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An axiomatic survey of diagram lemmas for non-abelian group-like structures

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ABSTRACT

It is well known that diagram lemmas for abelian groups (and more generally in abelian categories) used in algebraic topology, can be suitably extended to "non-abelian" structures such as groups, rings, loops, etc. Moreover, they are equivalent to properties which arise in the axiomatic study of these structures. For the five lemma this is well known, and in the present paper we establish this for the snake lemma and the 3×3 lemma, which, when suitably formulated, turn out to be equivalent to each other for all (pointed) algebraic structures, and also in general categories of a special type. In particular, we show that among varieties of universal algebras, they are satisfied precisely in the so-called (pointed) ideal determined varieties.

Introduction

In this paper we explore a connection between classical diagram lemmas of homological algebra and properties encountered in categorical and universal algebra, which arise there in the axiomatic study of "group-like" structures such as groups, rings, modules, algebras, topological groups, loops, etc. (many early investigations in this direction originate from [29,32,21]). It is well known that the five lemma in a pointed regular category [3] is equivalent to protomodularity in the sense of D. Bourn [7], which, for pointed varieties of universal algebras is equivalent to a property introduced by A. Ursini in [37]. In this paper we establish a similar link between the snake lemma and the so-called *normal subtractivity*, which for varieties is equivalent to a property introduced in universal algebra in [36] (see also [20]). Normal subtractivity is a conjunction of two separate properties: *normality*, which states that every coequalizer is a cokernel (in modern terminology, every regular epimorphism is

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a normal epimorphism), which is certainly a very old categorical property, and which is known in universal algebra under the name of 0-regularity, and *subtractivity* introduced for categories in [25] as a (pointed) categorical counterpart of universal-algebraic subtractivity, encountered for the first time in [20] (but named so later in [38]).

For a variety of universal algebras subtractivity states that the variety contains a binary term s and a nullary term 0 satisfying $s(x,x)=0$ and $s(x,0)=x$. It is not a new observation that these two identities satisfied by the subtraction in an abelian group is in many cases what is needed to carry out diagram chasing for abelian groups. This fact is certainly what leads to the "subtraction" rule in [31], which is used there for a proof of the snake lemma in an abelian category, in a way which avoids the use of an embedding theorem for abelian categories. As it follows from the result mentioned below, the subtraction rule of [31] is in some sense the same as the universal-algebraic subtractivity.

Categorical subtractivity introduced in [25] (see also [26]) is equivalent to the universal-algebraic one, in the context of pointed varieties, and can be stated there as follows:

(Subtractivity) for any two morphisms $s_1 : S \to X$ and $s_2 : S \to Y$, and for any two elements $a, b \in S$, if $s_1(a) = s_1(b)$ and $s_2(b) = 0$ then there exists an element $c \in S$ such that $s_2(a) = s_2(c)$ and $s_1(c) = 0$.

Intuitively, we may think of c as $c = a - b$, although formally c depends also on s_1 and s_2. There are also more economical equivalent reformulations of this property, one of which states that every reflexive homomorphic relation R satisfies $xR0 \Rightarrow 0Rx$; however, it is precisely the above formulation which is needed in diagram chasing. This property is formally weaker than the subtraction rule of [31] (the crucial difference being that the subtraction there does not depend on s_2), but we show that the two become equivalent in any pointed regular category with binary sums once the subtraction rule, which is stated in [31] in the abelian context, is suitably extended to this more general context. The proof is obtained by an application of the method of "categorical terms" introduced in a joint work of D. Bourn and the present author (see [12] and the references there).

The connection between subtractivity and the subtraction rule is studied in Section 2 below, while Section 1 contains main results of the paper, which relate the snake lemma and the 3×3 lemma with normal subtractivity. The last section (Section 3) points out some links to recent work in the literature, recalls some examples of normal subtractive categories, and outlines some areas for possible future investigation.

1. Diagram lemmas and normal subtractivity

The classical snake lemma in an abelian category (see e.g. [31]) can be stated as follows: for a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & A_0 \arrow[r, "m_0"] \arrow[d, "i"'] & B_0 \arrow[r, "e_0"] \arrow[d, "j"'] & C_0 \arrow[d, "k"] & \\
0 \arrow[r] & A \arrow[r, "m"] \arrow[d, "f"'] & B \arrow[r, "e"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "m'"] \arrow[d, "p"'] & B' \arrow[r, "e'"] \arrow[d, "q"'] & C' \arrow[r] \arrow[d, "r"] & 0 \\
 & A_1 \arrow[r, "m_1"'] \arrow[d] & B_1 \arrow[r, "e_1"'] \arrow[d] & C_1 \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{1}$$

where all three columns and the two middle rows are exact, there exists a morphism $\delta : C_0 \to A_1$ such that the sequence

$$0 \longrightarrow A_0 \xrightarrow{m_0} B_0 \xrightarrow{e_0} C_0 \xrightarrow{\delta} A_1 \xrightarrow{m_1} B_1 \xrightarrow{e_1} C_1 \longrightarrow 0 \tag{2}$$

is exact. Its elementary proof constructs δ using a composite of relations

$$\delta^+ = p^+ \circ m'^- \circ g^+ \circ e^- \circ k^+, \tag{3}$$

where for a morphism $u : X \to Y$ we write u^+ for the relation $(1_X, u) : X \to X \times Y$ and u^- for the opposite relation $(u, 1_X) : X \to Y \times X$. Then, a diagram chase shows exactness of (2). Accordingly, we formulate the snake lemma in an arbitrary pointed regular category $\mathbb{C}$ in two parts, where the first part only asserts existence of a morphism δ satisfying (3), while the second part asserts exactness of the sequence (2) *provided* such δ exist. We then investigate each of these parts separately.

Remark 1. Snake lemma is also often formulated for a truncated diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & A_0 \arrow[r] \arrow[d] & B_0 \arrow[r] \arrow[d] & C_0 \arrow[d] & \\
 & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[d] & \\
 & A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & C_1 & \\
 & 0 & 0 & &
\end{tikzcd}$$

and then (2) is no longer required to be exact at A_0 and C_1. A further truncated version is given by dropping the objects A_0 and C_1 in the above diagram, and then one only asks exactness at the domain and the codomain of the connecting morphism δ. Our results will not be affected if we choose any of these two formulations instead of the one above (although this would require a minor restructuring of proofs).

Throughout the paper we work with regular categories [3]. A quick introduction to regular categories can be found in e.g. Section 2 in [14]. For a basic background on categories, such as finite limits, subobjects, null objects, kernels, etc., see [31]. Recall that a category is said to be *pointed* if it contains a null object.

A good model example of a regular category is a (quasi-)variety of universal algebras. There, regular epimorphisms are the surjective homomorphisms, subobjects can be identified with subalgebra inclusions, regular images of morphisms are the ordinary images of homomorphisms, and pointedness means that the variety has unique constant (i.e. the free algebra over the empty set is a singleton).

As observed in [11], in a finitely complete pointed category $\mathbb{C}$ the following two conditions are equivalent:

- Every regular epimorphism in $\mathbb{C}$ is normal, i.e. every coequalizer is a cokernel.
- Every split epimorphism in $\mathbb{C}$ is normal, i.e. is a cokernel.

A pointed regular category $\mathbb{C}$ satisfying these conditions was called a *normal category* in [27]. Varieties of universal algebras which are normal categories are the same as the pointed "0-regular" varieties, well known in universal algebra, which were introduced (although under a different name) in [16].

As explained in [27], a pointed regular category $\mathbb{C}$ has a convenient "pointed subobject functor"

$$\mathcal{S} : \mathbb{C} \to \mathbf{Set}_*,$$

which maps every object of $\mathbb{C}$ to its pointed set $\mathcal{S}(X)$ of subobjects, whose base point is the zero subobject. This functor has certain preservation/reflection properties which allows one to use it for similar purposes as the more sophisticated Barr embedding [3]. In particular,

- $\mathcal{S}$ preserves and reflects the zero objects and the zero morphisms.
- $\mathcal{S}$ preserves and reflects regular epimorphisms.
- $\mathcal{S}$ preserves monomorphisms, kernels and weak pullbacks (recall that a weak pullback is defined similarly as a pullback, but with a weak universal property, where the canonical morphism into the universal object is required to exist but not necessarily to be unique, see e.g. [31]).
- The above implies that $\mathcal{S}$ preserves regular image decompositions of morphisms and preserves and reflects subobject inclusions.
- When $\mathbb{C}$ is normal (and more generally, when $\mathbb{C}$ is prenormal in the sense of Definition 3 below), the functor $\mathcal{S}$ reflects monomorphisms and isomorphisms.

In addition to the above properties, the functor $\mathcal{S}$ preserves and reflects *subtractive spans*. Recall from [27] that a span $s : S \to X \times Y$ is said to be subtractive if the relation $r : R \to X \times Y$ obtained as the regular image of s is subtractive, i.e. for any span $(x, y) : C \to X \times Y$, if (x, y) and $(x, 0)$ factor through r, then also $(0, y)$ factors through r. Thus in particular in $\mathbf{Set}_*$ a span $s : S \to X \times Y$ is subtractive when it satisfies the condition in the subtractivity property given in the introduction.

From the above preservation/reflection properties of $\mathcal{S}$, it follows that $\mathcal{S}$ preserves and reflects exact sequences defined in the following "non-symmetric" way: a sequence

$$X \xrightarrow{u} Y \xrightarrow{v} Z$$

is exact at Y when $\mathrm{im}(u) = \ker(v)$ (where $\mathrm{im}(u)$ denotes the regular image of u, and the equality is in fact equality of subobjects). When $\mathbb{C}$ is normal, this becomes the same notion of exactness as the more symmetric one used in e.g. [8,5].

As observed in [27], the pointed subobject functor $\mathcal{S}$ extends to categories of relations

$$\overline{\mathcal{S}} : \mathrm{Rel}(\mathbb{C}) \to \mathrm{Rel}(\mathbf{Set}_*).$$

Specifically, $\overline{\mathcal{S}}$ is defined as follows: for a relation $r = (r_1, r_2) : R \to X \times Y$,

$$\overline{\mathcal{S}}(r) = \overline{\mathcal{S}}(r_2^+ \circ r_1^-) = \mathcal{S}(r_2)^+ \circ \mathcal{S}(r_1)^-.$$

That $\overline{\mathcal{S}}$ above is well defined (i.e. the value $\overline{\mathcal{S}}(r)$ does not depend on the choice of the monomorphism r which represents the relation) follows from the fact that $\mathcal{S}$ preserves regular epimorphisms, while functoriality of $\overline{\mathcal{S}}$ follows from the fact that it preserves weak pullbacks.

The following will be useful:

Lemma 1. *In a pointed regular category $\mathbb{C}$, for a span $s = (s_1, s_2) : S \to X \times Y$ where s_1 is a regular epimorphism, and for any morphism $t : X \to Y$, we have the functional equality $ts_1 = s_2$ if and only if we have the relational equality $t^+ = s_2^+ \circ s_1^-$.*

Proof. Suppose first $ts_1 = s_2$. Then $t^+ \circ s_1^+ = s_2^+$ from which we get $t^+ \circ s_1^+ \circ s_1^- = s_2^+ \circ s_1^-$. Since s_1 is a regular epimorphism, $s_1^+ \circ s_1^-$ is the equality relation and so $t^+ = s_2^+ \circ s_1^-$, as desired.

Conversely, suppose $t^+ = s_2^+ \circ s_1^-$. To show $ts_1 = s_2$ it suffices to show $(ts_1)^+ \leqslant s_2^+$. We have:

$$(ts_1)^+ \leqslant (ts_1)^+ \circ s_2^- \circ s_2^+ = t^+ \circ (s_2^+ \circ s_1^-)^{\mathrm{op}} \circ s_2^+ = t^+ \circ t^- \circ s_2^+ \leqslant s_2^+. \qquad \square$$

Definition 1. The *snake lemma* in a pointed regular category $\mathbb{C}$ asserts that for any diagram (1), where all three columns and the two middle rows are exact,

(a) there exists a morphism δ such that we have the equality (3),
(b) when the above holds, the sequence (2) is exact for the same morphism δ.

We are now ready to formulate and prove our first main result:

Theorem 1. *For a pointed regular category $\mathbb{C}$, the following conditions are equivalent*:

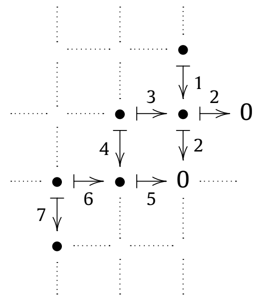
(a) *The condition* (a) *of Definition* 1 *holds for any diagram* (1)*, where all three columns and the two middle rows are exact.*
(b) $\mathbb{C}$ *is a normal category.*

Proof. (a) $\Rightarrow$ (b): Suppose (a) is satisfied. Let $e : B \to C$ be a regular epimorphism and let $g : B \to B'$ be any morphism such that $g \circ \ker(e) = 0$. We would like to show $\delta e = g$ for some morphism $\delta : C \to B'$. Without loss of generality we can assume that g is a regular epimorphism. Then we can construct a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathrm{Ker}(e) \arrow[r] \arrow[d, equal] & \mathrm{Ker}(g) \arrow[r] \arrow[d, "\ker(g)"] & C \arrow[d, equal] & \\
0 \arrow[r] & \mathrm{Ker}(e) \arrow[r, "\ker(e)"] \arrow[d, "0"'] & B \arrow[r, "e"'] \arrow[d, "g"'] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B' \arrow[r, equal] \arrow[d, equal] & B' \arrow[r] \arrow[d] & 0 \arrow[r, equal] \arrow[d, equal] & 0 \\
 & B' \arrow[r] \arrow[d] & 0 \arrow[r, equal] \arrow[d, equal] & 0 \arrow[d, equal] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where all three columns and the middle two rows are exact. By condition (a) of Definition 1, applied to the above diagram, there exists a morphism δ such that $\delta^+ = g^+ \circ e^-$. By Lemma 1, this gives $\delta e = g$, as desired.

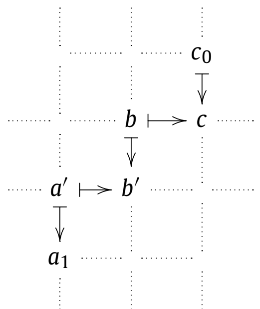
(b) $\Rightarrow$ (a): Suppose $\mathbb{C}$ is a normal category. Consider a commutative diagram (1) with exact columns and exact middle two rows. Let $(\delta_1, \delta_2) : R \to C_0 \times A_1$ be the relation obtained as the composite $p^+ \circ m'^- \circ g^+ \circ e^- \circ k^+$. First we show that δ_1 is a regular epimorphism. Via the pointed subobject functor we first transfer the problem to $\mathbf{Set}_*$, and then apply the following diagram chase:



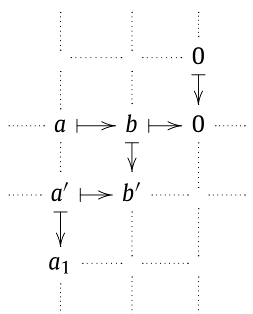
It remains to show that there exists a morphism δ such that $\delta\delta_1 = \delta_2$. Indeed, then we could apply Lemma 1 to conclude that for the same morphism δ we have

$$\delta^+ = \delta_2^+ \circ \delta_1^- = p^+ \circ m'^- \circ g^+ \circ e^- \circ k^+.$$

Since $\mathbb{C}$ is a normal category, the morphism δ_1 being regular is also normal and hence to get δ it suffices to show that $\delta_2 \circ \ker(\delta_1) = 0$. Again, the pointed subobject functor allows to transfer the problem to $\mathbf{Set}_*$. Consider a zigzag



We want to show that if $c_0 = 0$ then $a_1 = 0$. Suppose $c_0 = 0$. Then $c = 0$ and this yields the following:



Next, since $\mathbb{C}$ is normal and m' in (1) has a trivial kernel, it follows that m' is a monomorphism. Since the pointed subobject functor preserves monomorphisms, we get that a in the above diagram is mapped down to a', which forces $a_1 = 0$ by exactness at A'. This completes the proof. $\square$

Remark 2. Our method of diagram chasing in pointed regular categories, via the pointed subobject functor (which was first proposed in [27]) gives, in the case of abelian categories, the method of diagram chasing due to S. Mac Lane [30] (see also [31]).

Recall from [25] that a finitely complete pointed category is said to be a *subtractive category* when for any internal reflexive relation $r = (r_1, r_2) : R \to X \times X$ in $\mathbb{C}$, if $(1_X, 0) : X \to X \times X$ factors through r then $(0, 1_X)$ factors through r. As shown in [26], this is equivalent to every internal relation in $\mathbb{C}$ being subtractive, and when $\mathbb{C}$ is a regular category it is further equivalent to every span in $\mathbb{C}$ being subtractive.

Definition 2. In a pointed regular category $\mathbb{C}$ the *cross lemma* states that given a commutative diagram

$$\begin{tikzcd}
& 0 \arrow[d] & & 0 & \\
0 \arrow[r] & W_1 \arrow[rr, "w"] \arrow[dr, "w_1"'] & & W_2 \arrow[r] \arrow[u] & 0 \\
& & X \arrow[ur, "w_2"'] \arrow[dr, "y_2"] & & \\
0 \arrow[r] & Y_1 \arrow[rr, "y"'] \arrow[ur, "y_1"] & & Y_2 \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[u] & & 0 &
\end{tikzcd} \tag{4}$$

in which the cross consists of exact sequences, exactness of the bottom horizontal sequence implies exactness of the top horizontal sequence (by symmetry, we then have implication also in the opposite direction).

Theorem 2. *For a pointed regular category $\mathbb{C}$ the following conditions are equivalent*:

(a) *$\mathbb{C}$ is a subtractive category.*
(b) *The cross lemma holds true in $\mathbb{C}$.*

Proof. (a) $\Rightarrow$ (b): Via the pointed subobject functor, it suffices to show that in $\mathbf{Set}_*$, if in a diagram (4) the span (y_2, w_2) is subtractive, then exactness of the bottom horizontal sequence implies exactness of the top one, which can be done using elementary diagram chase (exactness at W_1 is automatic and subtractivity implies exactness at W_2).

(b) $\Rightarrow$ (a): Consider a reflexive relation $(r_1, r_2) : R \to X \times X$ and form the diagram

$$\begin{tikzcd}
& 0 \arrow[d] & & 0 & \\
0 \arrow[r] & \mathrm{Ker}(r_1) \arrow[rr, "r_2 \ker(r_1)"] \arrow[dr, "\ker(r_1)"'] & & X \arrow[r] \arrow[u] & 0 \\
& & R \arrow[ur, "r_2"'] \arrow[dr, "r_1"] & & \\
0 \arrow[r] & \mathrm{Ker}(r_2) \arrow[rr, "r_1 \ker(r_2)"'] \arrow[ur, "\ker(r_2)"] & & X \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[u] & & 0 &
\end{tikzcd} \tag{5}$$

Next, observe that

- Since the relation is reflexive, both r_1 and r_2 are split by 1_X, and hence they are regular epimorphisms; this implies exactness of the diagonal sequences at X.

- Since kernels are monomorphisms, we also have exactness of the diagonal sequences at $\mathrm{Ker}(r_1)$ and at $\mathrm{Ker}(r_2)$.
- Since r_1 and r_2 are jointly monomorphic, it follows that the top middle arrow as well as the bottom middle arrow are monomorphisms. In fact, the two squares

$$\begin{tikzcd} \mathrm{Ker}(r_1) \arrow[r, "\ker(r_1)"] \arrow[d, "r_2 \ker(r_1)"'] & R \arrow[d, "(r_1,r_2)"] \\ X \arrow[r, "(0,1_X)"'] & X \times X \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Ker}(r_2) \arrow[r, "\ker(r_2)"] \arrow[d, "r_1 \ker(r_2)"'] & R \arrow[d, "(r_1,r_2)"] \\ X \arrow[r, "(1_X,0)"'] & X \times X \end{tikzcd}$$

are pullbacks, and so the top horizontal sequence in (5) is exact if and only if $(0, 1_X)$ factors through (r_1, r_2), and symmetrically, the bottom horizontal sequence in (5) is exact if and only if $(1_X, 0)$ factors through (r_1, r_2).

The above shows that we can apply the cross lemma to deduce that if $(1_X, 0)$ factors through (r_1, r_2) then $(0, 1_X)$ factors through (r_1, r_2). $\square$

As observed in [27], in a pointed category with pullbacks (and in particular, in any pointed regular category) the following conditions are equivalent:

- Any morphism having a trivial kernel is a monomorphism.
- Any regular epimorphism having a trivial kernel is an isomorphism.
- Any split epimorphism having a trivial kernel is an isomorphism.

Further, for pointed regular categories, normality is equivalent to the above conditions together with the requirement that every kernel has a cokernel (see Proposition 3.12 in [27]). Dropping this last requirement, we define:

Definition 3. A pointed regular category $\mathbb{C}$ is said to be *prenormal* if any split epimorphism in $\mathbb{C}$ having a trivial kernel is an isomorphism.

Theorem 3. *For a pointed regular category $\mathbb{C}$ the following conditions are equivalent*:

(a) *The upper* 3×3 *lemma holds true in $\mathbb{C}$, i.e. given a commutative diagram*

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A_1 \arrow[r, "u_1"] \arrow[d, "f_1"'] & B_1 \arrow[r, "v_1"] \arrow[d, "g_1"'] & C_1 \arrow[r] \arrow[d, "h_1"] & 0 \\ 0 \arrow[r] & A_2 \arrow[r, "u_2"] \arrow[d, "f_2"'] & B_2 \arrow[r, "v_2"] \arrow[d, "g_2"'] & C_2 \arrow[r] \arrow[d, "h_2"] & 0 \\ 0 \arrow[r] & A_3 \arrow[r, "u_3"'] \arrow[d] & B_3 \arrow[r, "v_3"'] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd} \tag{6}$$

if all columns and the bottom two rows are exact, then the top row is also exact.

(b) $\mathbb{C}$ *is a prenormal subtractive category.*
(c) *The condition* (b) *of Definition* 1 *holds for any diagram* (1), *where all three columns and the two middle rows are exact.*

Proof. (a) $\Rightarrow$ (b): Let $f : X \to Y$ be a split epimorphism with a trivial kernel, and with right inverse $g : Y \to X$. Then the diagram

$$\begin{tikzcd}
 & 0 \arrow[d, equal] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r, equal] & 0 \arrow[r] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r, equal] & 0 \arrow[r] \arrow[d, equal] & Y \arrow[r, equal] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r, equal] & 0 \arrow[r, equal] \arrow[d, equal] & 0 \arrow[r, equal] \arrow[d, equal] & 0 \arrow[r, equal] \arrow[d, equal] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

commutes and has all columns and the bottom two rows exact. The upper 3×3 lemma gives that the top row is exact, and hence g is a regular epimorphism. Since g is a right inverse of f, this implies that f is an isomorphism. This proves prenormality. To prove subtractivity, we verify that the cross lemma can be deduced from the upper 3×3 lemma. Indeed, a diagram (4) can be rearranged into the diagram

$$\begin{tikzcd}
 & 0 \arrow[d, equal] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r, equal] & 0 \arrow[r] \arrow[d] & W_1 \arrow[r, "w"] \arrow[d, "w_1"'] & W_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y_1 \arrow[r, "y_1"] \arrow[d, equal] & X \arrow[r, "w_2"] \arrow[d, "y_2"'] & W_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y_1 \arrow[r, "y"'] \arrow[d] & Y_2 \arrow[r] \arrow[d] & 0 \arrow[r, equal] \arrow[d, equal] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

(b) $\Rightarrow$ (c): To get (b) of Definition 1, we can apply the pointed subobject functor and carry out the standard diagram chase in $\mathbf{Set}_*$. Subtractivity of spans will be used during this chase in a similar way as in the proof of the 3×3 lemma presented in detail in [27]. In particular,

- for exactness at A_0 we need to use exactness of the row in (1) passing through A and exactness of the column passing through A_0;
- for exactness at B_0 we need to use exactness of the row passing through B, the column passing through C_0, the column passing through B, the row passing through A', the column passing through A, and the fact that j is a monomorphism, which follows from exactness of the column passing through B_0 and prenormality;
- for exactness at C_0 we need to use exactness of the column passing through B, the column passing through A', the row passing though B, subtractivity of the span (g, e), and the fact that k is a monomorphism, which follows from exactness of the column passing through C_0 and prenormality;

- for exactness at A_1 we need to use exactness of the column passing through B', the column passing through A_1, the row passing through B', and exactness of the column passing through C;
- for exactness at B_1 we need to use exactness of the column passing through A_1, the row passing through B', the column passing through B_1, the column passing through C', the row passing through C, and subtractivity of the span (e', q);
- for exactness at C_1 we need to use exactness of the column passing through C_1 and the row passing through C'.

(c) $\Rightarrow$ (a): Diagram (6) gives rise to the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & A_1 \arrow[r, "u_1"] \arrow[d, "f_1"'] & B_1 \arrow[r, "v_1"] \arrow[d, "g_1"'] & C_1 \arrow[d, "h_1"] & \\
0 \arrow[r] & A_2 \arrow[r, "u_2"] \arrow[d, "f_2"'] & B_2 \arrow[r, "v_2"] \arrow[d, "g_2"'] & C_2 \arrow[r] \arrow[d, "h_2"] & 0 \\
0 \arrow[r] & A_3 \arrow[r, "u_3"'] \arrow[d] & B_3 \arrow[r, "v_3"'] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r, equal] \arrow[d, equal] & 0 \arrow[r, equal] \arrow[d, equal] & 0 \arrow[d, equal] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Let $(\delta_1, \delta_2) : R \to C_1 \times A_3$ be the relation obtained as the composite $0^+ \circ u_3^- \circ g_2^+ \circ v_2^- \circ h_1^+$. Then $\delta_2 = 0$. As we already know from the proof of Theorem 1, δ_1 is a regular epimorphism. Then $0 \circ \delta_1 = \delta_2$ which by Lemma 1 implies $0^+ = \delta_2^+ \circ \delta_1^-$ and so we can set $\delta = 0$. By (b) of Definition 1, the sequence

$$0 \longrightarrow A_1 \xrightarrow{u_1} B_1 \xrightarrow{v_1} C_1 \xrightarrow{\delta} 0 = 0 = 0 = 0$$

is exact and hence so is the top row in (6). $\square$

Theorems 1 and 3 together give:

Corollary 1. *In a pointed regular category $\mathbb{C}$, the snake lemma holds if and only if $\mathbb{C}$ is a normal subtractive category.*

In [27] it was shown that in a pointed regular category subtractivity is equivalent to the lower 3×3 lemma, which states that for a commutative diagram (6), if all columns and the top two rows are exact, then the bottom row is also exact. The lower 3×3 lemma can be also deduced directly from (c) of Theorem 3, using a similar argument as in the proof of (c) $\Rightarrow$ (a) in Theorem 3. This, together with Theorem 3 gives:

Corollary 2. *In a pointed regular category $\mathbb{C}$, the 3×3 lemma holds if and only if $\mathbb{C}$ is a prenormal subtractive category.*

2. André–Mac Lane subtraction

In this section we show that the subtraction rule which is used in [31] in the proof of the snake lemma, being credited to M. André, can be extended to a pointed regular category $\mathbb{C}$, and moreover, it becomes equivalent to $\mathbb{C}$ being a subtractive category under the presence of binary sums. Before showing this, first we recall the exact formulation of the subtraction rule from [31] (we have slightly changed notation to emphasize analogy with our subtractivity property stated in the introduction):

(André–Mac Lane subtraction) Given $s_1 : S \to X$ and $a, b \in_{\mathrm{m}} S$ with $s_1 a \equiv s_1 b$, there exists $c \in_{\mathrm{m}} S$ with $s_1 c \equiv 0$; moreover, any $s_2 : S \to Y$ with $s_2 b \equiv 0$ has $s_2 a \equiv s_2 c$ and any $s_3 : S \to Z$ with $s_3 a \equiv 0$ has $s_3 b \equiv -s_3 c$.

The above property is understood in an abelian category as follows: capital letters denote objects while arrows and small letters represent morphisms of the abelian category; then, for an object A and a morphism x we write $x \in_{\mathrm{m}} A$ if the codomain of x is A, and for two morphisms $x, y \in_{\mathrm{m}} A$ we write $x \equiv y$ if the image of x is the same as the image of y. Note that the image of any morphism x in an abelian category is the same as the image of $-x$, and so the equivalence $s_3 b \equiv -s_3 c$ above can be replaced with $s_3 b \equiv s_3 c$, after which André–Mac Lane subtraction can be restated word-for-word in any pointed regular category.

Theorem 4. *The André–Mac Lane subtraction property adapted to a pointed regular category $\mathbb{C}$ is equivalent to $\mathbb{C}$ being a subtractive category, provided for any object X in $\mathbb{C}$ the coproduct $X + X$ exists.*

Proof. If the André–Mac Lane subtraction property holds, then it is easy to see that the pointed subobject functor $\mathcal{S}$ will carry every span in $\mathbb{C}$ to a subtractive span in $\mathbf{Set}_*$, and since $\mathcal{S}$ reflects subtractive spans, we can conclude that $\mathbb{C}$ is a subtractive category.

For the converse, we use the so-called "categorical terms" in the sense of Bourn and the present author: for any two parallel morphisms $f, g : X \to Y$ we will write $f - g$ to denote the composite

$$f - g = \langle f, g\rangle \circ \ker(\langle 1_X, 1_X\rangle),$$

where $\ker(\langle 1_X, 1_X\rangle)$ is the kernel of the codiagonal $\langle 1_X, 1_X\rangle : X + X \to X$. Then for any morphism f we have

$$f - f = \langle f, f\rangle \circ \ker(\langle 1_X, 1_X\rangle) = f \circ \langle 1_X, 1_X\rangle \circ \ker(\langle 1_X, 1_X\rangle) = 0.$$

At the same time, it is not difficult to see that

$$f - 0 \equiv 0 - f.$$

As shown in [10], a pointed regular category with binary sums is subtractive if and only if for any object X the morphism $1_X - 0$ is a regular epimorphism. This of course implies that for any morphism f we have:

$$f - 0 \equiv f \equiv 0 - f.$$

Now, consider $s_1 : S \to X$ and $a, b \in_{\mathrm{m}} S$ with $s_1 a \equiv s_1 b$. Then $s_1 a p_1 = s_1 b p_2$ for some regular epimorphisms p_1 and p_2 (namely, we can choose them to be the projections of the pullback of $s_1 a$ and $s_1 b$), and define

$$c = a p_1 - b p_2.$$

This gives $s_1 c = s_1(a p_1 - b p_2) = s_1 a p_1 - s_1 b p_2 = 0$. Further, any $s_2 : S \to Y$ with $s_2 b = 0$ has $s_2 c = s_2 a p_1 - s_2 b p_2 = s_2 a p_1 - 0 \equiv s_2 a p_1 \equiv s_2 a$, and similarly any $s_3 : S \to Z$ with $s_3 a = 0$ has $s_3 c \equiv s_3 b$. □

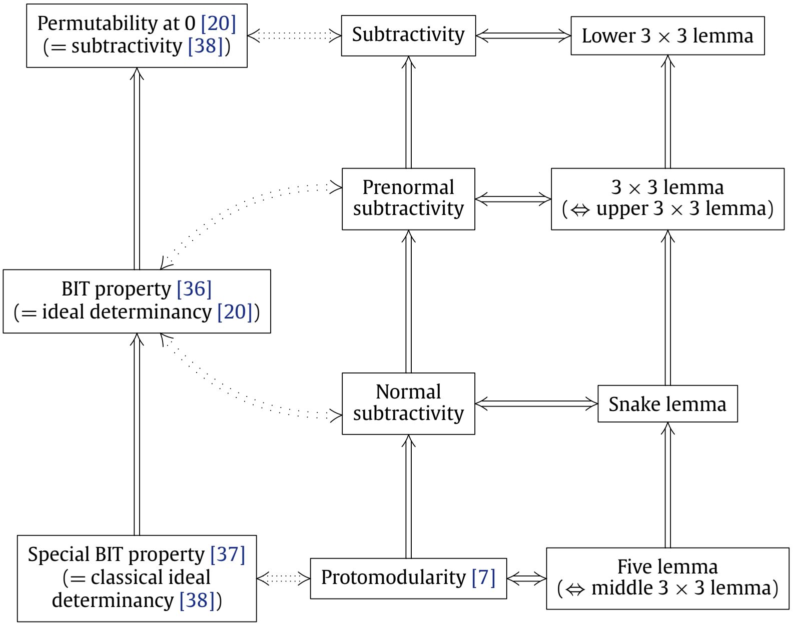
Remark 3. Condition (5) given in Proposition 1.9.4 in [4] is a weaker version of the André–Mac Lane subtraction where the assertion on s_3 is dropped, but it still implies subtractivity of spans, and hence by the above theorem it is still equivalent to subtractivity under the same assumptions on the category.

Remark 4. As we can see from the proof of Theorem 3, the only time we would need subtraction in the proof of the snake lemma in an abelian category, is to prove exactness of the sequence (2) at C_0 and B_1. However, by duality, exactness at C_0 follows from exactness at A_1, and exactness at B_1 follows from exactness at B_0, none of which require subtraction. This means that in an *abelian* category snake lemma can be proved by avoiding subtraction.

3. Conclusion

Theorems 1 and 3 give a somewhat unexpected result: once the "passive" aspect of the snake lemma, given by (b) in Definition 1, holds throughout the category, it implies the "active" aspect, given by (a) in Definition 1, provided every kernel has a cokernel (since under this condition normality becomes equivalent to prenormality).

Corollaries 1 and 2 allow to complete the table below which relates homological diagram lemmas to properties encountered in universal and categorical algebra. The solid implications and equivalences below hold in the context of pointed regular categories, while the dotted ones hold in the context of varieties of universal algebras, which is the context where the properties which appear in the left column were first introduced.



The context of pointed regular categories is certainly not the only one used in categorical algebra, and the properties above are encountered in other contexts as well. For instance, protomodularity in the context of pointed Barr exact categories having binary sums (which still includes all pointed varieties) gives the notion of a *semi-abelian category* introduced in [22], and as shown there, it can be also defined using categorical axioms studied in the early life of categorical approach to the axiomatic study of group-like structures (see [22] for a history and references).

Theorem 4 reveals that normal subtractivity is equivalent to an extension of "elementary rules for chasing diagrams" (i)–(vi) in abelian categories in the sense of [31] (see there Theorem 3 in the section on diagram lemmas in Chapter VIII), to pointed regular categories with suitable colimits. In such an extension, epimorphisms should be replaced with regular epimorphisms. Then, the first rule (i) becomes prenormality, the rules (ii)–(v) are satisfied in any pointed regular category, while the last rule (vi) is what we called above André–Mac Lane subtraction, which by Theorem 4 is equivalent to subtractivity.

Next, we list some examples of categories which have properties mentioned in the present paper. Pointed regular categories which are protomodular (which are the same as the *homological categories* in the sense of [5]) include all varieties of Ω-groups (which includes those of groups, rings, modules, algebras), loops, Heyting semilattices (see [28]), as well as their topological versions (e.g. topological groups, see [6]). A somewhat unexpected example is the dual of the category of pointed sets (more generally, the dual of the category of pointed objects in any topos is a homological category — see [5,9]). Thus, all diagram lemmas mentioned in this paper hold true in $\mathbf{Set}_*^{\mathrm{op}}$. This means that the usual diagram lemmas hold true in $\mathbf{Set}_*$ for the following notion of an exact sequence: call a sequence

$$X \xrightarrow{u} Y \xrightarrow{v} Z$$

of pointed sets *exact at* Y if $\mathrm{Im}(u) = \mathrm{Ker}(v)$ and in addition, the restriction of v on $Y \setminus \mathrm{Im}(u)$ is injective.

The above already gives a large collection of examples of normal subtractive categories. Further examples can be found by examining the syntactical condition which characterizes normal subtractive varieties, and in particular, an example which does not belong to the previous collection is given by the variety of so-called implication algebras (see [20], and see also [34]). This is the variety generated by the implication structure of a two-element Boolean algebra (see [1,2]). Among normal subtractive categories are also all ideal determined categories in the sense of [23].

As already mentioned in the introduction, subtractive varieties are those which contain a binary term s and constant 0 satisfying $s(x,0) = x$ and $s(x,x) = 0$. Then, pointedness forces the constant 0 to be the unique constant in the theory. In particular, the category of algebras $(A, s, 0)$ defined by these identities gives an example of a pointed regular subtractive category which is not normal. All pointed Mal'tsev varieties [32,35], and more generally, all pointed Mal'tsev categories in the sense of [15] are regular subtractive categories.

Note also that if a variety of universal algebras is protomodular, normal, or subtractive, then any quasi-variety of algebras in it (which is still a regular category) is protomodular, normal, and subtractive, respectively.

Thus, examples abound. However, categories of pointed sets, monoids, commutative monoids, and lattices with lower/upper bound are neither normal nor subtractive. In fact, regular subtractive categories are in some sense not too far from abelian categories: as shown in [13], an abelian category is the same as a regular subtractive category in which every monomorphism is a kernel (or in other words, every morphism is part of an exact sequence).

We conclude with some remarks on related work, open questions, and possible future generalizations:

- The fact that protomodularity implies the snake lemma and the 3×3 lemma was obtained in [8]. Our approach gives an alternative proof.
- Just like the notion of a pointed regular category is not self-dual, also our notion of an exact sequence in such a category is not self-dual, although for abelian categories it becomes the usual self-dual notion. A non-additive extension of homological algebra, which is based on a self-dual notion of an exact sequence, is described in [19]. See [33], where a unification of the two notions of exactness is proposed.
- All naturally occurring examples of regular categories have coequalizers, and hence all pointed ones have cokernels. The difference between normality and prenormality is therefore quite

esoteric. Moreover, the author does not know an example of a prenormal category which is not normal. It would be interesting to find such example, if it exists.
- In [5], Noether isomorphism theorems and the long exact homology sequence are derived from the 3×3 lemma and the snake lemma in the context of homological categories. It would be interesting to see whether the same can be done in normal subtractive categories.
- Our formulation of the snake lemma in two parts is similar to the one in [24]. There, a snake lemma is proved in a suitable "relative" context, which generalizes the context of homological categories. It would be interesting to know whether the results in the present paper can be similarly generalized.
- Exact sequences also appear in non-pointed regular categories in [3,15]. A first attempt to unify the pointed and non-pointed notions of exact sequences for regular categories was made in [17], based on the method proposed in [18]. Our snake lemma should be compared, from this new perspective, to the non-pointed snake lemma of [15].

References

[1] J.C. Abbott, Semi-boolean algebra, Mat. Vesnik 4 (19) (1967) 177–198.
[2] J.C. Abbott, Implicational algebras, Bull. Math. Soc. Sci. Math. R. S. Roumanie 11 (59) (1967) 3–23; (1968).
[3] M. Barr, P.A. Grillet, D.H. van Osdol, Exact Categories and Categories of Sheaves, Lecture Notes in Math., vol. 236, Springer, 1971.
[4] F. Borceux, Handbook of Categorical Algebra, vol. 2: Categories and Structures, Encyclopedia Math. Appl., vol. 51, Cambridge University Press, 1994.
[5] F. Borceux, D. Bourn, Mal'cev, Protomodular, Homological and Semi-Abelian Categories, Math. Appl., vol. 566, Kluwer, 2004.
[6] F. Borceux, M.M. Clementino, Topological semi-abelian algebras, Adv. Math. 190 (2005) 425–453.
[7] D. Bourn, Normalization equivalence, kernel equivalence and affine categories, in: Lecture Notes in Math., vol. 1488, Springer, 1991, pp. 43–62.
[8] D. Bourn, 3×3 lemma and protomodularity, J. Algebra 236 (2001) 778–795.
[9] D. Bourn, Protomodular aspect of the dual of a topos, Adv. Math. 187 (2004) 240–255.
[10] D. Bourn, Z. Janelidze, Subtractive categories and extended subtractions, Appl. Categ. Structures 17 (2009) 317–343.
[11] D. Bourn, Z. Janelidze, Pointed protomodularity via natural imaginary subtractions, J. Pure Appl. Algebra 213 (2009) 1835–1851.
[12] D. Bourn, Z. Janelidze, Categorical (binary) difference terms and protomodularity, Algebra Universalis 66 (2011) 277–316.
[13] D. Bourn, Z. Janelidze, A note on the abelianisation functor, Comm. Algebra, in press.
[14] A. Carboni, G.M. Kelly, M.C. Pedicchio, Some remarks on Maltsev and Goursat categories, Appl. Categ. Structures 1 (1993) 385–421.
[15] A. Carboni, J. Lambek, M.C. Pedicchio, Diagram chasing in Mal'cev categories, J. Pure Appl. Algebra 69 (1990) 271–284.
[16] K. Fichtner, Varieties of universal algebras with ideals, Mat. Sb. (N.S.) 75 (117) (1968) 445–453, English translation: Math. USSR Sb. 4 (1968) 411–418.
[17] M. Gran, Z. Janelidze, D. Rodelo, 3×3-lemma for star exact sequences, Homology Homotopy Appl. 14 (2) (2012) 1–22, http://dx.doi.org/10.4310/HHA.2012.v14.n2.a1.
[18] M. Gran, Z. Janelidze, A. Ursini, A good theory of ideals in regular multi-pointed categories, J. Pure Appl. Algebra 216 (2012) 1905–1919.
[19] M. Grandis, On the categorical foundations of homological and homotopical algebra, Cah. Topol. Géom. Différ. Catég. 33 (1992) 135–175.
[20] H.P. Gumm, A. Ursini, Ideals in universal algebras, Algebra Universalis 19 (1984) 45–54.
[21] P.J. Higgins, Groups with multiple operators, Proc. Lond. Math. Soc. 6 (1956) 366–416.
[22] G. Janelidze, L. Márki, W. Tholen, Semi-abelian categories, J. Pure Appl. Algebra 168 (2002) 367–386.
[23] G. Janelidze, L. Márki, W. Tholen, A. Ursini, Ideal determined categories, Cah. Topol. Géom. Différ. Catég. 51 (2010) 113–124.
[24] T. Janelidze, Snake lemma in incomplete relative homological categories, Theory Appl. Categ. 23 (2010) 76–91.
[25] Z. Janelidze, Subtractive categories, Appl. Categ. Structures 13 (2005) 343–350.
[26] Z. Janelidze, Closedness properties of internal relations, I: A unified approach to Mal'tsev, unital and subtractive categories, Theory Appl. Categ. 16 (2006) 236–261.
[27] Z. Janelidze, The pointed subobject functor, 3×3 lemmas, and subtractivity of spans, Theory Appl. Categ. 23 (2010) 221–242.
[28] P. Johnstone, A note on the semiabelian variety of heyting semilattices, in: Fields Inst. Commun., vol. 43, 2004, pp. 317–318.
[29] S. Mac Lane, Duality for groups, Bull. Amer. Math. Soc. 56 (1950) 485–516.
[30] S. Mac Lane, Homology, Springer, Berlin, Göttingen, Heidelberg, 1963.
[31] S. Mac Lane, Categories for the Working Mathematician, second edition, Grad. Texts in Math., vol. 5, Springer, New York, 1998.
[32] A.I. Mal'tsev, On the general theory of algebraic systems, Mat. Sb. (N.S.) 35 (77) (1954) 3–20 (in Russian); English translation: Amer. Math. Soc. Transl. Ser. 2 27 (1963) 125–341.
[33] F.I. Michael, A note on the five lemma, Appl. Categ. Structures, http://dx.doi.org/10.1007/s10485-011-9273-0 (online first).

[34] J.G. Raftery, G.D. Barboura, Ideal determined varieties have unbounded degrees of permutability, Quaest. Math. 20 (1997) 563–568.
[35] J.D.H. Smith, Mal'cev Varieties, Lecture Notes in Math., vol. 554, Springer, 1976.
[36] A. Ursini, Sulle varietà di algebre con una buona teoria degli ideali, Boll. Unione Mat. Ital. (4) 7 (1972) 90–95.
[37] A. Ursini, Osservazioni sulla varietà BIT, Boll. Unione Mat. Ital. 8 (1973) 205–211.
[38] A. Ursini, On subtractive varieties, I, Algebra Universalis 31 (1994) 204–222.