Generalized solutions of singular differential problems. Relationship with classical solutions

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Abstract

We discuss various methods of regularization of singular differential problems. Their common point is that we use the flexibility of the theories of nonlinear generalized functions for adapting the regularization to the singularity of the problem. We particularly underline the relationship between the generalized solutions and those classical or distribution, when they exist, giving a general result for the case of the regularization of data.

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1. Introduction

During the last three decades, various theories of nonlinear generalized functions have been developed by many authors (see [1,3,10,15,18,19,21,22],...). These theories have proved their efficiency to pose and solve differential problems with irregular data and operators (see the references quoted above and [2,11,16,19], for examples) or characteristic problems [8]. In all these works, the strategy to solve singular problems consists in replacing the given problem by a family of smooth problems (the regularized problems) depending on one or more small parameters. A first step is to refine classical estimates linking the data and the solution of the regularized problems. Starting from these estimates, the second step is to adapt the algebraic and asymptotic structure of an algebra of generalized functions, so that the family of solutions of the regularized problems lies in it as solution of the initial singular problem. In this paper, we use Colombeau type simplified algebras, with some refinements in the choice of the asymptotic scales [7].

Nevertheless, it seems that the natural question of the relationship between the classical solution of a differential problem, when it exists, and the generalized one, obtained by the above described method has been less investigated.

After recalling in Section 2 some results for the unidirectional wave equation as this example will be used throughout the paper, we distinguish two cases in our study. The first one, considered in Section 3, is related to regularized problems which depend in a moderate way on data. This case mostly covers linear and semilinear problems with irregular data for which the regularization is done only on data. The results are given in terms of \(C^p\) (resp. \(\mathcal{D}'\)) convergence of a family of...
solutions to the regularized problems toward a $C^p$-solution (resp. $D'$-solution) to the initial problem when it exists. They are based on functorial properties of algebras of nonlinear generalized functions [7,17].

Let us explain the second case of our study. Some singular problems are still out of the reach of the previous tools. The first situation is the case of a problem

$$P(D)(u) = F(u, \cdot),$$

where $P(D)$ is a differential operator and $F$ has a non-Lipschitz nonlinearity. A regularization, which uses a convenient cut-off technique, transforms the nonlinearity of $F$ into a Lipschitz one. This gives a technique to solve this kind of problem in the framework of nonlinear generalized functions. Moreover, when the nonregularized problem has a classical solution, then this solution and the generalized one are equal in a meaning explained in Proposition 13. The second situation is related to characteristic problems for which a regularization (which is a geometric transformation of the support of data) is used to de-characterize the problem. This permits us to solve the problem in a convenient algebraic framework. These situations are treated in Section 4. Of course, all the previous types of singularities can be mixed in same problem. This can be handled by using multi-parametric regularizations and will be treated in a separate paper.

Our hope is that this study will convince the reader that our theory of nonlinear generalized solutions is a natural continuation of the classical theory of smooth functions and of linear generalized function or distributions [9,13,23]. In addition, we refer the reader to [12,20] in which similar ideas of regularizing nonlinear problems within algebras of generalized functions are used. Finally, we hope that we showed that these theories (or similar ones) cannot be circumvented as soon as the data or the problems are too irregular.

2. Outline on the unidirectional wave equation

2.1. An initial value problem

Consider the initial value (or Cauchy) problem for the unidirectional wave equation

$$\begin{cases}
\frac{\partial u}{\partial t} + (\varphi' \otimes 1_x) \frac{\partial u}{\partial x} = 0, \\
u|_S = v,
\end{cases} \tag{IVP}$$

where $\varphi$ is a smooth function (for sake of simplicity), with $\varphi(0) = 0$ and $\varphi' > 0$ on $\mathbb{R}$, the initial curve $S$ is a smooth one-dimensional manifold, $u$ is the unknown function belonging at least to $C^1(\mathbb{R}^2)$, $u|_S$ is the restriction of $u$ to $S$, $v$ can be defined on $S$ by the restriction $v = w|_S$, with $w \in C^0(\mathbb{R}^2)$.

2.2. Non-characteristic condition and classical solution

We recall that the characteristic method is the most general one to solve the classical problem. However, when $S$ is the curve $\{t = 0\}$ and $v$ given by a function $f \in C^p(\mathbb{R})$, we can write (2) as

$$u(0, x) = f(x).$$

This leads easily to the solution

$$u(t, x) = f(x - \varphi(t)). \tag{3}$$

The regularity of $u$ given by (3) depends on that of the data. If $f$ is $C^p$ or $C^\infty$ on $\mathbb{R}$, the same holds for the solution $u$ on $\mathbb{R}^2$. If $f$ and $\varphi$ are analytic (in the small or in the large), the same holds for $u$, in agreement with the Cauchy–Kowaleska theorem.

2.3. The distributional framework

If the data are irregular, for example distributions, we can search the corresponding distribution solution $u$ to

$$\begin{cases}
\frac{\partial u}{\partial t} + (\varphi' \otimes 1_x) \frac{\partial u}{\partial x} = 0, \\
u|_S = v,
\end{cases} \tag{IVPD}$$

where $S$ is the curve $\{t = 0\}$, $v$ is a given distribution on $\mathbb{R}$, $u|_S$ the restriction of $u$ to $S$ (if it can be defined as a distribution on $\mathbb{R}$).
2.3.1. General distribution solution of the unidirectional wave equation

Let $U$ be the pullback of $u$ by the change $T = t, X = x - \varphi(t)$. Let $\Psi$ and $\psi \in \mathcal{D}(\mathbb{R}^2)$ be such that

$$\Psi(T, X) = \psi(t, x) = \psi(T, X + \varphi(T)).$$

It is easy to prove from (1) that

$$\frac{dU}{dt}, \Psi = 0.$$

Then we have $U = 1, \otimes R$, where $R$ is any distribution in $\mathcal{D}'(\mathbb{R})$. From

$$\langle U, \Psi \rangle = \left\langle R, X \mapsto \int \Psi(T, X) dT \right\rangle,$$

we deduce the general form of the solutions to (1)

$$\langle u, \psi \rangle = \left\langle R_{\Gamma_{\psi}}, \psi \right\rangle = \left\langle R, x \mapsto \int \psi(t, x + \varphi(t)) dt \right\rangle,$$

where $R_{\Gamma_{\psi}}$ is associated to $R$ and to the characteristic curve

$$\Gamma_{\psi} = \{(t, x) \in \mathbb{R}^2, x = \varphi(t)\}.$$

2.3.2. Pull back, restriction problem and Hörmander criterion

We have first to define the restriction $u_{|S}$. A sufficient condition is the Hörmander criterion: Let $N(S)$ be the normal bundle of $S$. For every distribution $u$ in $\mathbb{R}^2$ with $WFu \cap N(S) = \emptyset$, the restriction $u_{|S}$ is a well defined distribution on $S$, the pull back by the inclusion $S \hookrightarrow \mathbb{R}^2$ [13].

We recall that the wave front set $WF$ of $u \in \mathcal{D}'(\mathbb{R}^2)$ is here defined by

$$WFu = \{(x, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2, 0) \mid \xi \in \Sigma u\},$$

where $\Sigma u = (\mathbb{R}^2, 0) \cap \mathcal{O}u, \mathcal{O}u = \{\xi \in (\mathbb{R}^2, 0) \mid \exists V \in \mathcal{V}_\xi, \exists w \in \mathcal{V}_{\xi, \hat{w}} \forall \psi \in \mathcal{D}(V), \hat{\psi}w \in S\}$ and $\mathcal{V}_\xi$ is the filter of conic neighborhoods of $\xi$. If $u$ is the Fourier transform of $\psi$ and $S$ the algebra of rapidly decreasing functions on $\mathbb{R}^2$, $\Sigma u$ is the cone of high frequency components of $u$ causing the singularities.

2.3.3. Distribution solution to (IVPD)

For a given $v \in \mathcal{D}'(\mathbb{R})$, we know that $\nu_{R_{\Gamma_{\psi}}}$ defined by (4) when replacing $R$ by $v$, solves (1). Moreover $\nu_{R_{\Gamma_{\psi}}}$ is the unique solution to (IVPD). Indeed, according to a remark of G. Hörmann, we can refer to Section 2.3.1 in [14] to see that $WF\nu_{R_{\Gamma_{\psi}}} \cap N(\{t = 0\}) = \emptyset$ and that the restriction $\nu_{R_{\Gamma_{\psi}}}|_{t=0}$ is equal to $\nu$.

**Example 1.** We take $v = \delta \in \mathcal{D}'(\mathbb{R})$ in (IVPD), condition (2). Then the solution of (IVPD) is $\delta_{\Gamma_{\psi}}$, the Dirac distribution of $\Gamma_{\psi}$, defined by

$$\langle \delta_{\Gamma_{\psi}}, \psi \rangle = \left\langle \delta, x \mapsto \int \psi(t, x + \varphi(t)) dt \right\rangle = \int \psi(t, \varphi(t)) dt$$

for $\psi \in \mathcal{D}(\mathbb{R}^2)$.

3. Algebraic framework and regularization of data

3.1. Algebraic framework, extension principle and generalized solutions

Consider the problem (IVP). If the data are irregular, for example even not necessarily distributions, we cannot solve this problem in a classical way but it may admit a generalized solution obtained by regularization of the data. We shall discuss here this question in the framework of Colombeau type simplified algebra [7,10,18]. We begin by recalling the definition of this algebra and then introduce our main tool, continuously moderate (family of) maps.

Let $C^\infty$ be the sheaf of complex valued smooth functions on $\mathbb{R}^d (d \in \mathbb{N})$, equipped with the usual topology of uniform convergence. For any open subset $\Omega$ of $\mathbb{R}^d$, this topology can be described by the family $\mathcal{P}_\Omega = (P_K)_{K \in \Omega, \omega \in \mathbb{N}}$ of semi-norms given by

$$P_{K, l}(f) = \sup_{x \in K, |\alpha| \leq l} |D^\alpha f(x)|,$$

where the notation $K \in \Omega$ means that $K$ is a compact set included in $\Omega$ and $D^\alpha = \partial_{z_1}^{\alpha_1} \cdots \partial_{z_d}^{\alpha_d}$ for $z = (z_1, \ldots, z_d) \in \Omega$, $l \in \mathbb{N}$, $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$. 
A sequence \( a = (a_n)_{n \in \mathbb{Z}} \) of increasing functions from \((0, 1]\) to \((0, +\infty)\) is called an asymptotic scale [7] if for each \( n \in \mathbb{N} \) (except possibly \( n = 0 \)) \( \lim_{\varepsilon \to 0} a_n(\varepsilon) = 0 \) and if
\[
\begin{align*}
\forall n \in \mathbb{N} \setminus \{0\}, & \quad a_{-n} = 1/a_n, \\
\forall n \in \mathbb{N}, & \quad a_{n+1}(\varepsilon) = o(a_n(\varepsilon)) \quad \text{as } \varepsilon \to 0, \\
\forall n \in \mathbb{N}, & \quad \forall m \in \mathbb{N}, \exists N \in \mathbb{N}, \quad a_N(\varepsilon) = O(a_n(\varepsilon)a_m(\varepsilon)) \quad \text{as } \varepsilon \to 0.
\end{align*}
\] (6)

Set
\[
\begin{align*}
\mathcal{M}_d(\Omega) &= \{ (f_\varepsilon)_\varepsilon \in \mathbb{C}^\infty(\Omega)_{(0,1)} \mid \forall f \in \mathcal{D}, \forall K \subseteq \Omega, \exists q \in \mathbb{N}, \quad P_{K,1}(f_\varepsilon) = o(a_{-q}(\varepsilon)) \quad \text{as } \varepsilon \to 0 \}, \\
\mathcal{N}_d(\Omega) &= \{ (f_\varepsilon)_\varepsilon \in \mathbb{C}^\infty(\Omega)_{(0,1)} \mid \forall f \in \mathcal{O}, \forall \varepsilon \in \mathbb{R}_+, \exists p \in \mathbb{N}, \quad P_{K,1}(f_\varepsilon) = o(a_p(\varepsilon)) \quad \text{as } \varepsilon \to 0 \}.
\end{align*}
\]

We recall that \( \mathcal{M}_d(\cdot) \) (resp. \( \mathcal{N}_d(\cdot) \)) is a sheaf of subalgebras (resp. of ideals) of the sheaf \( \mathbb{C}^\infty(\Omega)_{(0,1)} \) (resp. of \( \mathbb{C}^\infty(\Omega) \)). The sheaf of factor algebras \( \mathcal{G}_d(\cdot) = \mathcal{M}_d(\cdot)/\mathcal{N}_d(\cdot) \) is called the sheaf of Colombou type simplified algebras or asymptotic algebras [7].

**Notation 1.** The class in \( \mathcal{G}_d(\Omega) \) of \( (f_\varepsilon)_\varepsilon \in \mathcal{M}_d(\Omega) \) is denoted by \( (f_\varepsilon)_\varepsilon \).

The sheaf \( \mathcal{G}_d(\cdot) \) turns out to be a sheaf of differential algebras (the derivatives are defined on \( \mathcal{M}_d(\cdot) \) component-wise). Moreover, \( \mathcal{G}_d(\cdot) \) is a sheaf of modules over the factor ring \( \mathcal{G}_d = \mathcal{M}_d(\mathcal{C})/\mathcal{N}_d(\mathcal{C}) \) where
\[
\begin{align*}
\mathcal{M}_{d,0}(\mathcal{K}) &= \{ (r_\varepsilon)_\varepsilon \in \mathbb{K}^{[0,1]} \mid \exists q \in \mathbb{N}, \quad |r_{\varepsilon}| = o(a_{-q}(\varepsilon)) \quad \text{as } \varepsilon \to 0 \}, \\
\mathcal{N}_{d,0}(\mathcal{K}) &= \{ (r_\varepsilon)_\varepsilon \in \mathbb{K}^{[0,1]} \mid \forall p \in \mathbb{N}, \quad |r_{\varepsilon}| = o(a_p(\varepsilon)) \quad \text{as } \varepsilon \to 0 \},
\end{align*}
\]
with \( \mathcal{K} = \mathbb{C} \) or \( \mathcal{K} = \mathbb{R}, \mathbb{R}_+ \). When \( a(\cdot) \) is the polynomial scale, defined by \( a_n(\varepsilon) = \varepsilon^n \), we recover the usual Colombou simplified sheaf of algebras [3]. We shall denote it by \( \mathcal{G}(\cdot) = \mathcal{M}(\cdot)/\mathcal{N}(\cdot) \) and the class of \( (f_\varepsilon)_\varepsilon \in \mathcal{M}(\Omega) \) by \( [f_\varepsilon] \). We shall use this algebra everywhere in the sequel, except in Section 4.1 where another scale is needed.

**Notation 2.**

(i) Let \( n \) be in \( \mathbb{N} \cup \{+\infty\} \) and set \( \mathbb{N}_n = \{0, \ldots, n\} \) if \( n \in \mathbb{N} \) and \( \mathbb{N}_\infty \equiv \mathbb{N} \).

(ii) We denote by \( \mathcal{R}_+(X) \) the set of polynomials in one variable with real positive coefficients and by \( \mathcal{R}_+(X, Y) \) the set of polynomials in two variables with no term of degree 0 in \( Y \).

Consider \( d_1 \) (resp. \( d_2 \)) an integer, \( \Omega_1 \) (resp. \( \Omega_2 \)) an open subset of \( \mathbb{R}^{d_1} \) (resp. \( \mathbb{R}^{d_2} \)).

**Definition 1.** Let \( p \) be in \( \mathbb{N} \cup \{+\infty\} \). A map \( \psi : \mathcal{C}^p(\Omega_1) \to \mathcal{C}^p(\Omega_2) \) is said to be continuously moderate if, for all \( K_2 \subseteq \Omega_2 \) there exist \( K_1 \subseteq \Omega_1 \), \( \lambda(\cdot) \), \( \mu(\cdot) \) in \( \mathbb{N}_+^{\mathbb{N}} \) two increasing sequences such that
\[
\begin{align*}
\forall f \in \mathcal{C}^p(\Omega_1), & \quad P_{K_2,l}(\psi(f)) \leq \lambda_l(P_{K_1,\lambda_l}(f)), \\
\forall f \in \mathcal{C}^p(\Omega_1), & \quad P_{K_2,l}(\psi(f + \eta) - \psi(f)) \leq \mu_l(P_{K_1,\lambda_l}(f), P_{K_1,\mu_l}(\eta)).
\end{align*}
\] (7) (8)

We say that the map \( \psi \) is uniformly continuously moderate (with respect to the \( \mathcal{C}^p \) topology) if \( \lambda(\cdot) \) and \( \mu(\cdot) \) do not depend on \( K_2 \). Set \( a_l = \max(\deg \lambda_l, \deg \mu_l) \), where \( \deg \lambda_l \) denotes the degree with respect to the variable \( X \) of \( \lambda_l \). Denote by \( b_l \) the highest power of \( Y \) which may be factorized in \( \mu_l \). Suppose that \( \psi \) is uniformly continuously moderate and that \( a_l \) (resp. \( b_l \)) do not depend on \( K_2 \) and \( l \). Then, we denote by \( a \) (resp. \( b \)) this common value and we say that \( \psi \) is \((a, b)\)-uniformly continuously moderate.

The extension principle (Proposition 3.2 in [7]) gives the next proposition.

**Proposition 1.** Any continuously moderate map \( \psi : \mathbb{C}^\infty(\Omega_1) \to \mathbb{C}^\infty(\Omega_2) \) admits a canonical extension \( \Psi : \mathcal{G}_d(\Omega_1) \to \mathcal{G}_d(\Omega_2) \) defined by \( \Psi(f) = [\psi(f_\varepsilon)]_{\varepsilon} \), where \( f \in \mathcal{G}_d(\Omega_1) \) and \( (f_\varepsilon)_\varepsilon \in f \).

**Remark 1.**

(i) We recall that condition (7) ensures the moderateness of the family \( \psi(f_\varepsilon)_\varepsilon \), whereas condition (8) ensures the independence of \( \psi(f_\varepsilon)_\varepsilon + \mathcal{N}(\Omega_2) \) with respect to the representative of \( f \). For the latter it is crucial that each monomial composing \( \Theta_l \) contains a power of the second variable.

(ii) In Section 4.1, we shall use an extension principle for family of maps. We say that a family \( \psi_\varepsilon : \mathcal{C}^p(\Omega_1) \to \mathcal{C}^p(\Omega_2)_\varepsilon \) is continuously moderate if condition (7) (resp. condition (8)) holds with \( \psi \) replaced by \( \psi_\varepsilon \), \( \Phi_l \) (resp. \( \Theta_l \)) replaced by \( \Phi_{l,\varepsilon} \) (resp. \( \Theta_{l,\varepsilon} \)). Moreover, we assume that the family \( \Phi_{l,\varepsilon} \) (resp. \( \Theta_{l,\varepsilon} \)) belongs to \( \mathcal{M}_{d,0}(\mathbb{R}_+)(X) \) the set of polynomial
with coefficients in $M_{a,0}(\mathbb{R}_+)$ (resp. $(\Theta_t, \epsilon)_\epsilon \in M_{a,0}(\mathbb{R}_+) (X, Y)$, the set of polynomials in two variables with no term of degree 0 in $Y$). This means, for example for $(\Phi_t, \epsilon)_\epsilon$, that $\Theta_t(X) = \sum_{j=0}^d a_{j, \epsilon} X^j$ with $(a_{j, \epsilon})_\epsilon \in M_{a,0}(\mathbb{R}_+)$ and $l$ not depending on $\epsilon$.

It can be easily checked that Proposition 1 still holds with $(\psi_\epsilon)_\epsilon$ instead of $\psi$.

Proposition 1 is used as follows. Consider

$$P(D)(u) = F(u), \quad u|_S = f.$$  \hfill(9)

We assume that for $f \in C^\infty(\mathbb{R}^d)$, there exists a unique solution $u_f \in C^\infty(\mathbb{R}^d_2)$ and that the map

$$\psi : C^\infty(\mathbb{R}^d) \rightarrow C^\infty(\mathbb{R}^d_2), \quad f \mapsto u_f$$

is continuously moderate. It follows from Proposition 1 that this map admits a canonical extension $\Psi$, which associates to $f \in \mathcal{G}(\mathbb{R}^d)$ the unique solution $u_f \in \mathcal{G}(\mathbb{R}^d_2)$ of (9).

**Example 2.** We consider the following slightly simplified version of (IVP)

$$\begin{cases}
(1) & \frac{\partial u}{\partial t} + (\varphi' \otimes 1_x) \frac{\partial u}{\partial x} = 0, \\
(2) & u|_{t=0} = f.
\end{cases}$$  \hfill(IVP0)

where $\varphi$ is, as before, a smooth function with $\varphi(0) = 0$ and $\varphi' > 0$ on $\mathbb{R}$. For $f \in \mathcal{G}(\mathbb{R})$, the following generalized function

$$u = ((t, x) \mapsto f_e(x - \varphi(t)))_e + \mathcal{N}(\mathbb{R}^2), \quad \text{where } (f_e)_e \in f,$$

is a solution to problem (IVP0).

Indeed, the map $\psi : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}^2)$ defined by $f \mapsto u_f = \{(t, x) \mapsto f(x - \varphi(t))\}$ is (1, 1)-uniformly continuously moderate. To see this, take a compact $K_2 \subset \mathbb{R}^2$. As $\varphi$ is continuous, there exists a compact $K_1 \subset \mathbb{R}$ such that $|x - \varphi(t)|$, $(t, x) \in K_2 \subset K_1$. Thus $P_{K_2}(u_f) \leq P_{K_1}(f)$ and condition (7) is satisfied with $\Phi_1(X) = X$ and $\lambda(l) = l$. From the definition of $\psi$, condition (8) is also obviously fulfilled with $\theta_1(X, Y) = Y$ and $\mu(l) = l$.

### 3.2. Generalized solutions and classical ones

#### 3.2.1. Introduction to the problem

The natural question which arises is the relationship between the generalized solution (10), the classical one (3) for $f \in C^\infty(\mathbb{R})$, and the distributional one (4) for $f \in \mathcal{D}'(\mathbb{R})$.

This question may be more precisely stated as follows. Starting from the sheaf $\mathcal{G}(\cdot)$ built over $\mathbb{R}^d$ ($d \in \mathbb{N}^*$), one shows [5,18] that there exist:

(i) A canonical sheaf embedding of $C^\infty(\cdot)$ into $\mathcal{G}(\cdot)$, through the morphism of algebra

$$\sigma^d_\Omega : C^\infty(\Omega) \rightarrow \mathcal{G}(\Omega), \quad f \mapsto (f)_\epsilon \quad (\Omega \text{ open subset of } \mathbb{R}^d);$$

(ii) A natural sheaf embedding of $\mathcal{D}'(\cdot)$ into $\mathcal{G}(\cdot)$, through the linear maps

$$\iota^d_\Omega : \mathcal{D}'(\Omega) \rightarrow \mathcal{G}(\Omega), \quad T \mapsto \{(T \ast \theta_{d,e})|_{\Omega}\}_e,$$

where $\theta_{d,e} = e^{-d} \rho_d(\cdot/\mathbb{R}) \chi_d(|\ln \mathbb{R}|)$ with

$$\rho_d \in \mathcal{S}(\mathbb{R}^d) \quad \text{with} \quad \int \rho_d(x) \, dx = 1, \quad \int x^m \rho_d(x) \, dx = 0 \quad \text{for all } m \in \mathbb{N}^d \setminus \{0\},$$

$$\chi_d \in \mathcal{D}(\mathbb{R}^d) \quad \text{with} \quad 0 \leq \chi_d \leq 1 \quad \text{and} \quad \chi_d = 1 \quad \text{on a neighborhood of } 0.$$

Moreover, the following diagram is commutative

$$\begin{CD}
C^\infty(\Omega) @>{\sigma^d_\Omega}>> \mathcal{D}'(\Omega) \\
@VV{\iota^d_\Omega}V \quad \quad \quad \quad \\
\mathcal{G}(\Omega)
\end{CD}$$

that is $\iota^d|_{C^\infty(\Omega)} = \sigma^d_\Omega$. 
Note that the spaces $C^p(\Omega)$ ($p \in \mathbb{N}$) are embedded into $G(\Omega)$ through the canonical embedding of $C^p(\Omega)$ into $D'(\Omega)$ and the restriction of $i_d$ to (the image of) $C^p(\Omega)$ in $D'(\Omega)$. This morphism is a linear one but not a morphism of algebra.

With this material we can state the generalized problem corresponding to a classical one and make the comparison between their respective solutions.

**Example 3.** Consider the problem (IVP0) with the data $f \in C^p(\mathbb{R})$. Since $C^p(\mathbb{R})$ is embedded into $G(\mathbb{R})$ through $i_1|C^p(\mathbb{R})$, the generalized problem associated to (IVP0) is the problem

$$
\begin{align*}
(1) \quad & \frac{\partial u}{\partial t} + (\varphi' \otimes 1_x) \frac{\partial u}{\partial x} = 0, \\
(2) \quad & u_{|t=0} = f_1,
\end{align*}
$$

where $f_1 = [f \ast \theta_{1,\varepsilon}]$. As in Example 2, the following generalized function

$$
u = \left((u_1, t, x) \right)_{\varepsilon} + \mathcal{N}(\mathbb{R}^2) = \left((t, x) \mapsto (f \ast \theta_{1,\varepsilon})(x - \varphi(t))\right)_{\varepsilon} + \mathcal{N}(\mathbb{R}^2)
$$

is a solution to this problem.

### 3.2.2. Asymptotic behavior of generalized solutions with respect to classical solutions

The notations are those of Section 3.1 and especially of Definition 1. The following proposition gives a general tool for our purpose, as detailed below.

**Proposition 2.** Consider a map $\psi : C^p(\Omega_1) \to C^q(\Omega_2)$ such that $\psi|_{C^\infty(\Omega_1)}$ maps $C^\infty(\Omega_1)$ into $C^\infty(\Omega_2)$. Furthermore, assume that $\psi$ is $(a, b)$-uniformly continuously moderate. Let $\tilde{q}$ be the greatest integer $q$ such that $p - \mu(q)$ is positive.

(i) For $f \in C^q(\Omega_2)$, we have $\lim_{\varepsilon \to 0} \psi(f \ast \theta_{d,\varepsilon}) = \psi(f)$ for the $C^{\min(p, \tilde{q})}$ topology.

(ii) More precisely, for all $l \in \mathbb{N}_{\min(p, \tilde{q})}$, we have

$$
\forall K \in \mathbb{R}^d, \forall f \in C^p(\Omega_1), \quad \left|P_{K, l}(\psi(f) - \psi(f \ast \theta_{d,\varepsilon}))\right| = O(e^{b(p-\mu(l))}) \quad \text{as } \varepsilon \to 0.
$$

Before proving this result, let us recall a classical result.

**Lemma 3.**

(i) For any $q \in \mathbb{N}$, we have

$$
\forall g \in C^q(\mathbb{R}^d), \forall K \in \mathbb{R}^d, \quad \left|P_{K, 0}(g - g \ast \theta_{d,\varepsilon})\right| = O(e^{bq}) \quad \text{as } \varepsilon \to 0.
$$

(ii) For any $g \in C^q(\mathbb{R}^d)$, $g \ast \theta_{d,\varepsilon} \to g$ as $\varepsilon \to 0$ for the usual $C^0$ topology.

The proof of the first assertion of Lemma 3 uses the Taylor expansion applied to $f \ast \theta_{d,\varepsilon} - f$ at the order $q$. (See [5] for the case $q = +\infty$.) The second one uses the following properties of $(\theta_{d,\varepsilon})_{\varepsilon}$

$$
\int |\theta_{d,\varepsilon}|(t) dt \quad \text{bounded,} \quad \int \theta_{d,\varepsilon}(t) dt = 1 + o(1) \quad \text{and} \quad \text{supp } \theta_{d,\varepsilon} \to \{0\} \quad \text{as } \varepsilon \to 0,
$$

and classical topological arguments.

**Proof of Proposition 2.** Take $f \in C^p(\mathbb{R}^d)$ and $l \in \mathbb{N}_{\min(p, \tilde{q})}$. Consider $K_2 \subset \Omega_2$. From the $(a, b)$-uniformly continuously moderate of $\psi$, we get the existence of $K_1 \subset \Omega_1$ such that (8) holds. Setting $\eta_f = f \ast \theta_{d,\varepsilon} - f$ for $f \in C^p(\Omega_1)$, we have

$$
P_{K_2, l}(\psi(f \ast \theta_{d,\varepsilon}) - \psi(f)) \leq \Theta_1(P_{K_1, \lambda(0)}(f), P_{K_1, \mu(a)}(\eta_f))
$$

and

$$
P_{K_2, l}(\psi(f \ast \theta_{d,\varepsilon}) - \psi(f)) \leq (P_{K_1, \mu(0)}(\eta_f))^b \Theta_1(P_{K_1, \lambda(0)}(f), P_{K_1, \mu(a)}(\eta_f)),
$$

where, in $\Theta_1 \in \mathbb{R}_+(X, Y)$, no power of $Y$ can be factorized. Using Lemma 3(i) for $g = \partial^\alpha f$ with $\alpha \in \mathbb{N}^d$, $|\alpha| \leq \mu(l)$, we get

$$
P_{K_1, \mu(0)}(\eta_f) = P_{K_1, \mu(l)}(f \ast \theta_{d,\varepsilon} - f) = O(e^{b(q-\mu(l))}) \quad \text{as } \varepsilon \to 0.
$$

Thus

$$
P_{K_2, l}(\psi(f \ast \theta_{d,\varepsilon}) - \psi(f)) \leq C \varepsilon \Theta_1(P_{K_1, \lambda(0)}(f), e^{b(q-\mu(l))}) \quad \text{with } C > 0.
$$
for ε small enough. As l ∈ N_{\min(p, 0)}, we have p − \mu(l) > 0 and the underlined term in the right-hand side of the above relation remains bounded as ε → 0. Thus, we obtain assertion (ii). Moreover, if l is such that p − \mu(l) > 0, we have ε^{(p−\mu(l))} → 0 as ε → 0 and assertion (i) holds. Finally, if l = \min(p, 0), we use this time Lemma 3(ii) to obtain that
\[ P_{K_1}(\mu(l))(\eta_f) \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0. \] (15)

Assertion (i) also holds in this case. □

**Remark 2.** For f ∈ C^\infty(\mathbb{R}^d), we get that
\[ \forall K_2 \subset \mathbb{R}^d, \quad \forall (l, m) \in \mathbb{N}^2, \quad P_{K_2}(\psi(f * \theta_{l,d}, \varepsilon) - \psi(f)) = O(\varepsilon^m) \quad \text{as} \quad \varepsilon \rightarrow 0. \]

It follows that \lim_{\varepsilon \rightarrow 0} \psi(f * \theta_{l,d}, \varepsilon) = \psi(f) for the C^\infty topology and that [\psi(f)] = [\psi(f * \theta_{l,d}, \varepsilon)]. This result follows also from the commutativity of diagram (11).

With this proposition, we can only compare the classical solution u of a problem (9) with a representative of the generalized solution \( u \) and not the image of u in \( \mathcal{G}(\mathbb{R}^2) \), with u. The following corollary gives a general answer for this problem.

**Corollary 4.** Consider a map \( \psi : C^\infty(\Omega_1) \rightarrow C^\infty(\Omega_2) \) such that \( \psi|_{C^\infty(\Omega_1)} \) maps \( C^\infty(\Omega_1) \) into \( C^\infty(\Omega_2) \). Furthermore, assume that \( \psi \) is (a, b)-uniformly continuous and moderate. Let \( \tilde{q} \) be the greatest integer \( q \) such that \( p + \mu(q) \) is positive. For \( f \in C^\infty(\Omega_1) \), \( t_2 \in [\mathcal{C}(\Omega_2)](f) \) and \( \psi(t_1|_{\mathcal{C}^\infty(\Omega_1)}(f)) \) are associated in the \( C^{\min(p, \tilde{q})}(R^2) \) sense, that is
\[ \forall l \in N_{\min(p, 0)}, \forall K \subset \mathbb{R}^d, \quad P_{K}(\psi(f * \theta_{l,d}, \varepsilon) - \psi(f_\varepsilon)) \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0. \]

The proof of this result uses similar tools as the one of Proposition 2 and is left to the reader. These results are used as follows. Suppose that the map \( \psi : C^\infty(\mathbb{R}^d) \rightarrow C^\infty(\mathbb{R}^d) \) which associates to the data \( f \in C^\infty(\mathbb{R}^d) \) the solution \( u_f \) of problem (9) satisfies the assumptions of Proposition 2. Denote by \( u_\varepsilon \) the generalized solution of problem (9), corresponding to a C^p data \( f_\varepsilon \). If the classical C^p solution \( u_f \) of the same problem exists, we have \( u_\varepsilon \rightarrow u_f \) for the \( C^{\min(p, \tilde{q})} \) topology, which is the best result one may expect.

**Example 4 (Continuation of Example 3).** Let \( (u_{1, \varepsilon})_\varepsilon \) be a representative of the solution \( u \) of generalized problem (IVP0) with a C^p data. Using Proposition 2, we get that
\[ \forall K \subset \mathbb{R}^2, \quad P_{K}(u - u_{1, \varepsilon}) \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0, \]
for all \( l \in N_p \). Indeed, we can take here \( \lambda(l) = \mu(l) = l, a = b = 1 \). (See Example 2.) Thus, any representative of the generalized solution (12) converges to the classical one (3) for the C^p-topology.

Moreover, Corollary 4 gives that \( t_2|_{\mathcal{C}^\infty(\mathbb{R}^2)}(u_\varepsilon) \) and \( u_\varepsilon \) are associated in the C^p sense \( [18] \), that is
\[ \forall K \subset \mathbb{R}^2, \quad P_{K}(u * \theta_{2, \varepsilon} - u_{1, \varepsilon}) \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0. \]

Note that Corollary 4 also implies that \( t_2|_{\mathcal{C}^\infty(\mathbb{R}^2)}(u_\varepsilon) \) and \( u_\varepsilon \) are associated in \( \mathcal{G}(\mathbb{R}^2) \) in the usual D' sense \( [3] \), that is
\[ u * \theta_{2, \varepsilon} - u_{1, \varepsilon} \rightarrow 0 \quad \text{in} \quad D'(\mathbb{R}) \quad \text{as} \quad \varepsilon \rightarrow 0. \]

For the case of a continuously moderate map, we still have a weaker result, which is valid only for C^\infty maps. (The proof is similar to the previous ones.)

**Proposition 5.** Take \( \psi : C^\infty(\Omega_1) \rightarrow C^\infty(\Omega_2) \) a continuously moderate map. For \( f \in C^\infty(\Omega_1) \), we have \( \lim_{\varepsilon \rightarrow 0} \psi(f * \theta_{1, \varepsilon}) = \psi(f) \) for the C^\infty topology.

3.3. Generalized solutions and distribution ones

Using the embedding \( t_1 \), we find that the generalized problem associated to (IVPD) is
\[
\begin{align*}
(1) \quad & \frac{\partial u}{\partial t} + (\phi' \otimes 1_x) \frac{\partial u}{\partial x} = 0, \\
(2) \quad & u|_{t=0} = f_1,
\end{align*}
\]
where \( f_1 = [v \ast \theta_{1, \varepsilon}] \). Following Proposition 2, this problem admits
\[ u = (u_{1,\varepsilon})_\varepsilon + N'(\mathbb{R}^2) = \left( t, x \mapsto (v \ast \theta_{1,\varepsilon})(x - \varphi(t)) \right)_\varepsilon + N'(\mathbb{R}^2) \]  
\[ (\text{IVPL}) \]

as solution. We shall study the asymptotic behavior \( v_{\Gamma_p} - u_{1,\varepsilon} \) for \( \varepsilon \) tending to 0 for the \( D'(\mathbb{R}^2) \) topology, where \( v_{\Gamma_p} \) is the distribution solution defined by (4).

For this, take \( \Phi \in D(\mathbb{R}^2) \), we have to evaluate \( \langle V_\varepsilon, \Phi \rangle = \langle v_{\Gamma_p} - u_{1,\varepsilon}, \Phi \rangle \). On one hand, we have

\[ \langle v_{\Gamma_p}, \Phi \rangle = \int (v, x \mapsto \int \Phi(t, x + \varphi(t)) \, dt) = \int (v, x \mapsto \Phi(t, x + \varphi(t)) \, dt) \, dx \]

On the other hand, we have

\[ \langle u_{1,\varepsilon}, \Phi \rangle = \int (v \ast \theta_{1,\varepsilon})(x - \varphi(t)) \Phi(t, x) \, dt \, dx = \int (v \ast \theta_{1,\varepsilon})(x) \Phi(t, x + \varphi(t)) \, dx \]

By setting \( \Phi_t : x \mapsto \Phi(t, x + \varphi(t)) \), we finally get

\[ \langle V_\varepsilon, \Phi \rangle = \int (v - v \ast \theta_{1,\varepsilon}, \Phi_t) \, dt \]  
\[ (\text{IVPL}) \]

Consider \( K_1 \) and \( K_2 \), two compacts intervals of \( \mathbb{R} \) such that \( \supp \Phi \subset K_1 \times K_2 \). As \( \varphi \) is continuous, there exists a compact interval \( K_3 \) of \( \mathbb{R} \) such that \( \supp((t, x) \mapsto \Phi_t(x)) \subset K_1 \times K_3 \). Thus, for all \( t \in K_1 \), the support of \( \Phi_t \) is included in the same compact \( K_3 \). Moreover, \( \sup_{t \in K_1} |\Phi_t(x)| \leq \sup_{(t, x) \in K_1 \times K_3} |\Phi(t, x)| < \infty \). Using the classical criterion [24], the family \( \{\Phi_t\} \) is bounded in \( D'(\mathbb{R}) \). As \( v - v \ast \theta_{1,\varepsilon} \to 0 \) as \( \varepsilon \to 0 \), for the strong dual topology [24], \( \langle v - v \ast \theta_{1,\varepsilon}, \Phi_t \rangle \to 0 \) uniformly in \( t \) as \( \varepsilon \to 0 \). Thus \( \langle V_\varepsilon, \Phi \rangle \to 0 \) as \( \varepsilon \to 0 \), since the integral in (17) is performed on a compact set. We have proved the:

**Proposition 6.** \( \lim_{\varepsilon \to 0} u_{1,\varepsilon} = v_{\Gamma_p} \) for the \( D' \) topology.

Thus the generalized solution \( u \) is associated to the distribution \( v_{\Gamma_p} \) in the usual \( D' \) sense [3].

### 3.4. The Lipschitz type nonlinearity

In this subsection we study the following nonlinear generalization of Example 3

\[
\begin{cases}
(1) & \frac{\partial u}{\partial t} + (\varphi' \otimes 1_x) \frac{\partial u}{\partial x} = F(\cdot, \cdot, u), \\
(2) & u|_{t=0} = f 
\end{cases} \tag{IVPL}
\]

for \( f \in \mathcal{G}(\mathbb{R}) \). We assume in this subsection that the right-hand side of (1) satisfies the following assumption

\[
\forall K \subset \mathbb{R}^2, \quad \sup_{(t, x) \in K, z \in \mathbb{R}} \left| \frac{\partial}{\partial z} F(t, x, z) \right| = m_K < +\infty, \tag{18}\]

which is little stronger than a Lipschitz condition. We shall use the method given in Section 3.1. We first show existence and uniqueness of the solution for the classical problem and then the moderneness of the map which associates the solution to the classical data (which is the difficult and technical part). Afterwards, Proposition 1 easily transfers the existence and uniqueness result to the problem with generalized data.

#### 3.4.1. Existence and moderneness results from regular data

We assume that \( f \in C^\infty(\mathbb{R}) \). Let \( U \) be the pullback of \( u \) by the change \( T = t, X = x - \varphi(t) \) and \( \varphi(0) = 0 \). Then the problem (IVPL) is changed into

\[
\begin{cases}
(1) & \frac{\partial U}{\partial T}(T, X) = F(T, X, U(T, X)), \\
(2) & U(0, X) = f(X) 
\end{cases} \tag{IVPLR}
\]

**Approximation techniques for the Cauchy problem**

**Proposition 7.** The problem (IVPLR) has a unique solution \( U_f \) verifying

\[
U_f(T, X) = f(X) + \int_0^T F(T, X, U_f(T, X)) \, dt. \tag{19}
\]
The classical proof goes as follows [8]. We introduce the sequence of approximations \((U_n)_n\) defined by

\[
\forall n \in \mathbb{N}, \quad U_{n+1}(T, X) = f(X) + \int_0^T \mathcal{F}(\tau, X, U_n(\tau, X)) \, d\tau,
\]

and \(U_0(T, X) = f(X)\). Using the auxiliary series \(\sum (U_n - U_{n-1})\), we show the uniform convergence of \((U_n)_n\) on every compact subset \(K = [-\lambda, \lambda] \times [-\lambda, \lambda] \subseteq \mathbb{R}^2\) toward a continuous function \(U\) satisfying (19). The uniqueness follows from Gronwall’s lemma, applied to the difference \(\Delta = W - U\), where \(W\) is another solution of (19), shown to be equal to \(U\). Note that we use, in an essential way, the hypothesis (18) in this proof.

**Remark 3.** The integral formula (19) and Gronwall’s lemma give easily the following estimate, useful in the sequel: For any compact subset \(K\) of \(\mathbb{R}^2\), there exists \(\lambda > 0\) such that \(K \subseteq [-\lambda, \lambda]^2\) and

\[
p_{K,0}(U) \leq c_{0,\lambda} + d_{0,\lambda} P_{1,-\lambda,\lambda,0}(f),
\]

where \(c_{0,\lambda} = \lambda \exp(\lambda m_K) \|\mathcal{F}(\cdot, 0)\|_{\infty, K}\) and \(d_{0,\lambda} = \exp(\lambda m_K)\).

**Moderation.** We assume in addition that

\[
\forall l \in \mathbb{N} \setminus \{0\}, \quad \forall K \subseteq \mathbb{R}^2, \exists M_{K,l} > 0, \quad \forall \alpha \in \mathbb{N}^3 \text{ with } |\alpha| = l, \quad \sup_{(t,x) \in K, z \in \mathbb{R}} |D^\alpha \mathcal{F}(t, x, z)| \leq M_{K,l}. \tag{20}
\]

**Proposition 8.** The map \(\psi : f \mapsto U_f\) where \(U_f\), given by (19), is continuously moderate.

**Proof.** First part. We begin by showing that condition (7) is fulfilled. Take \(K\) a compact subset of \(\mathbb{R}^2\). From Remark 3, there exist \(\lambda > 0\) and \((c_{0,\lambda}, d_{0,\lambda}) \in \mathbb{R}_+^2\) such that

\[
p_{K,0}(U) \leq c_{0,\lambda} + d_{0,\lambda} P_{1,-\lambda,\lambda,0}(f) \quad \text{with } I_3 = [-\lambda, \lambda]\]

giving the estimate of order 0 in view of (7).

For the estimates of higher order, we begin by estimating the derivatives of \(\mathcal{H}(T, X) = \mathcal{F}(T, X, U(T, X))\). For any \(n \geq 0\), \(m \geq 0\), \(\alpha = (n, m)\) with \(|\alpha| \geq 1\), the multivariate Faa di Bruno’s formula [4] implies that

\[
\frac{\partial^{n+m} \mathcal{H}}{\partial T^n \partial X^m}(T, X) = \sum_{|\beta| \leq n+m} c_{\beta} D^\beta \mathcal{F}(T, X, U(T, X)) \sum_{i=1}^{n+m} \sum_{(p_i(\alpha, \beta))} d_{i,\alpha,\beta} \prod_{j=1}^i (D^j U(T, X))^{k_j} + \frac{\partial \mathcal{F}}{\partial z}(T, X, U(T, X)) D(n,m)(U(T, X)),
\]

where \(\beta \in \mathbb{N}^3\), \(c_{\beta} \geq 0\) and \(c_{(0,0,1)} = 0\), \(d_{i,0,0,\beta} \geq 0\), \(\sum_{j=1}^i k_j = |\beta|\), \(\sum_{j=1}^i j_k = |\alpha|\) and \(p_i(\alpha, \beta)\) is a finite set. Note that in the first sum the derivatives of \(U\) which appear are of order less or equal to \(n+m-1\).

Using assumption (20) to get an upper bound of the terms \(D^\beta \mathcal{F}(T, X, U(T, X))\) on \(K_\alpha\), we obtain the existence of a polynomial \(\Psi_{K_\alpha,l}\) with positive coefficients depending only on \(F\), \(K_\alpha\) and \(l = n + m\) such that

\[
\left| \frac{\partial^{n+m} \mathcal{H}}{\partial T^n \partial X^m}(T, X, U(T, X)) \right| \leq \Psi_{K_\alpha,n+m-1}(P_{K_\alpha,n+m-1}(U)) + m K_\alpha \left| \frac{\partial^{n+m} U}{\partial T^n \partial X^m}(T, X) \right|. \tag{21}
\]

From this we deduce that, for any \(K \subseteq \mathbb{R}^2\),

\[
P_{K,n}(U) \leq \Psi_{K,n+m}(P_{K,n+m-1}(U)) + m K_\alpha P_{K,n+m}(U). \tag{22}
\]

We turn now to the estimates concerning the derivatives of \(U\) and proceed by induction. We assume that for every \(\alpha = (n, m)\) with \(n + m \leq l\) there exist \((c_{i,\alpha})_{0 \leq i \leq l}\) such that

\[
P_{K,l}(U) \leq \sum_{j=0}^l c_{j,\alpha} (P_{K,j}(f))^{a_j/l}. \tag{23}
\]

**Case 1.** Suppose that \(n \geq 1\). Then, we have \(\frac{\partial^{n+m+1}}{\partial T^n \partial X^m} U(T, X) = \frac{\partial^{n+m}}{\partial T^n \partial X^m} \mathcal{H}(T, X)\) (resp. \(\frac{\partial^{n+m+1}}{\partial T^n \partial X^m} U(T, X) = \frac{\partial^{n+m}}{\partial T^n \partial X^m} \mathcal{H}(T, X)\)). From (22), we deduce easily that \(P_{K,(n+1,m)}(U)\) and similarly \(P_{K,(n,m+1)}(U)\) can be bounded in a similar way as \(P_{K,l}(U)\) in (23).

**Case 2.** Suppose that \(n = 0\). We cannot proceed as before. Nonetheless, from the integral expression (19), we have
\[
\frac{\partial^{m+1} U}{\partial X^{m+1}}(T, X) = f^{(m+1)}(X) + \int_0^T \frac{\partial^{m+1} F}{\partial X^{m+1}}(\tau, X, U(\tau, X)) \, d\tau.
\]  

(24)

According to (21), we have, for \((T, X) \in K_\lambda\),

\[
\left| \frac{\partial^{m+1} F}{\partial X^{m+1}}(T, X, U(T, X)) \right| \leq \Psi_{K_\lambda, m+1}(P_{K_\lambda, m}(U)) + m_K \left| \frac{\partial^{m+1} U}{\partial X^{m+1}}(T, X) \right|.
\]

(25)

Using the induction hypothesis, we get a polynomial \(Q_m\) with positive coefficients such that

\[
\Psi_{K_\lambda, m+1}(P_{K_\lambda, m}(U)) \leq Q_m(P_{1, t}(f)) = q_m(f).
\]

Thus

\[
\left| \frac{\partial^{m+1} U}{\partial X^{m+1}}(T, X) \right| \leq \left| f^{(m+1)}(X) \right| + \int_0^T \left( q_m(f) + m_K \left| \frac{\partial^{m+1} U}{\partial X^{m+1}}(T, X) \right| \right) \, d\tau
\]

\[
\leq P_{1, t+1}(f) + \lambda q_m(f) + \int_0^T m_K \left| \frac{\partial^{m+1} U}{\partial X^{m+1}}(T, X) \right| \, d\tau.
\]

From Gronwall’s lemma, we get

\[
\left| \frac{\partial^{m+1} U}{\partial X^{m+1}}(T, X) \right| \leq (\lambda q_m(f) + P_{1, t+1}(f)) \exp \left( \int_0^T m_K \, d\tau \right).
\]

Thus \(P_{K_\lambda, t+1}(U) \leq (\lambda q_m(f) + P_{1, t+1}(f)) \exp(\lambda m_{K_\lambda})\), which gives a similar estimate as (23).

Summing up Cases 1 and 2, and replacing in the latter \(q_m(f)\) by its value, we finally get the existence of \((c_{j, t+1})_{0 \leq j \leq t+1, (\alpha_{j, t+1})_{0 \leq j \leq t+1}}\) such that

\[
P_{K, t+1}(U) \leq \sum_{j=0}^{t+1} c_{j, t+1}(P_{1, t}(f))^{\alpha_{j, t+1}}.
\]

Thus condition (7) is fulfilled. In addition, note that the coefficients \(c_{j, t+1}\) are polynomials in the variables \(\lambda, \exp(\lambda m_{K_\lambda})\) with coefficients depending on \(F\) and \(K_\lambda\).

**Second part.** We turn now to condition (8). Let \(V\) be another solution to (IVPLR) corresponding to the data \(f + \eta\). Set \(W = V - U\). Consider \((T, X) \in K_\lambda\). As

\[
V(T, X) = f(X) + \eta(X) + \int_0^T F(\tau, X, V(\tau, X)) \, d\tau
\]

we get

\[
W(T, X) = \eta(X) + \int_0^T (F(\tau, X, V(\tau, X)) - F(\tau, X, U(\tau, X))) \, d\tau
\]

\[
= \eta(X) + \int_0^T W(\tau, X) \left( \int_0^1 \frac{\partial F}{\partial z}(\tau, X, U(\tau, X) + \theta W(\tau, X)) \, d\theta \right) \, d\tau.
\]

Thus

\[
|W(T, X)| \leq m_K \int_0^T |W(\tau, X)| \, d\tau + P_{1, 0}(\eta).
\]

Set \(e(\tau) = \sup_{X \in [-\lambda, \lambda]} |W(\tau, X)|\). From the last inequality, we get

\[
\forall T \in [-\lambda, \lambda], \quad e(T) \leq m_K \int_0^T e(\tau) \, d\tau + P_{1, 0}(\eta).
\]
Thus, according to Gronwall’s lemma, we obtain
\[
\forall T \in [-\lambda, \lambda], \quad e(T) \leq P_{1,0}(\eta) \exp\left(\int_0^T m_{K,\lambda} \, d\tau\right) = P_{1,0}(\eta) \exp(m_{K,\lambda}).
\]

Consequently \(P_{K,0}(W) \leq \exp(m_{K,\lambda})P_{1,0}(\eta)\), which implies that the 0th-order estimate holds. For the higher order estimates, let us recall that
\[
W(T, X) = \eta(X) + \int_0^T W(\tau, X)\left(\int_0^1 \frac{\partial F}{\partial Z}(\tau, X, U(\tau, X) + \theta W(\tau, X)) \, d\theta\right) \, d\tau.
\]

Set
\[
L(T, X, \theta) = U(T, X) + \theta W(T, X) = (1 - \theta)U(T, X) + \theta V(T, X),
\]
\[
R(T, X) = \frac{1}{0} \frac{\partial F}{\partial Z}(T, X, L(T, X, \theta)) \, d\theta.
\]

Using very similar methods as in the first part, we show that, for every \(K_\lambda\) and \((n, m) \in \mathbb{N}^2\) with \(n + m \geq 1\), the analogue of (21) holds for \(R\), that is
\[
\frac{\partial^{n+m}R}{\partial T^n \partial X^m}(T, X) \leq \Psi(n, m, U, V) + M_{K,2} \left|\frac{\partial^{n+m}L}{\partial T^n \partial X^m}(T, X)\right|,
\]
where \(\Psi(n, m, U, V) = \Psi(P_{K,\lambda,n+m-1}(U), P_{K,\lambda,n+m-1}(V))\) is a polynomial with positive coefficients depending only of \(K_\lambda\) and \(L\). Thus, there exists a polynomial \(\Psi_{(n,m)} \in \mathbb{R}_+(X)\) with positive coefficients such that
\[
P_{K_\lambda,(n,m)}(R(\cdot, \cdot, L)) \leq \Psi_{(n,m)}(P_{K,\lambda,n+m-1}(f)).
\]

According to the Leibniz rule, we have
\[
\frac{\partial^{n+m+1}W}{\partial T^{n+1} \partial X^m}(T, X) = \frac{\partial^m}{\partial X^m}\left(\sum_{k=0}^n C_n^k \frac{\partial^k W}{\partial T^k}(T, X) \frac{\partial^{n-k}R(T, X)}{\partial T^{n-k}}\right)
\]
\[
= \sum_{k=0}^n C_n^k \left(\sum_{j=0}^m C_m^j \frac{\partial^{k+j} W}{\partial T^k \partial X^j}(T, X) \frac{\partial^{n-m-k-j}R(T, X)}{\partial T^{n-m-k-j} \partial X^{m-j}}\right).
\]

Thus
\[
\frac{\partial^{n+m+1}W}{\partial T^{n+1} \partial X^m}(T, X) \leq \sum_{k=0}^n C_n^k \left(\sum_{j=0}^m C_m^j \left|\frac{\partial^{k+j} W}{\partial T^k \partial X^j}(T, X)\right|P_{K_\lambda,(n-k,m-j)}(R)\right)
\]
and
\[
P_{K_\lambda,(n+1,m)}(W) \leq \sum_{k=0}^n C_n^k \left(\sum_{j=0}^m C_m^j P_{K_\lambda,(k,j)}(W)P_{K_\lambda,(n-k,m-j)}(R)\right).
\]

Remark that in the above inequality, the right-hand side consists of a sum of products. These products are formed by two terms, the first one depending on \(W\) and the other one on \(R\). Recalling that, in the 0th-order estimate, \(P_{K,0}(W)\) is bounded by a term depending on \(P_{1,0}(\eta)\) whereas the term \(P_{K_\lambda,(n-k,m-j)}(R)\) depends mainly on \(f\), we can prove by induction, with quite similar steps as above, the existence of a polynomial \(\Theta_l \in \mathbb{R}_+(X, Y)\) depending on \(l = n + m + 1\) and \(K_\lambda\) such that
\[
P_{K_\lambda,(n+1,m)}(W) \leq \Theta_l(P_{l,1}(f), P_{l,1}(\eta)).
\]
(Note that inequality (27) is essential to obtain the estimates for mixed derivatives, whereas (26) is used to obtain the estimates for the derivatives with respect to \(X\).) Thus, condition (8) is fulfilled. □
3.4.2. Nonregular data and generalized solutions

We return to the generalized problem (IVPL).

**Proposition 9.** For \( f \in \mathcal{G}(\mathbb{R}) \), the problem (IVPL) admits in \( \mathcal{G}(\mathbb{R}^2) \) a unique generalized solution \( u = [u_ε] \) verifying

\[
\begin{align*}
\frac{\partial u_ε}{\partial t}(t, x) &= f_ε(x - ϕ(t)) + \int_0^t F(τ, x, u_ε(τ, x)) \, dτ,
\end{align*}
\]

where \( (f_ε)_ε \) is a representative of \( f \).

**Proof.** When \( f \in C^\infty(\mathbb{R}) \), Proposition 7 gives the solution \( U_f \) of (IVPLR). It is easy to deduce that (IVPL) has a unique solution verifying

\[
\begin{align*}
\frac{\partial u_f}{\partial t}(t, x) &= f(x - ϕ(t)) + \int_0^t F(τ, x, u_f(τ, x)) \, dτ.
\end{align*}
\]

Then, Proposition 8 shows easily that the map \( f \mapsto u_f \) is continuously moderate and Proposition 1 leads to the result. \( \square \)

From Proposition 5, we deduce that when \( f \) is in \( C^\infty(\mathbb{R}) \), the generalized solution \( u = [u_ε] \) associated to the data \( f_1 = [f * δ_{1,ε}] \) and the classical smooth solution \( u_f \) associated to \( f \) are such that \( u_{1,ε} \rightharpoonup u_f \) for the \( C^∞ \) topology. We easily get the following:

**Proposition 10.** For a data \( f \) in \( C^\infty(\mathbb{R}) \), we have \( u = σ_2(u_f) \), where \( σ_2 \) is the canonical embedding of \( C^\infty(\mathbb{R}^2) \) into \( \mathcal{G}(\mathbb{R}^2) \).

4. Regularization of problems

As mentioned in the introduction, we are going to study two regularizing techniques, for singular problems admitting, in general, no classical solutions. Their common point is that the differential problem itself is regularized.

4.1. The non-Lipschitz nonlinearity

We start from the following problem

\[
\begin{align*}
\begin{cases}
(1) & \frac{\partial u}{\partial t} + (\varphi' \otimes 1_x) \frac{\partial u}{\partial x} = F(\cdot, \cdot, u), \\
(2) & u|_{(0)×\mathbb{R}} = f,
\end{cases}
\end{align*}
\]

(IVPnL)

with \( f \in C^\infty(\mathbb{R}) \). Here, \( F \) can be non-Lipschitz but still have polynomial growth. More precisely, we assume the existence of \( k \in \mathbb{N} \) such that

\[
\forall K \subseteq \mathbb{R}^2, \forall α \in \mathbb{N}^2, \exists C_{K, α} > 0, \quad \sup_{(t, x) \in K, z \in \mathbb{R}} |D^α F(t, x, z)| \leq C_{K, α}(1 + |z|)^k.
\]

(29)

The same change of variables as in Section 3.4.1 leads to

\[
\begin{align*}
\begin{cases}
(1) & \frac{\partial U}{\partial T}(T, X) = F(T, X, U(T, X)), \\
(2) & U(0, X) = f(X),
\end{cases}
\end{align*}
\]

(IVPnL2)

This problem has generally no global solution. It is easy to see that, when taking \( F(\cdot, \cdot, U) = U^2 \) and \( f = 1 \), the local solution \( U(T, X) = \frac{1}{1-T} \) exists in \( ]-∞, 1[×\mathbb{R} \) but blows up for \( T = 1 \).

4.1.1. Cut-off process

Let \( (r_ε)_ε \in \mathcal{M}_0(\mathbb{R}_+) \) be such that \( r_ε > 0 \) and \( \lim_{ε→0} r_ε = +∞ \). Consider a family of smooth one-variable functions \( (g_ε)_ε \) such that \( 0 \leq g_ε \leq 1 \) and

\[
g_ε(z) = 0 \quad \text{if} \quad |z| \geq r_ε; \quad g_ε(z) = 1 \quad \text{if} \quad -r_ε + 1 \leq z \leq r_ε - 1.
\]

Moreover assume that, for every integer \( n > 0 \), \( g_ε^{(n)} \) is bounded independently of \( ε \) and set

\[
\sup_{z \in [-r_ε, r_ε]} |g_ε^{(n)}(z)| = κ_n.
\]

(30)
Set $\phi_\epsilon(z) = zg_\epsilon(z)$. A routine checking shows that $(\phi_\epsilon)_\epsilon \in \mathcal{M}(\mathbb{R})$. We approximate the function $(T, X, z) \mapsto \mathcal{F}(T, X, z)$ by the family of functions

$$(T, X, z) \mapsto \mathcal{F}_\epsilon(T, X, z) = \mathcal{F}(T, X, \phi_\epsilon(z)).$$

So we can approximate problem (IVPnL3) by the family of regularized problems

$$\begin{cases}
(1) & \frac{\partial U_{\epsilon}}{\partial T}(T, X) = \mathcal{F}_\epsilon(T, X, U_{\epsilon}(T, X)), \\
(2) & U_{\epsilon}(0, X) = f(X).
\end{cases} \tag{IVPnL3_\epsilon}$$

4.1.2. Existence and uniqueness of a generalized solution

Lemma 11.

(i) For any $\epsilon$, the problem (IVPnL3_\epsilon) admits a unique solution $U_{f,\epsilon}$ verifying

$$U_{f,\epsilon}(T, X) = f(X) + \int_0^T \mathcal{F}_\epsilon(\tau, X, U_{f,\epsilon}(\tau, X)) \, d\tau.$$  

(ii) Moreover, for any compact $K \subseteq \mathbb{R}^2$ and $\lambda > 0$ such that $K \subseteq [-\lambda, \lambda]^2$, we have

$$p_{K,0}(U_{f,\epsilon}) \leq c_{\lambda,\epsilon} + d_{\lambda,\epsilon} p_{[-\lambda,\lambda],0}(f),$$

where $c_{\lambda,\epsilon} = \lambda \exp(\lambda m_{K_\epsilon,\epsilon}) \|\mathcal{F}(\cdot, 0)\|_{\infty, K_\epsilon}$ and $d_{\lambda,\epsilon} = \exp(\lambda m_{K_\epsilon,\epsilon})$.

Proof. Consider $K \subseteq \mathbb{R}^2$. We have

$$\left| \frac{\partial}{\partial z} \mathcal{F}_\epsilon(T, X, z) \right| \leq \sup_{(T, X) \in K, z \in \mathbb{R}} \left| \frac{\partial}{\partial z} \mathcal{F}(T, X, \phi_\epsilon(z)) \right| \sup_{z \in \mathbb{R}} |\phi_\epsilon'(z)| \leq CK_{\epsilon}(1 + r_{\epsilon})^k_1 = m_{K,\epsilon} < +\infty.$$  

Thus, assumption (18) is satisfied with a constant depending on $\epsilon$. Then, Proposition 7 implies the first claim. The second claim follows from Remark 3. \hfill \Box

Starting from Lemma 11, we can estimate the derivatives of $U_{f,\epsilon}$. To do so, we first have to search the analogue to assumption (20) in this new parametrized situation. We proceed by induction. For $n = 0$, we have

$$\sup_{(T, X) \in K, z \in \mathbb{R}} \left| \mathcal{F}_\epsilon(T, X, z) \right| \leq \sup_{(T, X) \in K, z \in \mathbb{R}} \mathcal{F}(T, X, \phi_\epsilon(z)) \leq C_{K,0}(1 + r_{\epsilon})^k = M_{K,0,\epsilon}.$$  

Let us assume that, for every $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq l$, there exists $C_{K,l} > 0$ such that

$$\sup_{(T, X) \in K, z \in \mathbb{R}} \left| D_\alpha \mathcal{F}_\epsilon(T, X, z) \right| \leq C_{K,l}(1 + r_{\epsilon})^k = M_{K,l,\epsilon}. \tag{33}$$

Take $\alpha = (n, m, p)$, with $n + m + p = l + 1$. For any $(n, m) \in \mathbb{N}^2$ with $n + m \leq l + 1$ we have

$$\forall (T, X, z) \in K \times \mathbb{R}, \left| \frac{\partial^{m+n}}{\partial T^m \partial X^n} \mathcal{F}_\epsilon(T, X, z) \right| = \left| \frac{\partial^{m+n}}{\partial T^m \partial X^n} \mathcal{F}(T, X, \phi_\epsilon(z)) \right| \leq C_{K,(m,n,0)}(1 + r_{\epsilon})^k, \tag{34}$$

according to the assumption (29). Thus, we obtain the result directly for all $\alpha = (n, m, p)$ with $n + m + p = n + 1$ and $p \geq 1$. According to Faà di Bruno's formula, we have

$$\forall (T, X, z) \in K \times \mathbb{R}, \left| \frac{\partial^{m+n+p}}{\partial T^m \partial X^n \partial z^p} \mathcal{F}_\epsilon(T, X, z) \right| = \left| \frac{\partial^{m+n+p}}{\partial T^m \partial X^n \partial z^p} \mathcal{F}(T, X, \phi_\epsilon(z)) \right| = \sum_{(m_1, \ldots, m_n, p)} \frac{\partial^{m_1+\cdots+m_n}}{\partial z^{m_1+\cdots+m_n}} \left| \frac{\partial^{m+n}}{\partial T^m \partial X^n} \mathcal{F}(T, X, \phi_\epsilon(z)) \right| \prod_{j: \phi_{(m_j)}(z) \neq 0} \phi_{(m_j)}(z),$$

where the sum is over all $n$-tuples $(m_1, \ldots, m_n)$ of non-negative integers satisfying the constraint $m_1 + 2m_2 + \cdots + pm_p = p$ and $\phi_{(m_j)}(z)$ are strictly positive integers. Using the induction hypothesis (33), which is possible since $n + m \leq l$, the relation (34) and the estimate (30), we get the existence of a constant $C_{K,(m,n,p)}$ such that

$$\forall (T, X, z) \in K \times \mathbb{R}, \left| \frac{\partial^{m+n+p}}{\partial T^m \partial X^n \partial z^p} \mathcal{F}_\epsilon(T, X, z) \right| \leq C_{K,(m,n,p)}(1 + r_{\epsilon})^k.$$
Thus, for all $\alpha$ with $|\alpha| = l + 1$, we have the existence of a constant $C_{K,l+1}$ such that
\[ \sup_{(T,X) \in K, z \in R^l} |D^\alpha F_\varepsilon(T, X, z)| \leq C_{K,l+1}(1 + \varepsilon r)^k = M_{K,l} \varepsilon, \]
which concludes the induction.

Thus, relation (20) is now replaced by
\[ \forall l \in \mathbb{N}, \forall K \in \mathbb{R}^2, \forall \alpha \in \mathbb{N}^3 \text{ with } |\alpha| = l, \sup_{(T,X) \in K, z \in R^l} |D^\alpha F_\varepsilon(T, X, z)| \leq C_{K,l}(1 + \varepsilon r)^k. \]

From the first step of the proof of Proposition 8, it follows that
\[ P_{K,l+1}(U_{f,\varepsilon}) \leq \sum_{j=0}^{l+1} C_{j,l+1}(P_{l,j}(f))^{a_{l,j+1}}, \]
(35)
where the coefficients $c_{j,l+1}$ are polynomials in the variables $\lambda$, $\exp(\lambda m_{K,l})$ with coefficients depending on $F$ and $K_{\lambda} = [-\lambda, \lambda]^2$. Recalling that $m_{K,l} = C(1 + \varepsilon r)^k$ with $C > 0$, we can now choose as asymptotic scale $a_0(\varepsilon) = (\exp(\varepsilon r))^k$ (Indeed $(U_{f,\varepsilon})_{\varepsilon}$ will be in $\mathcal{M}_a(\mathbb{R}^2)$.)

Moreover, one can easily check that the family of maps
\[ F_{1,\varepsilon} : C^\infty(\mathbb{R}^2) \rightarrow C^\infty(\mathbb{R}^2), \quad u \mapsto F_\varepsilon(u, u, \cdot) \]
is continuously moderate (see Remark 1). Thus
\[ F_1 : G(\mathbb{R}^2) \rightarrow G(\mathbb{R}^2), \quad U = [U_{f,\varepsilon}]_{\varepsilon} \mapsto [F_\varepsilon(U_{f,\varepsilon})]_{\varepsilon} \]
where $(U_{f,\varepsilon})_{\varepsilon} \in U$. This gives a meaning to the following:

**Proposition 12.** The problem
\[
\begin{cases}
(1) \quad \frac{\partial U}{\partial T} = F_1(U), \\
(2) \quad U|_{T=0} = f
\end{cases}
\] (IVPnL3)
admits a unique generalized solution in $G_0(\mathbb{R}^2)$.

**Proof.** We have to verify that the net of maps $(f \mapsto U_{f,\varepsilon})_{\varepsilon}$ is continuously moderate in the sense given in Remark 1(ii), in order to apply Proposition 1. From relation (35), it follows that condition (7) is satisfied, as the (nets) of coefficients $(c_{j,l+1})_{\varepsilon}$ belong to $\mathcal{M}_a(\mathbb{R}^2)$, thanks to the choice of the scale $a$. Furthermore, a close inspection of the second part of the proof of Proposition 8 shows the following: the nets (indexed by $\varepsilon$) of coefficients of the nets of polynomials $(\theta_{l,\varepsilon})_{\varepsilon}$, which replaces in the present case the polynomial $\theta_l$ in relation (28), also belong to $\mathcal{M}_a(\mathbb{R}^2)$. Thus, condition (8) holds, which ends the proof.

**Remark 4.** A slight generalization of the previous proofs allows to solve problem (IVPnL) with a $f$ as irregular as a distribution. In this case, we regularize $f$ by convolution with the mollifier $(\theta_{l,\varepsilon})_{\varepsilon}$ introduced in Section 3.2. This case, as well as other singular cases, will be treated in a separate paper.

4.1.3. Comparison

**Proposition 13.** Let $\Omega = \{-\mu, \mu\} \times \{-\nu, \nu\}$ be an open box of $\mathbb{R}^2$. Assume that problem (IVPnL2) admits a smooth solution $V$ on $\Omega$ and that $\Omega = \bigcup \Omega_\sigma$ where $(\Omega_\sigma)_\sigma$ is an increasing family of open boxes of $\Omega$ such that $\sup_{x,y \in \Omega_\sigma} |V(x, y)| < r_k - \varepsilon$, where $(r_k)_{\varepsilon}$ is defined in Section 4.1.1. Then, the generalized solution $U \in G_0(\mathbb{R}^2)$ to problem (IVPnL3), given by Proposition 12 coincides with $V$ on $\Omega$.

To be more precise, as recalled in Section 3.1, there exists a canonical sheaf embedding of $C^\infty(\cdot)$ into $G_0(\cdot)$ through the morphism of algebra
\[ \sigma_\gamma : C^\infty(\gamma) \rightarrow G_0(\gamma), \quad f \mapsto (f)_\varepsilon + N_0(\gamma) \quad (\text{where } \gamma \text{ is any open subset of } \mathbb{R}^2). \]
Proposition 13 asserts that we have $\sigma_\Omega(V) = U|_{\Omega}$ in $G_0(\Omega)$.

**Proof.** We can choose as representative of $\sigma_\Omega(V)$ the net $(V)_\varepsilon$. We clearly have
According to Section 2, for a fixed 

\[ V(T, X) = f(X) + \int_0^T \mathcal{F}(\tau, X, V(\tau, X)) \, d\tau, \]

as \([0, T] \subset ]-\mu, \mu[\). We take as representative of \( U \) the net \((U_\varepsilon)_\varepsilon\) given by Lemma 11. This net satisfies

\[ V(T, X) \in \Omega, \quad U_\varepsilon(T, X) = f(X) + \int_0^T \mathcal{F}_\varepsilon(\tau, X, U_\varepsilon(\tau, X)) \, d\tau. \]

Take \( K \in \Omega \). As \( \Omega \) is a box, there exists \( \lambda > 0 \) such that \( K \subset [-\lambda, \lambda] \times [-\lambda', \lambda'] \subset \Omega \). Moreover, there exists \( \varepsilon_0 \) such that, for all \( \varepsilon \leq \varepsilon_0 \), \([-\lambda, \lambda] \times [-\lambda', \lambda'] \in \Omega_\varepsilon \) as \( \Omega_\varepsilon \) is also a box. Note that, for \((\tau, X, z) \in \Omega_\varepsilon \times ]-r_\varepsilon + 1, r_\varepsilon - 1[,\) we have 

\[ \mathcal{F}(\tau, X, z) = \mathcal{F}_\varepsilon(\tau, X, z) \]

by construction of \( \mathcal{F}_\varepsilon \). Thus, \( V \), which values are in \([-r_\varepsilon + 1, r_\varepsilon - 1[,\) is solution of the same integral equation as \( U_\varepsilon \), which admits a unique solution since \( \mathcal{F}_\varepsilon \) is a smooth function of its arguments. Thus, for all \( \varepsilon \leq \varepsilon_0 \), \( V \) and \( U_\varepsilon \) are equal on \( \Omega_\varepsilon \). Setting \( W_\varepsilon = U_\varepsilon|_{\partial \Omega} - V \), it follows that, for all \( \varepsilon \leq \varepsilon_0 \), \( \sup_{(T, X) \in K} |W_\varepsilon(T, X)| = 0 \). This means that \((W_\varepsilon)_\varepsilon\) vanishes on \( K \). Thus \((W_\varepsilon)_\varepsilon \in \mathcal{N}_0(\Omega)\) and \( \sigma(\Omega)(V) = U|_{\partial \Omega} \) as claimed. \( \Box \)

**Remark 5.** The hypotheses made in Proposition 13 are satisfied for the example \( \mathcal{F}(\cdot, \cdot, U) = U^k \) \((k \geq 2)\) and \( f = 1 \), for which the local solution \( U(T, X) = (1 - (k - 1)T)^{-1} \) exists in \([-\infty, (k - 1)^{-1}] \times \mathbb{R} \). On \( \Omega_\varepsilon \) we have \( |U(T, X)| \leq \varepsilon^{-1} \). It suffices to take \( r_\varepsilon > \varepsilon^{-1} \), say \( r_\varepsilon = \varepsilon^{-1} \). We refer the reader to [6] for a spectral asymptotic analysis of the solution of the regularized problem in the case \( k = 2 \).

4.2. The characteristic case

The strategy developed to get the previous results is based on the continuous moderateness of the map \( f \mapsto u_f \) (regularization of data) or the net of maps \((f \mapsto U_{f, \varepsilon})_\varepsilon \) (regularization of non-Lipschitz nonlinearity). An adaptation of this concept is still possible in the following characteristic case, but would bring us too far and is left to a forthcoming paper. However, simple considerations allow us to give the asymptotic structure of the solution in the linear characteristic case.

4.2.1. A regularizing technique

We consider the following characteristic initial value (or Cauchy) problem

\[
\begin{align*}
(1) & \quad \frac{\partial u}{\partial t} + (\varphi' \otimes 1)_x \frac{\partial u}{\partial x} = 0, \\
(2) & \quad u|_{S} = v, 
\end{align*}
\]

where \( S = \Gamma_0 \) is the characteristic curve given by (5). We know that the problem is ill posed, and that the characteristic method is unable to solve it. We are trying to define a well posed generalized problem which approximate \((CIVP)\) in a natural way.

Let \((\Phi_\varepsilon)_\varepsilon\) be the family of functions in \( C^\infty(\mathbb{R}) \) defined by

\[ \Phi_\varepsilon(t) = \varphi(t) + \varepsilon t. \]

It is easy to see that the family \((R_\varepsilon)_\varepsilon\) defined by

\[ R_\varepsilon : C^{\infty}(\mathbb{R}^2) \to C^{\infty}(\mathbb{R}), \quad u \mapsto u(\cdot, \Phi_\varepsilon(\cdot)) \]

is continuously moderate. Consequently a generalized restriction mapping \( R : \mathcal{G}(\mathbb{R}^2) \to \mathcal{G}(\mathbb{R}) \) can be defined by

\[ R : u = [u_{\varepsilon}] \mapsto [u_{\varepsilon}(\cdot, \Phi_\varepsilon(\cdot))]. \]

We can associate to \((CIVP)\) the generalized problem

\[
\begin{align*}
(1) & \quad \frac{\partial u}{\partial t} + (\varphi' \otimes 1)_x \frac{\partial u}{\partial x} = 0, \\
(2) & \quad R(u) = v, 
\end{align*}
\]

where \( u \) is searched in \( \mathcal{G}(\mathbb{R}^2) \) when \( v \) is in \( C^\infty(\mathbb{R}) \), canonically embedded into \( \mathcal{G}(\mathbb{R}) \). To solve \((CIVP_\varepsilon)\) we begin to solve the one parameter family of regularized problems

\[
\begin{align*}
(1) & \quad \frac{\partial u_{\varepsilon}}{\partial t} + (\varphi' \otimes 1)_x \frac{\partial u_{\varepsilon}}{\partial x} = 0, \\
(2) & \quad u_{\varepsilon}(t, \varphi(t) + \varepsilon t) = v(t). 
\end{align*}
\]

According to Section 2, for a fixed \( \varepsilon \), the solution of \((CIVP_\varepsilon)\) is given by \( u_{\varepsilon}(t, x) = F_\varepsilon(x - \varphi(t)) \) where \( F_\varepsilon \) is determined by
\[ v(t) = u_\varepsilon(t, \varphi(t) + \varepsilon t) = F_\varepsilon(\varepsilon t). \]

Then
\[ u_\varepsilon(t, x) = \psi \left( \frac{x - \varphi(t)}{\varepsilon} \right). \tag{36} \]

4.2.2. Generalized solution and association with a distributional object

To solve \((\text{CIVP}_\varepsilon)\) it suffices to prove that the family \((u_\varepsilon)_\varepsilon\) defined by (36) is in \(\mathcal{M}(\mathbb{R}^2)\). This will be the case if, for example, we assume that \(v\) is slowly increasing. Then
\[ \mathbf{u} = \left( u_\varepsilon(t, x) \right)_\varepsilon + \mathcal{N}(\mathbb{R}^2) \tag{37} \]

is a solution to \((\text{CIVP}_\varepsilon)\) in \(\mathcal{G}(\mathbb{R}^2)\) where \(u_\varepsilon(t, x)\) is defined by (36). However, following a remark of Michael Oberguggenberger (private communication), uniqueness is lost here, since the arguments of Proposition 1 (continuous moderateness) are no longer valid in this situation.

**Example 5.** Take \(\rho \in \mathcal{D}(\mathbb{R})\) with \(\rho(0) = 1\) and set \(\rho_\varepsilon(t) = \rho(t - 1/\varepsilon)\). For any \(K \subseteq \mathbb{R}\), we have \(\sup \rho_\varepsilon \cap K = \emptyset\) for \(\varepsilon\) small enough. It follows that \((\rho_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})\). Set \(w_\varepsilon(t, x) = \rho \left( (x - \varphi(t))/\varepsilon - 1/\varepsilon \right)\). We have \((w_\varepsilon)_\varepsilon \in \mathcal{M}(\mathbb{R}^2)\) and
\[ \frac{\partial w_\varepsilon}{\partial t}(t, x) = -\frac{1}{\varepsilon} \varphi'(t) \rho' \left( \frac{x - \varphi(t)}{\varepsilon} - \frac{1}{\varepsilon} \right) = -\varphi'(t) \frac{\partial w_\varepsilon}{\partial x}(t, x). \]

Furthermore, \((w_\varepsilon, \cdot, \varphi(\cdot), \cdot))_\varepsilon = (\rho_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})\). Thus \(\mathbf{u} + [w_\varepsilon]\) is a solution to \((\text{CIVP}_\varepsilon)\) in \(\mathcal{G}(\mathbb{R}^2)\). However \((w_\varepsilon)_\varepsilon \notin \mathcal{N}(\mathbb{R}^2)\) since \(w_\varepsilon(t, \varphi(t) + 1) = \rho(0) = 1\). Finally the problem \((\text{CIVP}_\varepsilon)\) admits at least two solutions (and indeed an infinity) in \(\mathcal{G}(\mathbb{R}^2)\).

We recover uniqueness by working in the algebras \(\mathcal{G}_\varepsilon(\mathbb{R}^d) = \mathcal{M}_\varepsilon(\mathbb{R}^d)/\mathcal{N}_\varepsilon(\mathbb{R}^d)\) \((d = 1, 2)\) [10, 18] of temperate generalized functions, with
\[ \mathcal{M}_\varepsilon(\mathbb{R}^d) = \{ (f_\varepsilon)_\varepsilon \in \mathcal{O}_{\mathcal{M}}(\mathbb{R}^d)^{[0, 1]} \mid \forall \varepsilon \in \mathbb{N}, \exists q \in \mathbb{N}, \exists m \in \mathbb{N}, (1 + |x|^2)^{-q} f_\varepsilon(x) = o(\varepsilon^{-m}) \}, \]
\[ \mathcal{N}_\varepsilon(\mathbb{R}^d) = \{ (f_\varepsilon)_\varepsilon \in \mathcal{O}_{\mathcal{M}}(\mathbb{R}^d)^{[0, 1]} \mid \forall \varepsilon \in \mathbb{N}, \exists q \in \mathbb{N}, \forall m \in \mathbb{N}, (1 + |x|^2)^{-q} f_\varepsilon(x) = o(\varepsilon^{-m}) \}. \]

Indeed a straightforward calculation shows that \((u_\varepsilon)_\varepsilon \in \mathcal{M}_\varepsilon(\mathbb{R}^2)\). Thus the class of \((u_\varepsilon)_\varepsilon \in \mathcal{G}_\varepsilon(\mathbb{R}^2)\) is a solution to \((\text{CIVP}_\varepsilon)\) in \(\mathcal{G}_\varepsilon(\mathbb{R}^2)\). The uniqueness is the consequence of the following property: If \((i_\varepsilon)_\varepsilon \in \mathcal{M}_\varepsilon(\mathbb{R}^2)\) and if
\[ \left( \frac{\partial i_\varepsilon}{\partial t} + (\varphi' \otimes 1_x) \frac{\partial i_\varepsilon}{\partial x} \right)_\varepsilon \in \mathcal{N}_\varepsilon(\mathbb{R}^2), \quad (i_\varepsilon(\cdot), \varphi(\cdot), \cdot)_\varepsilon \in \mathcal{N}(\mathbb{R}) \]

then \((i_\varepsilon)_\varepsilon \in \mathcal{N}_\varepsilon(\mathbb{R}^2)\), which holds true for \(\mathcal{G}_\varepsilon(\mathbb{R}^2)\). (But is false for \(\mathcal{G}(\mathbb{R}^2)\) as shown by Example 5).

In \(\mathcal{G}(\mathbb{R}^2)\) and in \(\mathcal{G}_\varepsilon(\mathbb{R}^2)\), some association processes of generalized functions with distributions are defined. For example, for \(w = [w_\varepsilon]\) in \(\mathcal{G}(\mathbb{R}^2)\) or \(\mathcal{G}_\varepsilon(\mathbb{R}^2)\), \(T \in \mathcal{D}(\mathbb{R}^2)\), \(\Psi\) a mapping from \(\mathbb{R}_+^* \rightarrow \mathbb{R}_+^*\) such that the class of \((\Psi(\varepsilon))_\varepsilon\) is a generalized number, we define
\[ w \sim T \Rightarrow \lim_{\varepsilon \rightarrow 0, \mathcal{D}(\mathbb{R}^2)} \Psi(\varepsilon) w_\varepsilon = T. \]

(The case \(\Psi = 1\) gives the classical association in the distribution sense already considered previously [3].) We assume in addition that \(v\) is in \(L^1(\mathbb{R})\) with \(\int v(x) \, dx = 1\) and set
\[ v_\varepsilon(x) = (1/\varepsilon) v(x/\varepsilon). \]

We know that \(\lim_{\varepsilon \rightarrow 0, \mathcal{D}(\mathbb{R}^2)} u_\varepsilon = \delta\). Analogously, one can show that
\[ \lim_{\varepsilon \rightarrow 0, \mathcal{D}(\mathbb{R}^2)} \left( \left( t, x \right) \mapsto \frac{1}{\varepsilon} u_\varepsilon(t, x) \right) = \delta_{T_\varepsilon}, \]
where \(u_\varepsilon(t, x)\) is defined by (36). Indeed, for \(\psi \in \mathcal{D}(\mathbb{R}^2)\), we have
\[ \int \int \frac{1}{\varepsilon} \psi \left( \frac{x - \varphi(t)}{\varepsilon} \right) \varphi(t, x) \, dx \, dt = \int \int \frac{1}{\varepsilon} \psi \left( \frac{x}{\varepsilon} \right) \varphi(t, x + \varphi(t)) \, dx \, dt. \]

Then, by classical results of Berge and Lebesgue, we can show that \(\int v_\varepsilon(x) \varphi(t, x + \varphi(t)) \, dx\) is smaller than an integrable function. Thus
\[ \lim_{\varepsilon \rightarrow 0} \int \int \frac{1}{\varepsilon} u_\varepsilon(t, x) \varphi(t, x) \, dx \, dt = \int \int \psi(t, \varphi(t)) \, dt = \langle \delta_{T_\varepsilon}, \psi \rangle. \]

We have proved:
Proposition 14. With the previous hypotheses, the solution $u$ of the problem (CIVP), defined by (37) satisfies

$$u \sim \varepsilon \delta_{r^*}.$$ 

In other words, $u$ have a bidimensional soliton structure, and $\text{supp } u = \text{supp } \delta_{r^*} = \Gamma_{r^*}$: The solution in $G_t(\mathbb{R}^2)$ of the characteristic Cauchy problem for the unidirectional wave equation is associated to a bidimensional soliton whose support is the characteristic curve. Of course, in this case, no classical solution exists.

References


