{PG}(3,K)$.

(3) *A translation plane $\pi$ with spread in* PG$(3,K)$ *corresponds to a flock of an elliptic quadric if and only if there is a Pappian plane $\Sigma$, a set $D$ of mutually disjoint reguli and at most two components $L$, $M$ such that the spread for $\Sigma$ is $D \cup \{L,M\}$ and the plane $\pi$ is the plane obtained by multiply deriving $D$.*

### *3.2. Generalized quadrangles and flocks of quadratic cones*

Thas [65] noted that the algebraic requirements for a flock of a finite quadratic cone in PG$(3,q)$ are exactly those which define a generalized quadrangle of type $(q^2,q)$ constructed by use of a $q$-clan via the coset geometry method of Kantor (see [51] or [46]).

When $q$ is odd, this connection is also explained by the results of Knarr [52].

However, when $q$ is even only the algebraic connections are presently available.

When there is an infinite flock of a quadratic cone, there does not necessarily correspond a generalized quadrangle (see [19]).

### *3.3. Projective planes with a single incident point-line transitivity*

Given a flock of a quadratic cone, there is a corresponding translation plane $\pi$ with spread in PG$(3,K)$, where $K$ is a field, that admits an elation group $E$ whose orbits define reguli. The natural affine restriction of the dual of the projective extension of $\pi$ is derivable and the derived plane admits an incident point-line transitivity.

In the finite case, if the original translation plane is not a semifield plane then results on the derivation of finite translation planes (see [34,36]) show that the derived plane from the dual translation plane actually admits exactly one point-line transitivity which forces the plane to be in Lenz-Barlotti class II-1.

In the infinite case, this result is probably still valid but nothing much is known about such planes. We leave this as an open question:

*If a flock of a quadratic cone in* PG$(3,K)$, *for $K$ an infinite field, corresponds to a translation plane which is not a semifield plane, is a derived plane of a natural dual translation plane of Lenz-Barlotti class* II-1?

### *3.4. Herds of ovals and flocks of quadratic cones*

When $q$ is even, given a flock of a quadratic cone in PG$(3,q)$ there is associated a set of $q^2-1$ ovals mutually sharing two points and their knots. Such a family is called a herd of ovals. The ovals are defined via polynomial functions related with the equations of the planes of the flock. Conversely, using similar functions to define a herd, there is a corresponding flock of a quadratic cone. The reader is referred to Cherowitzo et al. [18] for a development of these concepts.

Since it is still possible to define infinite flocks of quadratic cones over perfect fields $K$ of characteristic 2, we may ask the following question:

*For infinite flocks of quadratic cones over fields of characteristic* 2, *is there a corresponding herd of ovals*?

### *3.5. Herds of ovals and flocks of translation oval cones*

In the finite case, Cherowitzo [16] shows that, for many classes of flocks of translation oval cones, it is possible to associate a family (herd) of ovals. Moreover, the general study of such flocks provides a convenient vehicle to analyze ovals in finite Desarguesian planes. In particular, Cherowitzo is able to show that the 'Cherowitzo' ovals form an infinite class using this technique.

*Do herds of ovals exist in any infinite flock of an oval type Laguerre plane*?

### *3.6. Flocks of Minkowski planes and translation planes covered by subplane covered nets*

Recall that a flock of a Minkowski plane of type $M(q)^{\sigma}$ is a sharply transitive subset of a particular subset of $\mathrm{P\Gamma L}(2,q)$. Knarr [53] showed that corresponding to any sharply transitive subset of $\mathrm{P\Gamma L}(2,q)$ is a translation plane whose spread is the union of a set of derivable nets sharing two components. Furthermore, in [39], it is shown that corresponding to any sharply transitive set of $\mathrm{P\Gamma L}(n,q)$ acting on the points of $\mathrm{PG}(n-1,q)$, there is a translation plane which is covered by subplane covered nets sharing two components.

The idea connecting the correspondence is as follows: Let a sharply transitive set of $\mathrm{P\Gamma L}(n,q)$ be realized by mappings of the form $z \to z^{\sigma}M$ where $M$ is a $n \times n$ matrix over $\mathrm{GF}(q)$ and $z=(x_1,x_2,\ldots,x_n)$ for $x_i$ in $\mathrm{GF}(q)$, $i=1,2,\ldots,n$ and $z^{\sigma}=(x_1^{\sigma},x_2^{\sigma},\ldots,x_n^{\sigma})$ for $\sigma$ an automorphism of $\mathrm{GF}(q)$. Then it is shown in [39] that

$$z \to z^{\sigma}MuI_2 \quad \text{for all } u \text{ in } K-\{0\}$$

is a sharply transitive set of $\mathrm{GL}(n,q)$ acting on a corresponding $n$-dimensional vector space over $\mathrm{GF}(q)$. It is well known that such a subset of $\mathrm{GL}(n,q)$ is equivalent to a translation plane of order $q^n$ whose kernel is the intersection of the fixed fields of the automorphisms associated with the original sharply transitive set. Note that this also provides a simple algebraic proof that flocks of hyperbolic quadrics correspond to translation planes whose spread is a union of reguli sharing two lines.

The flocks $E \cup Eg$ of Bonisoli of $M(q)^{\sigma}$ thus produce a class of translation planes of order $q^2$ admitting two affine homology groups of order $(q^2-1)/2$ coming from the mappings $EuI_2$ for all $u$ in $K-\{0\}$. Actually, it is shown in Johnson [38] that these translation planes are certain André planes.

Note that corresponding to sharply $k$-transitive sets are certain geometric structures and if considered within $P\Gamma L(n,K)$, for $K$ a field, then flocks of such structures might actually correspond to a class of translation planes.

## 4. Partial flocks and translation planes admitting Baer groups

**Definition 4.1.** A partial flock of a circle plane is a set of mutually disjoint circles. A partial flock of $t$ circles of a finite circle plane of order $n$ has deficiency $(n-1-t),(n-t)$, or $(n+1-t)$ respectively as the circle plane is Möbius, Laguerre, or Minkowski.

In order to define deficiency for infinite flocks, we consider only circle planes which have a nontrivial point parallelism and define a sort of regularity condition.

**Definition 4.2.** A partial flock $F$ of deficiency $t$ of a circle plane with a set of nontrivial point parallelisms is a partial flock such that for each point parallelism equivalence class $c$ then

$$t = \mathrm{card}\{c - c \cap F\}.$$

When $t$ is finite, exactly $t$ points on each point parallelism equivalence class are not covered by circles of the partial flock. In this case, the previous two definitions are equivalent.

**Definition 4.3.** A partial flock of a circle plane shall be said to be maximal if and only if no extension of the set is a partial flock. A partial flock of a circle plane with nontrivial point parallelism is said to be a class-covered partial flock if and only there exists a point equivalence class $c$ which is covered by the circles of the partial flock.

### *4.1. Deficiency one*

It turns out that deficiency one partial flocks of certain circle planes correspond to translation planes admitting Baer groups. This was considered for finite flocks in [35] and more generally for arbitrary flocks in [31,30].

We remind the reader that a Baer subplane $\pi_o^+$ of a projective plane $\pi^+$ is a subplane with the properties that given a point $P$ of the plane, there is a line of the subplane $\pi_o^+$ incident with $P$ and given a line $\ell$ of the plane, there is a point $Q$ of the subplane $\pi_o^+$ which is incident with $\ell$. Affine Baer subplanes are affine subplanes whose projective extensions are Baer.

**Theorem 4.4** (Johnson [35] and Jha and Johnson [31,30]).

(1)(a) *The set of partial flocks of deficiency one of a quadratic cone in* $PG(3,K)$, *for* $K$ *a field, is equivalent to the set of translation planes with spread in* $PG(3,K)$ *that admit a Baer collineation group* $B$ *which acts regularly on the* 1*-dimensional* $K$*-subspaces not on* $\mathrm{Fix}\,B$ *of any component of* $\mathrm{Fix}\,B$.

(1)(b) *A partial flock of deficiency one of a quadratic cone may be extended to a flock if and only if in the corresponding translation plane the net defined by the components of* Fix $B$ *is a K-regulus net.*

(2)(a) *The set of partial flocks of deficiency one of a hyperbolic quadric in* $\mathrm{PG}(3,K)$ *for K a field, is equivalent to the set of translation planes with spread in* $\mathrm{PG}(3,K)$ *that admit a Baer collineation group B which fixes another Baer subplane called the* coFix $B$ *in the net defined by* Fix $B$ *such that B is regular on the* 1*-dimensional K-subspaces not on* $\mathrm{Fix}\,B \cup \mathrm{coFix}\,B$ *on components of* Fix $B$.

(2)(b) *A partial flock of deficiency one of a hyperbolic quadric in* $\mathrm{PG}(3,K)$, *for K a field, may be extended to a flock if and only if in the corresponding translation plane the net defined by* Fix $B$ *is a K-regulus net.*

With partial flocks of deficiency one of Minkowski planes $M(q)^{\sigma}$, it also possible to associate a translation plane. More generally,

**Theorem 4.5** (Johnson [39]). *There is a* 1–1 *correspondence between partially sharp subsets of* $\mathrm{P\Gamma L}(2,q)$ *of deficiency one and translation planes of order* $q^2$ *that admit two distinct Baer groups of order* $q-1$ *which have the same component orbits.The partially sharp set can be extended to a flock* (*sharply transitive set*) *if and only if in the corresponding translation plane the net defined by one of the Baer groups is derivable.*

There are partial flocks of deficiency one of hyperbolic quadrics in $\mathrm{PG}(3,q)$ where $q=4,5$, or 9 due respectively to Johnson [33], Johnson and Pomareda [47], and Biliotti and Johnson [10].

We thus may pose some questions:

*Given an infinite field K, do partial flocks of deficiency one of quadratic cones in* $\mathrm{PG}(3,K)$ *exist which cannot be extended to flocks*?

*Given an infinite field K, are there infinite partial flocks of deficiency one of hyperbolic quadrics in* $\mathrm{PG}(3,K)$ *which cannot be extended to flocks*?

There are some plane representations which might be mentioned here. If a partial flock of deficiency one of a quadratic cone is projected via a ‘missing’ point onto a plane not containing the missing point, there is an associated Desarguesian affine plane $\pi$ whose points of $\pi - \mathscr{M}$ are covered by a set of mutually affinely disjoint conics that share an infinite point $(\infty)$ and where $\mathscr{M}$ is not necessarily a line of $\pi$ but every affine line of $(\infty)$ contains exactly one point of $\mathscr{M}$. Hence, we may pose a question?

*Let* $\pi$ *be an affine plane and a set* $\mathscr{M}$ *of points and a set of tangential ovals to* $(\infty)$ *such that the set partitions the points of* $\pi-\mathscr{M}$ *and each line of* $(\infty)$ *is incident with a unique point of* $\mathscr{M}$ *and* $\mathscr{M}$ *is defined by these intersections. We call such a cover a partial elliptic flock of deficiency one. Is* $\mathscr{M}$ *a line*? *Equivalently*, *are partial elliptic flocks of deficiency one flocks*?

### 4.2. Class-covered partial flocks and quasifibrations

Of course, a finite class-covered partial flock is a flock. However, in the infinite case, it is possible to have a proper class-covered partial flock.

We have noted that it is possible to associate with certain partial flocks various partial spreads. Thus, we may also consider 'class-covered' partial spreads or even more generally 'class-covered' nets both finite and infinite.

**Definition 4.6.** Let $N$ be an arbitrary net. Let $P$ be an affine point and $L_P$ the set of lines which are incident with $P$. Let $\alpha$ be any parallel class and let $M$ be a line of $\alpha$ which is not incident with $P$. The net $N$ shall be said to be class-covered with respect to $M$ and $P$ if and only if the points of $M$ are contained in the intersections of $M$ with $L_P$.

In [25], a study of such nets is considered. In particular, either the net is an affine plane or is a maximal net in the sense that there are no transversals containing $P$. Furthermore, this may be considered for translation nets and/or vector space nets.

**Definition 4.7.** Let $V$ be a vector space of the form $W \oplus W$ where $W$ is a $K$-space for a skewfield $K$. A partial spread is a set of subspaces $K$-isomorphic to $W$ such that the direct sum of any two distinct subspaces is $V$.

Let $B$ be a basis for $W$ and choose a fixed vector $e$ from $W \oplus 0$. A quasifibration $Q$ is a set of mutually disjoint subspaces $K$-isomorphic to $W$ which contains $W \oplus 0$ and $0 \oplus W$ and which has the property that for each vector of the form $e + w$ for $w$ in $0 \oplus W$, there is a subspace of $Q$ which contains $e + w$. Furthermore, $K$ is called the kernel of the quasifibration if and only if $K$ is maximal among skewfields $L$ such that the subspaces indicated are $L$-spaces.

Then

**Theorem 4.8** (Jha and Johnson [25]). *A quasifibration is either a spread or a proper maximal partial spread.*

Class-covered partial flocks of quadratic cones or of hyperbolic quadrics in $\mathrm{PG}(3,K)$ correspond to certain quasifibrations.

**Theorem 4.9.** (1) *The set of class-covered partial flocks of a quadratic cone in* $\mathrm{PG}(3,K)$, *for $K$ a field, is equivalent to the set of quasifibrations whose corresponding translation nets admit an elation group such that any component orbit union the axis forms a regulus in* $\mathrm{PG}(3,K)$.

(2) *The set of class-covered partial flocks of a hyperbolic quadric in* $\mathrm{PG}(3,K)$ *is equivalent to the set of quasifibrations whose corresponding translation net admits a*

*homology group such that any orbit of components union the axis and coaxis is a regulus in* PG$(3,K)$.

There are quite a variety of examples of proper class-covered partial flocks in [25]. For example,

**Example 4.10.** Let $K$ be a field in which the function $C$ defined by $C(x)=x^3$ is $1-1$ but not onto. Let $V_4$ denote a 4-dimensional vector space over $K$. Then the following equations define a quasifibration which is a proper maximal partial spread and corresponds to a class-covered partial flock of a quadratic cone:

$$x=0,\quad y=0,\quad y=x\begin{bmatrix} u-s^2 & -s^3/3 \\ s & u \end{bmatrix} \text{ for all } u,s \text{ in } K.$$

As the above example over the reals provides a topological projective plane of Betten, we call this example a generalized Betten quasifibration.

We have seen that we may associate certain partial spreads and partial flocks of circle planes. For example, a partial flock of a quadratic cone corresponds to a partial spread which is made up of a set of reguli that share a component. We may ask if there is a spread containing a partial spread of this type.

*If $\pi$ is a translation plane with spread in* PG$(3,q)$ *which contains a set of $t$ reguli that share a line or a set of $k$ reguli that share two lines, what is a bound on $t$ or $k$ so that $\pi$ is not a translation plane associated with a flock of a quadratic cone or associated with a flock of a hyperbolic quadric*?

## 5. Extension theory and derivation

Bader et al. [3] have shown that given a flock of a quadratic cone of odd order in PG$(3,q)$ there is actually a set of $q+1$ flocks of a quadratic cone. Furthermore, the argument is valid for partial flocks of $t$ conics then producing a set of $t+1$ partial flocks of $t$ conics.

The construction, called 'derivation', is as follows:

Let $q$ be odd and let $\Sigma_3 \simeq \mathrm{PG}(3,q)$ and contained in $\Sigma_4 \simeq \mathrm{PG}(4,q)$ in such a way so that there is a quadric $Q_4$ in $\Sigma_4$ such that $\Sigma_3 \cap Q_4$ is a quadratic cone $H_o$ with vertex $p_o$. Let $\{C_i$ for $i=1,2,\dots,q\}$ be a flock of $H_o$ and let $\pi_i$ denote the plane of $H_o$ containing $C_i$. Let $\perp$ denote the polarity of $\Sigma_4$ associated with $Q_4$. Then $\pi_i^{\perp}=\langle p_o\rangle \oplus \langle p_i\rangle$ as vector spaces where $p_i$ is in $Q_4$. Then $p_i^{\perp}=H_i$ is a 3-dimensional projective space and for each $i=1,2,\dots,q$,

$$\{H_i \cap H_j \text{ for } j=1,2,\dots,q \text{ and } j\neq i\} \cup \pi_i$$

determines a flock of the quadratic cone $Q_4 \cap H_i$ with vertex $p_i$.

The construction is valid for partial flocks (also see [37]) and so a partial flock of a quadratic cone of deficiency $t$ in $\mathrm{PG}(3,q)$ produces a set of $q+1-t$ partial flocks of deficiency $t$ of various $\mathrm{PG}(3,q)$'s in $\mathrm{PG}(4,q)$.

Payne and Thas [58] use this construction to show that any partial flock of deficiency one of a quadratic cone in $\mathrm{PG}(3,q)$, for $q$ odd, may be uniquely extended to a flock.

Furthermore, Payne and Thas also show that a partial flock of deficiency one of even order may be extended uniquely to a flock. Obviously, the argument for $q$ even cannot involve the Bader–Lunardon–Thas construction and, actually, involves the knots of the associated $q$ conics.

More generally, Liu [54] shows that any partial flock of deficiency one of any oval cone in $\mathrm{PG}(3,2^a)$ may be uniquely extended to a flock.

### 5.1. *Derivation and generalized quadrangles*

Payne and Rogers [57] show that the $q+1$ flocks associated with a given flock using the above derivation process correspond to the same generalized quadrangle or more exactly correspond to recoordinatizations of the same generalized quadrangle. Using the idea that recoordinatizations can lead to a set of flocks, one may define a set of $q+1$ flocks associated with a given flock of even order even though there is no known supporting geometry.

Moreover, De Clerck and Van Maldeghem [19] consider infinite derivation using this coordinate approach and show

**Theorem 5.1** (De Clerck and Van Maldeghem [19]). *A generalized quadrangle is associated with a flock of a quadratic cone in* $\mathrm{PG}(3,K)$ *if and only if all of the derivations are flocks.*

Actually, it turns out that not all flocks produce generalized quadrangles. In [12] as there are examples of nonlinear flocks whose planes share one of two lines defined over ordered fields $K$. It is possible to find such fields $K$ such that such bilinear flocks exist in $\mathrm{PG}(3,K)$ but do not correspond to generalized quadrangles.

Note that something similar can be accomplished with translation planes admitting Baer groups.

**Theorem 5.2.** *Let* $\pi$ *be a translation plane with spread in* $\mathrm{PG}(3,K)$ *that admits a Baer group* $B$ *which acts regularly on the* 1*-dimensional* $K$*-subspaces on components of* $\mathrm{Fix}\,B$. *Let* $F$ *be the associated partial flock of deficiency one in* $\mathrm{PG}(3,K)$ *for* $K$ *a field.*

*Then, for each derivation which is a partial flock, there is an associated translation plane with spread in* $\mathrm{PG}(3,K)$ *that admits a Baer group* $B$ *which acts regularly on the* 1*-dimensional* $K$*-subspaces on components of* $\mathrm{Fix}\,B$.

We may pose the following question:

*Assume the hypothesis of the above result and further assume that all derivations are partial flocks. Is it possible to extend the partial flock to a flock*?

Also considering derivation, we may formulate questions on the nature of spreads containing various sets of reguli.

*Assume F is a partial flock of a quadratic cone in* $\mathrm{PG}(3,K)$ *such that each derivation is a partial flock. Assume that each such partial flock produces a partial spread that can be embedded in a spread in* $\mathrm{PG}(3,K)$. *Classify the spreads.*

### 5.2. Herds of ovals and extensions

*As mentioned previously, for each partial flock of k conics of a quadratic cone in* $\mathrm{PG}(3,2^a)$, *there is a set of* $2^{2a}-1$ $(k+2)$*-arcs that share two points and their knots. If k is large enough, such arcs can be uniquely extended to hyperovals and if these hyperovals are shown to have appropriate properties then there is a corresponding flock of a quadratic cone.*

This approach is used in [62] to prove the following result.

**Theorem 5.3** (Storme and Thas [62]). *A partial flock of a quadratic cone in* $\mathrm{PG}(3,2^a)$ *of k conics may be extended uniquely to a flock provided*
(1) *for a even,* $k>2^a-2^{a/2}-1$ *and*
(2) *for a odd,* $k>2^a-2^{(a+1)/2}$.

For a more general setting, a partial flock of $k$ conics of a translation oval cone in $\mathrm{PG}(3,2^a)$ may or may not give rise to a herd of $(k+2)$-arcs (see [16]) but when it does, essentially the same argument shows:

**Theorem 5.4** (Cherowitzo et al. [17]). *Let F be a partial flock of a translation oval cone in* $\mathrm{PG}(3,2^a)$ *of k conics such that there is an associated herd of* $(k+2)$*-arcs. If k satisfies the conditions of the previous result then F may be uniquely extended to a flock.*

### 5.3. Partial flocks and spreads

If one assumes that a partial flock of $k$ conics is associated with a partial spread which is contained within a spread then there are upper bounds for $k$ so that the associated spread is not, in fact, associated with a flock. Roughly speaking, for “nice” sets of reguli in spreads within $\mathrm{PG}(3,q)$ the bound is $3q/4$. The exact formulation is as follows:

**Theorem 5.5** (Johnson and Storme [48]). Elliptic case : *Let* $F^t$ *for* $1\leqslant t\leqslant q-1$ *be a partial spread of cardinality* $t(q+1)$ *in* $\mathrm{PG}(3,q)$ *consisting of t reguli which share* *two conjugate lines $L$ and $L^q$ of* PG$(3,q^2)$ − PG$(3,q)$ *and furthermore assume that $F^t$ is contained in a spread $S$ of* PG$(3,q)$.

*If either*

(1) *$q$ is even and $t \geqslant (3q-1)/4$,*

(2) *$q \equiv 1$ mod 4 and $t \geqslant (3q-3)/4$, or*

(3) *$q \equiv 3$ mod 4 and $t \geqslant (3q-5)/4$*

*then $S$ is subregular; that is, $S$ can be obtained by multiply deriving a set of mutually disjoint reguli in a Desarguesian spread.*

*Hyperbolic case: Let a spread $S$ in* PG$(3,q)$ *contain at least $k$ reguli that mutually share two lines. When $q=3,4$, or 5 assume that $k \geqslant (q-1)$. Otherwise, assume*

(1) *if $q$ is even that $k \geqslant (3q/4)+1$,*

(2) *if $q \equiv 1$ mod 4 then $k \geqslant (3q+5)/4$ or*

(3) *if $q \equiv 3$ mod 4 then $k \geqslant (3q+3)/4$.*

*Then $S$ consists of exactly $(q+1)$ reguli that mutually share two lines and corresponds to a flock of a hyperbolic quadric.*

*Quadratic case: Let a spread $S$ in* PG$(3,q)$ *contain at least $k$ reguli that mutually share exactly one line. When $q=3,4$, or 5 assume that $k \geqslant (q-2)$. Otherwise, assume*

(1) *if $q$ is even that $k \geqslant (3q/4)$,*

(2) *if $q \equiv 1$ mod 4 then $k \geqslant (3q+1)/4$ or*

(3) *if $q \equiv 3$ mod 4 then $k \geqslant (3q-1)/4$.*

*Then $S$ consists of exactly $q$ reguli that mutually share a line and corresponds to a flock of a quadratic cone.*

Payne and Thas [58] show that a flock of a quadratic cone in PG$(3,q)$, for $q$ odd, which contains a linear subflock of at least $(q-1)/2$ conics may be uniquely extended to a flock. More generally, one may ask the following question:

*If a partial flock $F$ of a quadratic cone in* PG$(3,q)$ *properly contains a linear subflock of at least $(q-1)/2$ conics, is it possible to extend $F$?*

Similarly, one notes that the regular near-field planes admit two linear sets of $(q+1)/2$ reguli that share two lines and so the corresponding Thas flock contains two linear partial flocks of $(q+1)/2$ conics.

*If a partial flock $F$ of a hyperbolic quadric in* PG$(3,q)$ *properly contains a linear subflock of at least $(q+1)/2$ conics, is it possible to extend $F$?*

### 5.4. Nests of reguli and flocks

Recall that a *$t$-nest of reguli* is a set of $t$ reguli in PG$(3,q)$ such that each line of the union is in exactly two reguli of the set (see [6]). Baker and Ebert (see e.g. [6,7], and the references to Baker and Ebert in [42]) construct translation planes admitting various configurations of $t$-nests where $t=(q-1), q, (q+1)$, and $2(q-1)$.

In particular, Payne [56] showed that the translation planes corresponding to the $q$-nests of Baker and Ebert turn out to be associated with flocks of quadratic cones. (Payne also extended the original construction of Baker and Ebert from planes of order $p^2$ for $p$ a prime to all odd prime powers $p^{2r}$.)

What actually occurs is that one finds a Baer subplane $\pi_o$ in a Desarguesian affine plane of order $q^2$ which sits in exactly $(q+1)/2$ of the set of $q$ reguli that share a line of a corresponding linear flock of a quadratic cone. Then the set of images $\pi_o G$, where $G=EH$ where $E$ is the regulus inducing elation group mentioned previously and $H$ is the kernel homology subgroup of squares, defines a replacement partial spread for the intersection set of reguli (with the exception of the common line). The $q$-nest of reguli is then $N_{\pi_o}G$ where $N_{\pi_o}$ is the unique regulus net containing $\pi_o$. The corresponding flocks are called the Fisher flocks.

Using this general ideal, one can actually extend any partial flock which contains a linear subflock of at least $(q-1)/2$ conics.

**Theorem 5.6** (Payne and Thas [58], Johnson [40]). *Let $F$ be any partial flock of a quadratic cone in* PG$(3,q)$ *for $q$ odd.*

*If $F$ properly contains a linear subflock of $(q-1)/2$ conics then $F$ may be uniquely extended to a flock. Moreover, the flock is either linear or a Fisher flock.*

## 6. Classification results

We have mentioned the general result of the classification of flocks of finite hyperbolic quadrics.

One of the first classification results for finite quadratic cones is a theorem of Thas [65].

**Theorem 6.1** (Thas [65]). *If the planes of a flock of a quadratic cone in* PG$(3,q)$ *share a common point then either the flock is linear or $q$ is odd and the flock is a Kantor–Knuth flock.*

Actually, when $q$ is even, Storme and Thas [62] give a short proof based on herds of ovals. If one considers flocks of translation oval cones which possess an assumed herd of ovals then the same proof works more generally.

**Theorem 6.2** (Storme-Thas [62]; Cherowitzo et al. [17]). If the planes of a flock of a translation oval cone in PG$(3,2^a)$ share a common point and there is an associated herd of ovals then the flock is linear.

### *6.1. Rigidity*

A group of a partial flock of $k$ conics which fixes each plane containing the conics in question is said to be a locally rigid group. If the group acts as an automorphism group of a flock containing the partial flock, we use simply the term $k$-rigid. It turns out that there is a fundamental connection between locally rigid groups and partial flocks whose planes share a common point.

**Theorem 6.3** (Jha and Johnson [27]). *The planes of a partial flock $P$ of $s$ conics of a quadratic cone in* $\mathrm{PG}(3,q)$ *share a common point if and only if there exists a nontrivial linear $s$-locally rigid group on $P$.*

It is possible to characterize the Kantor–Knuth flocks and Fisher flocks using rigidity.

**Theorem 6.4** (Jha and Johnson [27]). (1) *If a flock of a quadratic cone admits a nontrivial rigid group then the flock is either linear or of Kantor–Knuth type.*

(2) *If a flock of a quadratic cone admits a nontrivial* $(q-1)/2$*-rigid automorphism group of order* $>2$ *then the flock is either linear or a flock of Fisher.*

Actually, using this technique, it is possible to improve the classification of flocks whose planes share a point as follows:

**Theorem 6.5** (Jha and Johnson [27]). *If at least* $q/2$ *planes of a flock of a quadratic cone in* $\mathrm{PG}(3,q)$, *for $q$ even, share a point then the flock is linear.*

Since we have a herd of ovals, we may ask the following question:

*Is there a way to prove the above result with herds of ovals assuming that at least half of the planes of the flock share a point*?

## 6.2. *Semifield flocks*

Considering planes associated with flocks of quadratic cones, it may be possible that the plane is a semifield plane as is the case for example when the flock is a Kantor–Knuth flock of odd order.

In Johnson [32], it is shown that there are no 'semifield' flocks of even order except the linear ones. Note that any such generalized quadrangle corresponding a semifield flock of either even or odd order is a translation generalized quadrangle.

Also in Storme and Thas [62], there is a proof based upon herds of ovals which shows that any translation generalized quadrangle of even order must be classical. Corresponding to any flock in $\mathrm{PG}(3,K)$ of a quadratic cone $x_ox_1=x_2^2$ representing the points homogeneously as $(x_o,x_1,x_2,x_3)$ and vertex by $(0,0,0,1)$, the planes have the following form:

$$tx_o - f(t)x_1 + g(t)x_2 + x_3 = 0 \quad \text{for all } t \text{ in a field } K.$$

More generally, the planes of a flock in $\mathrm{PG}(3,K)$ of a translation oval cone $x_o^{\sigma-1}x_1=x_2^{\sigma}$ where $\sigma$ is an automorphism of $K$ and the mapping $z\to z^{\sigma-1}$ is bijective may be similarly represented.

A semifield flock of a translation oval cone is one where the mappings $f(t)$ and $g(t)$ are both additive. In the case of quadratic cones, the corresponding translation plane is a semifield plane.

A proof following Storme and Thas [62] in the finite case may be given if one assumes that a semifield flock has an associated herd of ovals. Moreover, even without this assumption, one may use trace arguments to show the flocks are always linear in the even order case.

**Theorem 6.6** (Storme and Thas [62]; Cherowitzo et al. [17]). *Let $F$ be a semifield flock of a translation oval cone in* $\mathrm{PG}(3,2^a)$. *Then $F$ is linear.*

### 6.3. Maximal partial spreads and flocks

Another direction that one may take is to consider the construction of maximal partial spreads from spreads in $\mathrm{PG}(3,q)$ that correspond to flocks. The idea goes back to Bruen and roughly is as follows. If $\pi$ is a nonDesarguesian translation plane with spread in $\mathrm{PG}(3,K \simeq GF(q))$, choose a Baer subplane $\pi_o$ which is a $K$-subspace such that the net determined by $\pi_o$ is not a $K$-regulus net. It turns out that there are either one or two Baer $K$-subspaces in this net. One may then define a maximal partial spread by taking the spread components to be the spread lines not intersecting $\pi_o$ and the possibly one other $K$-Baer subplane of the net defined by $\pi_o$. The question then becomes whether there are actually “two” Baer subplanes in the net. The following uses the results on “rigidity” in the proofs.

**Theorem 6.7** (Johnson and Lunardon [45]). *Let $F$ be a flock of a quadratic cone of* $\mathrm{PG}(3,q)$ *and let $S_F$ denote the corresponding spread in* $\mathrm{PG}(3,q)$. *The reguli of the corresponding spread that correspond to the conics of the flock are called base reguli. Suppose there are two lines of* $\mathrm{PG}(3,q)$ *which are not transversal lines of a base regulus which share the same lines of the spread $S_F$. Then $F$ is either linear or a Kantor–Knuth flock.*

**Corollary 6.8.** *If a flock of a finite quadratic cone corresponds to a translation plane which is derivable by a net which is not a base regulus net then the flock is linear or a Kantor–Knuth flock.*

### 6.4. The no 4-theorem

There is a class of translation planes due to Walker [70], and Betten [9] which correspond to a class of flocks in $\mathrm{PG}(3,q)$ for $q$ odd and even respectively (actually, Betten shows that such planes exist not only for $q$ odd and even but for over the field of real numbers as well). The flocks have the property that no four planes of the flock share a common point. These flocks are called the Fisher–Thas–Walker flocks but perhaps one might also include the name Betten to this designation. Upon dualizing the projective space, one obtains a set of $q$ points no four of which are coplanar; i.e. a $q$-arc. Using results on the extension of $q$-arcs, Thas proved the following result:

**Theorem 6.9** (Thas [68]). *Let F be a flock of a quadratic cone in* PG$(3,q)$ *with the property that no four of the planes of the flock share a point.*

*Assume either*

(a) *$q$ is even or*

(b) *$q$ is odd and $q > 83$ or $q < 17$ or $q \varepsilon \{27, 81\}$.*

*Then F is a Betten–Fisher–Thas–Walker flock.*

Actually, the above examples can be generalized over translation oval cones and this Thas did in Fisher and Thas [21]. Since $q+1$ arcs in PG$(3,q)$ have been completely classified by Casse and Glynn [15], it is possible to extend the above theorem to arbitrary oval cones.

**Theorem 6.10** (Jha and Johnson [28]). *Let F be a flock of any oval cone in* PG$(3,2^a)$ *with the property that no four of the planes of the flock share a point. Then the oval cone is a translation oval cone and F is isomorphic to one of the translation oval cones of Thas.*

*When the oval cone is a quadratic cone, F is a Betten–Fisher–Thas–Walker flock.*

### *6.5. Doubly transitive results*

#### *6.5.1. BLT sets*

One question that constantly arises when dealing with flocks of finite quadratic cones is whether the derived flocks are isomorphic to the original. To distinguish between derivation of an affine plane and derivation of a flock, we use the term “skeleton” to refer to the set of derived flocks or their corresponding translation planes.

Doubly transitivity arises as follows: Corresponding to a skeleton of a flock of a quadratic cone is a set of $q+1$ points arising as vertices $p_i$ of the $q+1$ cones where $p_i^{\perp} = H_i$ for $i = 1, 2, 3, \ldots, q$, listed previously in the construction of Bader–Lunardon–Thas. The set of vertex points is generally referred to as a BLT set and can be combinatorially characterized.

Isomorphisms of translation planes within the skeleton of a given one induce permutations of the $q+1$ vertex points. For semifield planes, there is a group which acts regularly on the reguli sharing a line. This translates into the statement that a corresponding permutation group on the BLT set has a one point stabilizer subgroup which acts transitively on the remaining points. Now if the planes of the skeleton are also isomorphic then there is a corresponding doubly transitive group on the BLT set. Using this technique, the following may be proved:

**Theorem 6.11** (Johnson et al. [44]). (1) *Let $\pi$ be a semifield plane of flock type and odd order $q^2$. Then the planes of the skeleton of $\pi$ are semifield planes if and only if $\pi$ is the Knuth semifield plane of flock type.*

(2) *Let* $\pi$ *be a flock of a quadratic cone in* PG$(3,q)$ *for* $q$ *odd. If* $F$ *admits an automorphism group which acts transitively on the conics of* $F$ *and the flocks of the skeleton are all isomorphic then* $F$ *either corresponds to the Walker plane or to the Knuth semifield plane of flock type.*

(3) *Let* $\pi$ *and* $\rho$ *be nonisomorphic semifield planes of flock type and odd order* $q^2$ *which are not Knuth semifield planes. Then no plane in the skeleton of* $\pi$ *is isomorphic to any plane of the skeleton of* $\rho$.

*6.5.2. Doubly transitive flocks*

Also, one can determine the set of flocks of a quadratic cone in PG$(3,q)$ that admit a doubly transitive group on the planes of the flock. Furthermore, one can consider this problem more generally for flocks of arbitary oval cones.

**Theorem 6.12** (Jha and Johnson [29,26]). *Let* $F$ *be a flock of a finite oval cone in* PG$(3,q)$ *which admits a linear doubly transitive group on the planes of the flock.*(1) *If* $q$ *is odd then the flock is either linear*, *a Fisher–Thas–Walker–Betten or a Kantor–Knuth flock.*

(2)(a) *If* $q$ *is even then either the flock is linear or the oval cone is a translation oval cone.*

(b) *If the translation oval cone is a quadratic cone then the flock is linear or a Fisher–Thas–Walker–Betten flock.*

(c) *If the translation oval cone is not a quadratic cone then the flock is linear or a translation oval flock of Thas.*

We end this section with a problem.

*Determine the flocks* $F$ *of an infinite oval cone in* PG$(3,K)$, *for* $K$ *an infinite field*, *which admit a linear doubly transitive group on the planes of the flock.*

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## References

[1] L. Bader, Some new examples of flocks of $Q^+(3,q)$, Geom. Dedicata 27 (1988) 213–218.

[2] L. Bader, G. Lunardon, On the flocks of $Q^+(3,q)$, Geom. Dedicata 29 (1989) 177–183.

[3] L. Bader, G. Lunardon, J.A. Thas, Derivation of flocks of quadratic cones, Forum Math. 2 (1990) 163–174.

[4] L. Bader, S.E. Payne, On infinite $K$-clan geometry, J. Geom., to appear.

[5] R.D. Baker, G.L. Ebert, A nonlinear flock in the Minkowski plane of order 11, Congr. Numer. 58 (1987) 75–81.

[6] R.D. Baker, G.L. Ebert, A new class of translation planes, Ann. Discrete Math. 39 (1988) 7–20.

[7] R.D. Baker, G.L. Ebert, Nests of size $(q-1)$ and another family of translation planes, J. London Math. Soc. 2 (38) (1988) 341–365.
[8] A. Barlotti, Un'estensione della teorema di Segre-Kustaanheimo, Boll. Un. Mat. Utal. 10 (1955) 498–506.
[9] D. Betten, 4-Dimensionalen Translationsebenen mit 8-Dimensionalen Kollineationsgruppen, Geom. Dedicata 2 (1973) 327–329.
[10] M. Biliotti, N.L. Johnson, Maximal Baer groups in translation planes and compatibility with homology groups, Geom. Dedicata 59 (1996) 65–101.
[11] M. Biliotti, N.L. Johnson, Variations on a theme of Dembowski. Mostly finite geometries, Lecture Notes in Pure and Applied Math. vol. 190, Marcel Dekker, New York, l997, pp. 139–168.
[12] M. Biliotti, N.L. Johnson, Bilinear flocks of quadratic cones, J. Geom., to appear.
[14] A. Bonisoli, On resolvable finite Minkowski planes, J. Geom. 36 (1989) 1–7.
[15] R.L.A. Casse, D.G. Glynn, On the uniqueness of $(q+1)_4$ of PG$(4,q)$, $q=2^h,h\geqslant 3$, Discrete Math. 48 (1984) 173–186.
[16] W. Cherowitzo, $\alpha$-flocks and hyperovals, Preprint.
[17] W. Cherowitzo, N.L. Johnson, X. Liu, L. Storme, Flocks of translation oval cones, Preprint.
[18] W. Cherowitzo, T. Penttila, I. Pinneri, G.F. Royle, Flocks and ovals, Geom. Dedicata 60 (1996) 17–37.
[19] F. De Clerck, H. Van Maldeghem, Flocks of an infinite quadratic cone, Bull. Belg. Math. Soc. Simon Stevin (1994) 399–415.
[20] P. Dembowski, Finite Geometries, Springer, Berlin, l967.
[21] J.C. Fisher, J.A. Thas, Flocks in PG$(3,q)$, Math. Zeit. 169 (1979) 1–11.
[22] H. Gevaert, N.L. Johnson, Flocks of quadratic cones, generalized quadrangles, and translation planes, Geom. Dedicata 27 (1988) 301–317.
[23] H. Gevaert, N.L. Johnson, J.A. Thas, Spreads covered by reguli, Simon Stevin 62 (1988) 51–62.
[24] V. Jha, N.L. Johnson, Infinite flocks of quadratic cones, J. Geom. 57 (1996) 123–150.
[25] V. Jha, N.L. Johnson, Quasifibrations, Bull. Belg. Math. Soc. Simon Stevin 3 (1996) 313–324.
[26] V. Jha, N.L. Johnson, Automorphism groups of oval cones, Geom. Dedicata 61 (1996) 71–85.
[27] V. Jha, N.L. Johnson, Rigidity in conical flocks, J. Combin. Theory Ser. A 73 (1) (1996) 60–76.
[28] V. Jha, N.L. Johnson, Flocks of oval cones and extensions of theorems of Thas, Note Mat. 2 (1994) 189–197.
[29] V. Jha, N.L. Johnson, The doubly transitive flocks of quadratic cones, European J. Combin. 18 (1997) 763–778.
[30] V. Jha, N.L. Johnson, Infinite Baer nets, J. Geom., submitted for publication.
[31] V. Jha, N.L. Johnson, Conical, ruled and deficiency one translation planes, Bull. Belg. Math. Soc. Simon Stevin, submitted for publication.
[32] N.L. Johnson, Semifield flocks of quadratic cones, Simon Stevin 61 (1987) 313–326.
[33] N.L. Johnson, Flocks of hyperbolic quadrics and translation planes admitting affine homologies, J. Geom. 34 (1989) 50–73.
[34] N.L. Johnson, The derivation of dual translation planes, J. Geom. 36 (1989) 63–90.
[35] N.L. Johnson, Translation planes admitting Baer groups and partial flocks of quadric sets, Simon Stevin 63 (3) (1989) 163–187.
[36] N.L. Johnson, Ovoids and translation planes revisited, Geom. Dedicata 38 (1991) 13–57.
[37] N.L. Johnson, Derivation of partial flocks of quadratic cones, Rend. Mat. 12 (1992) 817–848.
[38] N.L. Johnson, Translation planes covered by subplane covered nets, Simon Stevin 66 (1992) 221–239.
[39] N.L. Johnson, Sharply transitive sets in $P\Gamma L(n,q)$, London Math. Soc. Lecture Notes #191, Finite Geometry and Combinatorics, 1993, pp. 217–223.
[40] N.L. Johnson, Extending partial flocks containing linear subflocks, J. Geom. 55 (1996) 99–106.
[41] N.L. Johnson, Flocks of infinite hyperbolic quadrics, J. Algebra Combin. 6 (1997) 27–51.
[42] N.L. Johnson, Infinite nests of reguli, Geom. Dedicata 70 (1998) 221–267.
[43] N.L. Johnson, X. Liu, Flocks of quadratic and semi-elliptic cones. Mostly finite geometries, Lecture Notes in Pure and Applied Math., vol. 190, Marcel Dekker, New York, l997 pp. 275–304.
[44] N.L. Johnson, G. Lunardon, F.W. Wilke, Semifield skeletons of conical flocks, J. Geom. 49 (1991) 105–112.
[45] N.L. Johnson, G. Lunardon, Maximal partial flocks and flocks of quadratic cones, Des. Codes Cryptogr. 10 (1997) 193–202.

[46] N.L. Johnson, S.E. Payne, Flocks of Laguerre planes and associated geometries. Mostly finite geometries, Lecture Notes in Pure and Applied Math., vol. 190, Marcel Dekker, New York, l997, pp. 51–122.
[47] N.L. Johnson, R. Pomareda, A maximal partial flock of deficiency one of the hyperbolic quadric in PG(3, 9), Simon Stevin 64 (1990) 169–177.
[48] N.L. Johnson, L. Storme, Spreads corresponding to flocks of quadrics. Mostly finite geometries, Lecture Notes in Pure and Applied Math., vol. 190, Marcel Dekker, New York, l997, pp. 315–324.
[49] M.J. Kallaher, A note on Bol projective planes, Arch. Math. 20 (1969) 329–333.
[50] M.L. Kallaher, A note on finite bol quasifields, Arch. Math. 23 (1972) 164–166.
[51] W.M. Kantor, Some generalized quadrangles with parameters $q^2$,$q$, Math. Z. 192 (1986) 45–50.
[52] N. Knarr, Sharply transitive subsets of $P\Gamma L(2,F)$ and spreads covered by derivable partial spreads, J. Geom. 40 (1991) 121–124.
[53] N. Knarr, A geometric construction of generalized quadrangles from polar spaces of rank three, Resultate Math. 21 (1992) 332–334.
[54] X. Liu, Flocks of oval and quadratic cones, Ph.D. Thesis, University of Iowa, l996.
[55] W.F. Orr, The Miquelian Inversive plane $IP(q)$ and the associated projective planes, Thesis, Univ. Wisconsin, l973.
[56] S.E. Payne, Spreads, flocks and generalized quadrangles, J. Geom. 33 (1988) 113–128.
[57] S.E. Payne , L.A. Rogers, Local group actions on generalized quadrangles, Simon Stevin 64 (1990) 249–284.
[58] S.E. Payne, J.A. Thas, Conical flocks, partial flocks, derivation, and generalized quadrangles, Geom. Dedicata 38 (1991) 229–243.
[59] R. Riesinger, Faserungen, die aus Reguli mit einem gemeinsamen Garadenpaar zusammengesetzt sind, J. Geom. 45 (1992) 137–157.
[60] R. Riesinger, Spreads admitting elliptic regulizations, Preprint.
[61] A.E. Schroth, Topological circle planes and topological quadrangles, Pittman Research Notes in Mathematics Series # 337, Longman, New York, 1995.
[62] L. Storme, J.A. Thas, k-arcs and partial flocks, Linear Algebra Appl., to appear.
[63] J.A. Thas, Flocks of finite egglike inversive planes, in Finite Geometric Structures and their Applications, Bressanone, l972, Cremonese, Roma, pp. 189–191.
[64] J.A. Thas, Flocks of nonsinglular ruled quadrics in $PG(3,q)$, Atti Accad. Naz. Lincei Rend. 59 (1975) 83–85.
[65] J.A. Thas, Generalized quadrangles and flocks of cones, European J. Combin. 8 (1987) 441–452.
[66] J.A. Thas, Solution of a classical problem on finite inversive planes, in: W.M. Kantor, R.A. Liebler, S.E. Payne, E.E. Shult (Eds.), Buildings and Related Topics, Clarendon Press, Oxford, 1990, pp. 145–159.
[67] J.A. Thas, Flocks, maximal exterior sets and inversive planes, Contemp. Math. 111 (1990) 187–218.
[68] J.A. Thas, A characterization of the Fisher–Thas–Walker flocks, Simon Stevin 67 (1993) 219–226.
[69] J.A. Thas, The affine plane $AG(3,q), q$ odd, has a unique one point extension, Invent. Math. 118 (1994) 133–139.
[70] M. Walker, On translation planes and their collineation groups, Ph.D Thesis, Univ. London, l973.