Cyclomatic numbers of connected induced subgraphs

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Received 15 June 1989
Revised 19 March 1990

Abstract


We give an upper bound for $\omega(A)$, the minimum cyclomatic number of connected induced subgraphs containing a given independent set $A$ of vertices in a given graph $G$. We also give an upper bound for $\omega(A)$ when $G$ is triangle-free. We show that these two bounds are best possible. Similar results are obtained for $A$ to be a matching of $G$.

1. Introduction

In this paper, we only deal with simple, finite graphs. We usually do not distinguish between a vertex set (or an edge set) and its induced subgraph. The readers are referred to [2] for other terminology not specified in the rest of the paper.

We say that a set $A$ of vertices (edges) of a graph $G$ is an independent set (a matching) if no two vertices (edges) of $A$ are adjacent.

The cyclomatic number of a graph $H$, denoted by $\text{cy}(H)$, is $|E(H)| - |V(H)| + 1$.

Let $A$ be an independent set of vertices of a graph $G$. Let $C(A)$ be the collection of all connected induced subgraphs of $G$ which contain $A$. Define

$$\omega(A) = \min\{\text{cy}(H) : H \in C(A)\}.$$ 

In [1], Alspach and Oral asked the following question: what can be said about $\omega(A)$ for various classes of graphs? Of particular interest are the cases when $A$ is a maximal independent set of vertices in $G$ or $A$ is a color class in a proper vertex coloring of $G$ with number of colors equal to the chromatic number of $G$.

Note that if $A$ is not contained in a component of $G$, then $C(A)$ is empty. So we only consider connected graphs.
In what follows, we will get upper bounds for $\omega(A)$ for the case that $G$ is a general graph and for the case that $G$ is a triangle-free graph (a graph is said to be triangle-free if it has no induced subgraph isomorphic to $K_3$). In addition, an edge version of the above question is considered in Section 5 and similar results are obtained.

2. Lemmas

In order to obtain our main results, we need several lemmas. The first lemma is obvious and the proof is omitted.

**Lemma 2.1.** Let $H$ be a graph. If $v \in V(H)$ is such that $\deg_H(u) \geq 1$, then $cy(H) \geq cy(H - u)$ with equality iff $\deg_H(u) = 1$.

**Lemma 2.2.** Let $H$ be a connected graph and $H'$ be a proper connected induced subgraph of $H$. Then $cy(H) \geq cy(H')$.

**Proof.** Let $V(H) \setminus V(H') = \{u_1, \ldots, u_m\}$ and let $H_i = H[V(H') \cup \{u_1, \ldots, u_m\}]$, $i = 1, \ldots, m$. Moreover, we can assume that $\deg_{H_i}(u_i) \geq 1$, $i = 1, \ldots, m$. For example, we can choose $u_i$ to be a vertex adjacent to a vertex in $V(H_{i+1})$, $i = 1, \ldots, m$, where $H_{m+1} = H'$. Then by Lemma 2.1, we have

$$cy(H) = cy(H_1) \geq cy(H_2) \geq \cdots cy(H_m) \geq cy(H').$$


Note that Lemma 2.2 may not be true in general, for example, if we let $H$ be obtained from $H'$ by adding an isolated vertex, then $cy(H) = cy(H') - 1$.

We say that $H$ is minimal in $C(A)$ if $H \in C(A)$ but any proper induced subgraph of $H$ is not in $C(A)$. A vertex in $A$ is said to be an $A$-end in $H$ if it has degree one in $H$. It is obvious that $H$ is minimal in $C(A)$ iff each vertex of $V(H) \setminus A$ is a cutvertex of $H$. Though we can prove our results without using Lemma 2.2, Lemma 2.2 says that $\omega(A) = \min\{cy(H): H \text{ is minimal in } C(A)\}$, and so we only need to look at those connected induced subgraphs of $G$ that are minimal in $C(A)$.

Notice that if $|A| = 1$, then $\omega(A) = 0$, so we assume that $|A| \geq 2$.

**Lemma 2.3.** Let $A$ be an independent set of vertices of a connected graph $G$ and let $H$ be minimal in $C(A)$. Then $\deg_H(u) \leq |A|$ for each $u \in V(H)$.

**Proof.** Since $H$ is in $C(A)$, $|V(H) \setminus A| \geq 1$. Let $N(u)$ be the neighborhood of $u$ in $H$. If $N(u) \subseteq A$, then $\deg_H(u) \leq |A|$, and so, let $N(u) \setminus A = \{u_1, \ldots, u_s\}$. Each $u_i$
is a cutvertex of $H$ by the choice of $H$. So let $C_i$ be a component of $H \setminus u$, which does not contain $u$. Also let $B_j$ be the component of $H \setminus u$, that contains $u$. Thus $u_j \in B_j$ for $j \neq i$, and so if $j \neq i$, $1 \leq i, j \leq s$, then $C_i \cap C_j = \emptyset$ (since $C_i$ is contained in $B_j$). Also $C_i \cap A \neq \emptyset$, otherwise $H \setminus C_i$ is a proper connected induced subgraph of $H$ which is also in $C(A)$, a contradiction.

Since $C_i \cap C_j = \emptyset$ for $i \neq j$, and $C_i \cap A \neq \emptyset$, we can choose $a_i$ from $C_i \cap A$ for $i = 1, 2, \ldots, s$. It is obvious that $\{a_1, a_2, \ldots, a_s\}$ is a subset of $A$ and $N(u) \cap \{a_1, a_2, \ldots, a_s\} = \emptyset$. Therefore, we have

$$\deg_H(u) = |\{a_1, \ldots, a_s\} \cup (N(u) \cap A)| \leq |A|.$$ 

**Lemma 2.4.** Under the conditions of Lemma 2.3, there are at least two $A$-ends in $H$.

**Proof.** We use the same notation as in the first paragraph of the proof of Lemma 2.3. We claim that for each $i$, $1 \leq i \leq s$, $C_i$ contains an $A$-end, say $v_i$, in $H$.

If $|C_i| = 1$, say $C_i = \{u_i\}$, then $v_i$ is an $A$-end. So we assume that $|C_i| \geq 2$. Since $A$ is independent, and $C_i$ is connected, $C_i \setminus A \neq \emptyset$. We also notice that each vertex, say $w$, in $C_i \setminus A$ is a cutvertex of $H$ and all vertices outside $C_i$ are in one component of $H \setminus w$. Hence we can pick such a vertex $w$ in $C_i \setminus A$ that $H \setminus w$ has a component, say $C$, within $C_i$ having as few vertices as possible. That forces $|C| = 1$, for otherwise, there is a $w'$ in $C \setminus A$ and we can use the same argument as we did for $C_i$ to produce a smaller component within $C$ (hence within $C_i$) in $H \setminus w'$, a contradiction. So let $C = \{v_i\}$, and $v_i$ is what we want. Since $s \geq 2$, $H$ has at least two $A$-ends. \qed

**Lemma 2.5.** Let $A$ be an independent set of vertices of $G$ and let $H$ be minimal in $C(A)$. If $u$ and $w$ are two adjacent vertices in $V(H)$ that have no common neighbors, then $\deg_H(u) + \deg_H(w) \leq |A| + 2$.

**Proof.** Let $C = N(u) \setminus (A \cup \{w\})$ and $D = N(w) \setminus (A \cup \{u\})$.

(a) If $C = \emptyset = D$, then $N(u) \setminus \{w\}$ and $N(w) \setminus \{u\}$ are disjoint subsets of $A$, and so, Lemma 2.5 follows.

(b) If $C \neq \emptyset$ and $D \neq \emptyset$, then let $C = \{u_1, \ldots, u_t\}$ and $D = \{w_1, \ldots, w_t\}$. Similar to the proof for Lemma 2.3, let $C_i$ (or $D_j$) be a component of $H \setminus u_i$ (or $H \setminus w_j$) which does not contain $u$ (or $w$), where $1 \leq i \leq s$ (or $1 \leq j \leq t$). By the proof of Lemma 2.4, we have an injection from $C$ (or $D$) to $A$-ends contained in $C_i$, $1 \leq i \leq s$ (or in $D_j$, $1 \leq j \leq t$). Since $u$ and $w$ have no common neighbors, $C_i \cap D_j = \emptyset$ for each pair of $i, j$. Since all $C_i$ are pairwise disjoint and all $D_j$ are pairwise disjoint, $\deg_H(u) + \deg_H(w) \leq |A| + 2$.

(c) If exactly one of $C$ and $D$ is empty, say $C$, then a similar argument of (b) can be used except that we do not need $C_i$ or $D_j$. \qed
3. General graphs

In this section, we assume that $A$ is an independent set of vertices of a connected graph $G$ and that $H$ is minimal in $C(A)$.

**Theorem 3.1.** $\text{cy}(H) \leq \binom{|A|}{2} - |A| + 1$, if $|A| \geq 2$.

**Proof.** If $|A| = 2$, then $H$ is an induced path of $G$ connecting the two vertices of $A$, so $\text{cy}(H) = 0$, i.e., the equality holds. Let $|A| \geq 3$. Suppose that we have proved the above result for all independent sets of $G$ with fewer than $|A|$ vertices. According to Lemma 2.4, we assume that $v$ is an $A$-end which is only adjacent to a vertex $u$ in $V(H) \setminus A$.

**Case 1: $H \setminus u$ has at least three components.**

Let $A' = A \setminus v$ and $H' = H \setminus v$. Then each vertex of $V(H') \setminus A'$ is a cutvertex of $H'$. So $H'$ is minimal in $C(A')$. By induction, we have

$$\text{cy}(H') \leq \binom{|A'|}{2} - |A'| + 1$$

$$= \binom{|A|}{2} - |A| + 1 + (2 - |A|)$$

$$< \binom{|A|}{2} - |A| + 1 \quad \text{since } |A| \geq 3.$$

But

$$\text{cy}(H) = |E(H)| - |V(H)| + 1$$

$$= (|E(H')| + 1) - (|V(H')| + 1) + 1$$

$$= \text{cy}(H').$$

So

$$\text{cy}(H) < \binom{|A|}{2} - |A| + 1.$$

**Case 2: $H \setminus u$ has exactly two components (hence one of them is $\{v\}$).**

$H \setminus \{u, v\}$ is minimal in $C(A \setminus v)$ unless $\deg_H(u, v) = 2$ and the other neighbor of $u$ in $V(H) \setminus A$. So we may assume that there is a path $P = v \ u_0 \ \cdots \ u_q$ such that $u_i \in V(H) \setminus A$ ($0 \leq i \leq q - 1$, $u_0 = u$), $\deg_H(u_i) = 2$ if $0 \leq i \leq q - 1$, and $\deg_H(u_q) \geq 3$ if $u_q \in V(H) \setminus A$. Thus let $A' = A \setminus v$ and $H' = H \setminus \{v, u_0, \ldots, u_{q - 1}\}$.

(2a) $H'$ is minimal in $C(A')$. So by induction, we have

$$\text{cy}(H) = |E(H)| - |V(H)| + 1$$

$$= (|E(H')| + q + 1) - (|V(H')| + q + 1) + 1$$

$$= \text{cy}(H')$$

$$< \binom{|A|}{2} - |A| + 1.$$
(2b) $H'$ is not minimal in $C(A')$. Then $u_q$ must be in $V(H) \setminus A$ with $\deg_H(u_q) \geq 3$ and $H' \setminus u_q$ is connected. Let $H'' = H' \setminus u_q$. Then $H''$ is minimal in $C(A')$. Hence
\[
\text{cy}(H'') \leq \left( \frac{|A'|}{2} \right) - |A'| + 1
\]
\[
= \left( \frac{|A|}{2} \right) - |A| + 1 + (2 - |A|).
\]
So
\[
\text{cy}(H) = |E(H)| - |V(H)| + 1
\]
\[
= (|E(H')| + \deg_H(u_q) + q) - (|V(H')| + q + 2) + 1
\]
\[
= \text{cy}(H') + \deg_H(u_q) - 2
\]
\[
\leq \text{cy}(H') + |A| - 2 \quad (\text{by Lemma 2.3})
\]
\[
\leq \left( \frac{|A|}{2} \right) - |A| + 1.
\]

Hence, we have the following.

**Theorem 3.1'.** $\omega(A) \leq \left( \frac{|A|}{2} \right) - |A| + 1$, if $|A| \geq 2$.

**4. Triangle-free graphs**

Now we turn our attention to triangle-free graphs. In this section, we assume that $A$ is an independent set of vertices of a connected triangle-free graph $G$ and that $H$ is minimal in $C(A)$.

**Theorem 4.1.** $\text{cy}(H) \leq |A|^2 / 4 - |A| + 1$, if $|A| \geq 1$.

**Proof.** We prove the theorem by way of contradiction. Suppose that the claim is not true. Select a triple $(G, A, H)$, where $A$ is an independent set of a triangle-free graph $G$ and $H$ is minimal in $C(A)$, such that the claim fails and $s(G, A, H) = |V(G)| + |A| + |V(H)|$ is minimal. We start with $|A| \geq 3$. Since $s(G, A, H)$ is minimal and $H$ is an induced subgraph, we conclude that $G = H$.

**Claim 1.** $H = G$ has no degree 2 vertex in $V(H) \setminus A$.

**Proof.** For otherwise, let $v$ be a degree 2 vertex in $V(H) \setminus A$ with $u, w$ as its two neighbors. If one of $u$ or $w$ is not in $A$, then the graph $(H \setminus v) + uw$ is minimal in $C(A)$ where $A$ is independent in $(H \setminus v) + uw$ which is also triangle-free. Since
\[
s((H \setminus v) + uw, A, (H \setminus v) + uw) < s(H, A, H),
\]
we have
\[ \text{cy}(H) = \text{cy}((H \setminus u) + uw) \leq |A|^2/4 - |A| + 1, \]
a contradiction. So \( u, w \in A \). Let \( C_u \) and \( C_w \) be the two components of \( H \setminus u \) containing \( u \) and \( w \) respectively, and let \( A_u = C_u \cap A \) and \( A_w = C_w \cap A \). Then \( C_u \)
is minimal in \( C(A_u) \) and \( C_w \) is minimal in \( C(A_w) \). Since
\[ s(C_u, A_u, C_u) < s(H, A, H), \quad s(C_w, A_w, C_w) < s(H, A, H), \]
and \( C_u \) and \( C_w \) are triangle-free, we have
\[ \text{cy}(C_u) \leq |A_u|^2/4 - |A_u| + 1 \quad \text{and} \quad \text{cy}(C_w) \leq |A_w|^2/4 - |A_w| + 1. \]
Thus
\[ \text{cy}(H) = \text{cy}(C_u) + \text{cy}(C_w) \]
\[ \leq (|A_u|^2 + |A_w|^2)/4 - (|A_u| + |A_u|) + 2 \]
\[ = |A|^2/4 - |A| + 1 - (|A_u| |A_w| - 2)/2 \]
\[ \leq |A|^2/4 - |A| + 1, \quad \text{since} \ |A| \geq 3. \]
But this contradicts the choice of \( (H, A, H) \), so the claim is proved. \( \square \)

By Claim 1, we can easily see that each \( A \)-end is adjacent to a vertex (in \( V(H) \setminus A \)) of degree \( \geq 3 \).

**Claim 2.** For each \( u \in V(H) \setminus A \), \( u \) is adjacent to at most one \( A \)-end.

**Proof.** For suppose that \( v, w \) are two \( A \)-ends adjacent to \( u \in V(H) \setminus A \). Then \( H \setminus v \)
is minimal in \( C(A \setminus v) \). Since \( s(H \setminus v, A \setminus v, H \setminus v) < s(H, A, H) \), we have
\[ \text{cy}(H) = \text{cy}(H \setminus v) \leq (|A| - 1)^2/4 - (|A| - 1) + 1 < |A|^2/4 - |A| + 1. \]
This is, again, a contradiction. So Claim 2 is true. \( \square \)

**Claim 3.** For each vertex \( u \) in \( V(H) \setminus A \), if there is an \( A \)-end, say \( v \), adjacent to \( u \), then \( \deg_H(u) \geq (|A| + 3)/2. \)

**Proof.** Otherwise, \( \deg_H(u) \leq (|A| + 2)/2. \) By a similar argument as in proof of Claim 2, \( H \setminus u \)
is not minimal in \( C(A \setminus u) \). So \( H \setminus (u, v) \) is minimal in \( C(A \setminus u) \), and thus
\[ \text{cy}(H) = \text{cy}(H \setminus (u, v)) + \deg_H(u) - 2 \]
\[ \leq (|A| - 1)^2/4 - (|A| - 1) + 1 + (|A| + 2)/2 - 2 \]
\[ = |A|^2/4 - |A| + 1 + \frac{1}{4}. \]
So \( \text{cy}(H) \leq |A|^2/4 - |A| + 1 \) since \( \text{cy}(H) \) is an integer. This contradicts the choice of \( (H, A, H) \), so Claim 3 is proved. \( \square \)
Claim 4. The vertex $u$ in Claim 3 is adjacent to a vertex, say $w$, in $A \setminus v$.

Proof. Suppose that is not the case. Then $H \setminus v$ is minimal in $C((A \setminus v) \cup \{u\})$. Since

$$s(H \setminus v, (A \setminus v) \cup \{u\}, H \setminus v) < s(H, A, H),$$
$$cy(H) = cy(H \setminus v) \leq \frac{|A|^2}{4} - |A| + 1,$$

a contradiction. So Claim 4 is true. \[\square\]

By Lemma 2.5 and Claims 3 and 4, $\deg_H(w) \leq (|A| - 1)/2$.

Claim 5. The vertex $w$ in Claim 4 is a cutvertex of $H$.

Proof. Otherwise, $H \setminus w$ is minimal in $C(A \setminus w)$, since (by Claim 2) $w$ can not be an $A$-end. Therefore, we have

$$cy(H) = cy(H \setminus w) + \deg_H(w) - 1$$
$$\leq (\frac{|A| - 1}{2} - (|A| - 1) + 1 + (|A| - 1)/2 - 1$$
$$< |A|^2/4 - |A| + 1,$$

a contradiction. So $w$ is a cutvertex of $H$. \[\square\]

Let $C_k, k = 1, \ldots, p$, be the components of $H \setminus w$. Note that $p \geq 2$, $C_k \cap A = \emptyset$, and $C_k \setminus A \neq \emptyset$. Let $A_k = C_k \cap A$ and assume that $|A_1| \leq |A_2| \leq \cdots \leq |A_p|$. By Claim 1, $|A_k| \geq 2$. Let $a_k = |C_k \cap N(w)|$.

If $a_k > 1$, then $C_k$ is minimal in $C(A_k)$.

If $a_k = 1$, let $w_k$ be the only neighbor of $w$ in $C_k$, then either $C_k$ or $C_k \setminus w_k$ is minimal in $C(A_k)$. Since $cy(C_k) = cy(C_k \setminus w_k) + \deg_H(w_k) - 2$, by the choice of $(I, A, II)$, we have

$$cy(C_k) \leq \frac{|A_k|^2}{4} - |A_k| + 1 + b_k,$$

where $b_k = \deg_H(w_k) - 2$ if $a_k = 1$ and $b_k = 0$ if $a_k > 1$. Then by Lemma 2.3, $\deg_H(w) \leq |A_k| + 1$ and $a_k \leq |A_k|$. Thus

$$cy(H) = \left(\sum_{k=1}^{p} |E(C_k)|\right) - \left(\sum_{k=1}^{p} |V(C_k)|\right) + 1 + (\deg_H(w) - 1)$$
$$- \sum_{k=1}^{p} cy(C_k) + \deg_H(w) - p$$
$$\leq \sum_{k=1}^{p} (\frac{|A_k|^2}{4} - |A_k| + 1 + b_k) + \sum_{k=1}^{p} a_k - p$$
$$= \left(\sum_{k=1}^{p} |A_k|^2\right)/4 - (|A| - 1) + \sum_{k=1}^{p} (a_k + b_k)$$
\[
\frac{(|A| - 1)^2 - 2 \sum_{k<l} |A_k| |A_l|)}{4 - (|A| - 1) + \sum_{k=1}^n |A_k|}
\]

\[
= \frac{|A|^2 - 2|A| + 1}{4} - \left( \frac{\sum_{k<l} |A_k| |A_l|}{2} - (|A| - 1) + (|A| - 1) \right)
\]

\[
\leq \frac{|A|^2/4 - |A| + 1}{2 - \frac{1}{4}}
\]

The last inequality follows because

\[
\left( \sum_{k<l} |A_k| |A_l| \right) - (|A| - 1) \geq |A_1|(|A| - 1) - |A_1| - (|A| - 1)
\]

\[
\geq (|A_1| - 1)(|A| - 1) - |A_1|^2
\]

\[
\geq 2|A_1|(|A_1| - 1) - |A_1|^2
\]

\[
= |A_1|(|A_1| - 2) \geq 0.
\]

Hence we have a contradiction. Therefore the theorem is true. □

**Theorem 4.1.’** \(\omega(A) \leq |A|^2/4 - |A| + 1\), if \(|A| \geq 1\).

5. Related problems and the edge version

In this section, we consider the edge version of the question.

Let \(M\) be a matching of a connected graph \(G\) and \(C(M)\) be the collection of all connected induced subgraphs of \(G\) that contain \(M\). Define

\[
\omega'(M) = \min \{\chi(H) : H \in C(M)\}.
\]

What can be said about \(\omega'(M)\)? Of particular interest are the cases when \(M\) is a maximal matching and when \(M\) is a color class of a proper edge coloring with number of colors equal to the chromatic index.

Here we give similar results for this question as we obtained for the original one. \(H\) is said to be minimal in \(C(M)\) if \(H \in C(M)\) but any proper induced subgraph of \(H\) does not belong to \(C(M)\). Notice that \(\omega'(M) = 0\) when \(|M| = 1\). So we assume that \(|M| \geq 2\). By using the same argument (replacing \(A\) by \(V(M)\) if appropriate) as for the original problem, we have the following.

**Theorem 5.1.** \(\omega'(M) \leq 2|M|^2 - 3|M| + 1\), and \(\omega'(M) \leq |M|^2 - 2|M| + 1\) if \(G\) is triangle-free.
Proof. Let $H$ be an induced subgraph of $G$ that is minimal in $C(M)$. If $|V(H)| = 2|M|$, then

$$|E(H)| \leq \binom{2|M|}{2}, \quad \text{and} \quad |E(H)| \leq (2|M|)^2/4 = |M|^2$$

by Turan’s Theorem [2] if $G$ is triangle-free.

Hence, it is easy to check that the two inequalities are true if $|V(H)| = 2|M|$. Let $|V(H)| > 2|M|$. Then we can pick a vertex $u \in V(H) \setminus V(M)$. Since $H$ is minimal, $H \cup u$ is disconnected. Let the components of $H \cup u$ be $C_i$, $i = 1, \ldots, s$ and $s > 1$. Let $M_i = M \cap C_i$ and $m_i = |M_i|$. Let $a_i = |C_i \cap N(u)|$. We claim that

$$a_i \leq 2m_i$$

and that if $G$ is triangle-free, then $a_i = m_i$. The proof is similar to that of Lemma 2.3 by establishing an injection from $N(u) \cap C_i$ to $V(M_i)$ (by the same minimizing technique). Hence we have

$$\deg_{H}(u) \leq 2|M|,$$

and

$$\deg_{H}(u) \leq |M| \quad \text{if} \quad G \text{ is triangle-free.}$$

Now everything will be similar to the last part of the proof of Theorem 4.1. $\square$

The upper bounds in Theorem 5.1 are best possible, since $G = K_{2|M|}$ and $G = K_{|M_1|, |M_1|}$ attain these bounds respectively.

The upper bounds in Theorems 3.1’ and 4.1’ are also best possible, since we can construct a graph $G$ and the set $A$ such that the upper bounds are attained.

Let $A$ be a set of $n$ vertices and let $H = K_n$. Let $G$ be obtained from $H$ by connecting $A$ to $V(H)$ through an $n$-matching. Then it is easy to check that $\omega(A) = \binom{n}{2} - |A| + 1$ for $|A| \geq 2$. So the upper bound in Theorem 3.1’ is best possible.

Let $A$ be a set of $2n$ vertices and let $H = K_{n,n}$. Construct $G$ from $H$ by connecting $A$ to $H$ through a $2n$-matching. Then

$$\omega(A) = n^2 - n + 1 = |A|^2/4 - |A| + 1.$$  

So the upper bound in Theorem 4.1’ is best possible.

Upon finishing this paper, the author was informed that the Alspach–Oral problem is related to Barnette’s conjecture that every 3-connected cubic bipartite planar graph has a hamiltonian cycle. So it is worthwhile to mention the following:

1. The upper bound for $\omega(A)$ might decrease if we increase the girth of $G$. So the classes of graphs with girth $\geq 4$ are also of interest.

2. The upper bound for $\omega(A)$ might be lowered by increasing the connectivity of $G$.

3. It is likely that if $G$ is planar, then $\omega(A) = 2|A| - 5$.

4. It is also interesting to consider 3-connected bipartite planar graphs.
Acknowledgement

The author thanks the referee for a simpler proof of Lemma 2.3 and a better argument for Lemma 2.5.

References