Quintic nonpolynomial spline method for the solution of a second-order boundary-value problem with engineering applications

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Abstract

Nonpolynomial quintic spline functions are used to develop a numerical algorithm for computing an approximation to the solution of a system of second order boundary value problems associated with heat transfer. We show that the approximate solutions obtained by our algorithm are better than those produced by other spline and domain decomposition methods. A comparison of our algorithm with nonpolynomial quadratic spline method is discussed with the help of two numerical examples.

1. Introduction

In this paper, nonpolynomial quintic spline functions are applied to obtain a numerical solution of the following second order two-point boundary value problem

\[ y''(x) + f(x)y = g(x), \quad x \in [a, b], \]  

subject to Neumann boundary conditions given by

\[ y'(a) - A_1 = y'(b) - A_2 = 0, \]

where \( A_i, \ i = 1,2 \) are finite real constants. The functions \( f \), \( g \in C[a, b] \), the set of all continuous functions on the interval \([a, b]\). The analytic solution of (1) and (2) cannot be obtained for arbitrary choices of functions \( f(x) \) and \( g(x) \). Such problems are known to arise in heat transfer and deflection in cables [1,2].

Al-Said [3] has solved the system of second-order boundary value problem of the type

\[ u'' = \begin{cases} 
  f(x), & a \leq x \leq c, \\
  g(x)u(x) + f(x) + r, & c \leq x \leq d \\
  f(x), & d \leq x \leq b
\end{cases} \]

with the Dirichlet boundary conditions

\[ u(a) = \alpha_1 \quad \text{and} \quad u(b) = \alpha_2, \]
assuming the continuity conditions of \( u \) and \( u' \) at \( c \) and \( d \) with \( f \) and \( g \) being continuous functions on \([a, b]\) and \([c, d]\), respectively. Here the parameters \( r, \alpha_1, \alpha_2 \) are finite real constants. In [3], a cubic spline function was used to develop a numerical method for computing smooth approximations to the solution and its derivatives for the system of second-order boundary-value problems of the type \((3)\) subject to \((4)\). Khan and Aziz [4], applied parametric cubic spline functions to develop a new numerical scheme to obtain smooth approximations to the solutions of \((3)\) and \((4)\). Siraj-ul-Islam and Ikram Tirmizi [5] have applied nonpolynomial spline functions to devise a new numerical method for obtaining smooth approximations to the solution of \((3)\) and \((4)\). Arshad Khan [6] has obtained a uniformly convergent uniform mesh difference scheme using parametric cubic splines for the solution of the two-point boundary value problem with Dirichlet boundary conditions of the type

\[
y''(x) = f(x)y(x) + g(x), \quad a \leq x \leq b
\]

\[
y(a) = \alpha_0, \quad y(b) = \alpha_1,
\]

where \( f(x) \) and \( g(x) \) are in \( C[a, b] \) and \( a, b, \alpha_0, \alpha_1 \) are arbitrary, real, finite constants. Such problems arise in the theory of deflection of plates and a number of other applications (see [6] for details). In general, it is difficult to obtain the analytical solution of \((5)\) for arbitrary choices of functions \( f(x) \) and \( g(x) \). The standard numerical method for solving \((5)\) consists of finite difference schemes outlined in [7–9]. Ramadan et al. [10] have used both polynomial and nonpolynomial spline functions to obtain approximate solutions for second order, two point boundary value problems subject to Neumann boundary conditions. Albasiny and Raghavarao [11, 12] solved a linear second order two-point boundary problem \((3)\) subject to Dirichlet boundary conditions using cubic splines. Blue [13] solved this problem using quintic polynomial splines, while, Caglar et al. [14] solved the same problem using cubic \( B \)-splines. Zahra [15] has also solved this problem using quadratic polynomial splines at mid-knots.

In the present paper, we apply nonpolynomial spline functions with polynomial and trigonometric parts [16, 17] to obtain numerical solutions of a second-order differential equation. The spline functions used for the solution of the system \((1)\) are nonpolynomial quintic splines [18]. The paper is organized as follows: in Section 2, we have discussed the Physical Description of the Differential Equations based on Heat Transfer; in Section 3, we have given a brief introduction of nonpolynomial quintic splines; in Section 4, we give a brief derivation of this nonpolynomial quintic spline, in Section 5; we present our numerical method for the solution of the boundary-value problems; in Section 6, numerical evidences are included to compare and demonstrate the efficiency of the method used in this paper. Finally, in Section 7, we show that our algorithm performs better than the known methods like collocation, finite difference and polynomial spline methods and in Section 8, we make some remarks based on our investigation.

2. Physical description of the differential equations based on heat transfer

The problem of finding the time required for a trunnion to cool down in a refrigerated chamber requires the solution of certain differential equations associated with the model given in [1,2]. It is assumed that the internal conduction in the trunnion is such that the temperature throughout the trunnion is uniform. This allows us to make the assumption that the temperature is only a function of time and not of the position in the trunnion. This means that if a differential equation governs this physical problem, it would be an ordinary differential equation for a lumped system and a partial differential equation for a non-lumped system. We know from the heat transfer theory that the system considered for this purpose can be lumped or non-lumped. In simplistic terms, this distinction is based on the material, geometry, and heat exchange factors of the ball with its surroundings.

In the study of heat transfer, problems of the deflection of plates and in a number of other scientific applications, we encounter a system of differential equations of different orders with different types of boundary conditions. Many problems are formulated mathematically as boundary value problems for second order differential equations as in heat transfer and deflection in cables. In this paper we consider a lumped system forming an ordinary differential equation with Neumann boundary conditions.

3. Nonpolynomial quintic spline

Throughout the paper, let \( a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \). Each \( x_j, j = 0, 1, \ldots, n \), is called a knot for the spline function which we shall use.

A quintic spline function \( S_\Delta(x) \), interpolating a function \( u(x) \) defined on \([a, b]\) is such that

(i) In each subinterval \([x_{j-1}, x_j]\) of \([a, b]\), \( S_\Delta(x) \) is a polynomial of degree at most five; and

(ii) The functions \( S_\Delta(x) \in C^4[a, b] \).

To be able to deal effectively with such problems we introduce ‘spline functions’ containing a parameter. These are ‘nonpolynomial splines’ defined in each subinterval of the domain of definition of the solution. The arbitrary constants are chosen to satisfy certain smoothness conditions at the joints. These ‘splines’ belong to the class \( C^4[a, b] \) and reduce
From algebraic manipulation, we get the following expression:

that depend on a parameter \( \tau \) with nodal points \( x \).

To develop the numerical method for approximating solutions of a differential equation, we consider a uniform mesh \( \Delta \) with nodal points \( x_i \) on \( [a, b] \) such that

\[
\Delta : a = x_0 < x_1 < x_2 < x_3 \ldots \ldots < x_N = b
\]

with \( x_i = a + ih, \ i = 0, 1, 2, \ldots, N \), where \( h = \frac{b-a}{N} \).

A nonpolynomial quintic spline function \( S\Delta(x) \) of class \( C^4[a, b] \) which interpolates \( y(x) \) at the mesh points \( x_i, \ i = 0, 1, 2, \ldots, N \), depends on a parameter \( \tau \), and reduces to ordinary quintic spline \( S\Delta(x) \) in \( [a, b] \) as \( \tau \to 0 \).

For each segment \( [x_i, x_{i+1}], \ i = 0, 1, 2, \ldots, N - 1 \), the nonpolynomial \( S\Delta(x) \) is defined by

\[
S\Delta(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3 + e_i(x-x_i) + f_i \cos \tau (x-x_i),
\]

\[
i = 0, 1, 2, \ldots, N - 1,
\]

where \( a_i, b_i, c_i, d_i, e_i, \) and \( f_i \) are constants and \( \tau \) is an arbitrary parameter.

Let \( y_i \) be an approximation of \( y(x_i) \), obtained by the segment \( S\Delta(x) \) of the mixed splines function passing through the points \( (x_i, y_i) \) and \( (x_{i+1}, y_{i+1}) \), to obtain the necessary conditions for the coefficients introduced in (3), we not only require that \( S\Delta(x) \) interpolates \( x_i \) and \( x_{i+1} \), but also has continuous first derivative at the common nodes \( (x_i, y_i) \).

To derive an expression for the coefficients of (6) in terms of \( y_i, y_{i+1}, M_i, M_{i+1}, F_i \) and \( F_{i+1} \) we first denote:

\[
S\Delta(x_i) = y_i, \quad S\Delta(x_{i+1}) = y_{i+1}, \quad S\Delta''(x_i) = M_i, \quad S\Delta''(x_{i+1}) = M_{i+1},
\]

\[
S\Delta^{(3)}(x_i) = F_i, \quad S\Delta^{(4)}(x_{i+1}) = F_{i+1}.
\]

From algebraic manipulation, we get the following expression:

\[
a_i = y_i - \frac{F_i}{\tau^4},
\]

\[
b_i = \frac{y_{i+1} - y_i}{h} + \frac{F_i - F_{i+1}}{\theta \tau^3} - \frac{h}{6} (2M_i + M_{i+1}) - \frac{h}{6\tau^2} (2F_i + F_{i+1}),
\]

\[
c_i = \frac{1}{2} \left( \frac{M_i + F_i}{\tau^2} \right),
\]

\[
d_i = \frac{M_{i+1} - M_i}{6h} + \frac{F_{i+1} - F_i}{6\theta \tau},
\]

\[
e_i = \frac{F_{i+1} - F_i \cos \theta}{\tau^4 \sin \theta},
\]

\[
f_i = \frac{F_i}{\tau^4}.
\]
where $\theta = \tau h$ and $i = 0, 1, 2, \ldots, N - 1$. We may write (1) in the discrete form as

$$M_i + f_i y_i = g_i, \quad 0 \ll x \ll 1. \tag{9}$$

Using the continuity of the first and second derivatives at $(x_i, y_i)$, that is $S'_{\Delta i}(x_i) = S'_{\Delta i}(x_i)$ and $S''_{\Delta i}(x_i) = S''_{\Delta i}(x_i)$, we obtain the following relations:

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2} (y_{i-1} - 2y_i + y_{i+1}) - 6h^2[\alpha_1 F_{i-1} + 2\beta_1 F_i + \alpha_1 F_{i+1}]$$

$$M_{i-1} - 2M_i + M_{i+1} = h^2[\alpha_1 F_{i-1} + 2\beta_1 F_i + \alpha_1 F_{i+1}], \quad i = 1(1)N - 1. \tag{10}$$

The operator $\Lambda$ is defined by

$$\Lambda w_i = p(w_{i+2} + w_{i-2}) + q(w_{i+1} + w_{i-1}) + sw_i$$

for any function $w$ evaluated at the mesh points. Then we have the following relations connecting $y$ and its derivatives:

(i) $\Lambda M_i = \frac{1}{h^2} [\alpha (y_{i+2} + y_{i-2}) + 2(\beta - \alpha)(y_{i+1} + y_{i-1}) + (2\alpha - 4\beta) y_i]$  

(ii) $\Lambda T_i = \frac{1}{h^3} [(\alpha + \beta)(y_{i+2} + y_{i-2}) + (2\alpha - 4\beta)(y_{i+1} - y_{i-1})]$  

(iii) $\Lambda F_i = \frac{1}{h^4} \delta^4 y_i,$

where $p = \alpha_1 + \frac{q}{6},$

$$q = 2 \left[ \frac{1}{6} (2\alpha + \beta) - (\alpha_1 - \beta_1) \right],$$

$$s = 2 \left[ \frac{1}{6} (\alpha + 4\beta) + (\alpha_1 - 2\beta_1) \right],$$

$$\alpha = \left( \frac{1}{\theta^2} \right) (\theta \csc \theta - 1),$$

$$\beta = \left( \frac{1}{\theta^2} \right) (1 - \theta \cot \theta),$$

$$\alpha_1 = \frac{1}{\theta^2} \left( \frac{1}{6} - \alpha \right),$$

$$\beta_1 = \frac{1}{\theta^2} \left( \frac{1}{3} - \beta \right),$$

$$\theta = \tau h,$$

$$M_i = S''_{\Delta i}(x_i),$$

$$T_i = S''_{\Delta i}(x_i) \quad \text{and} \quad F_i = S^{(4)}_{\Delta i}(x_i).$$

5. Description of the method and development of boundary conditions

At the mesh point $x_i$ the proposed differential equation (1) may be discretized by

$$M_i + f_i y_i = g_i, \tag{12}$$

where $M_i = S''_{\Delta i}(x_i), \ g_i = g(x_i), \ f_i = f(x_i)$ and $y_i = y(x_i)$.

Using the spline relation (11)[i], in (12) we have

$$(-\alpha - ph^2 f_{i-1}) y_{i-2} + (-2(\beta - \alpha) - h^2 q f_{i-1}) y_{i-1} + (-2\alpha + 4\beta - sh^2 f_i) y_i + (-2(\beta - \alpha) - h^2 q f_{i+1}) y_{i+1}$$

$$+ (-\alpha - ph^2 f_{i-1}) y_{i+2} = -h^2 (p g_{i-2} + q g_{i-1} + s g_i + q g_{i+1} + p g_{i+2}), \quad i = 2(1)N - 2. \tag{13}$$

To obtain a unique solution we need two more equations to be associated with (13) so that we use the following boundary conditions:

(a) To obtain the second-order boundary formula we define:

$$y_1 - 2y_2 + y_3 = \frac{h^2}{6} (y_1'' + 4y_2'' + y_3''), \quad i = 1,$$

$$y_{N-3} - 2y_{N-2} + y_{N-1} = \frac{h^2}{6} (y_{N-3}''' + 4y_{N-2}''' + y_{N-1}''), \quad i = N - 1. \tag{14}$$
using Eq. (1) we have:
\[
\begin{align*}
&\left(-1 - \frac{h^2}{6} f_1\right) y_1 + \left(2 - \frac{4h^2}{6} f_2\right) y_2 + \left(-1 - \frac{h^2}{6} f_3\right) y_3 = -\frac{h^2}{6} [g_1 + 4g_2 + g_3], \\
&\left(-1 - \frac{h^2}{6} f_{N-1}\right) y_{N-1} + \left(2 - \frac{4h^2}{6} f_{N-2}\right) y_{N-2} + \left(-1 - \frac{h^2}{6} f_{N-1}\right) y_{N-1} = -\frac{h^2}{6} [g_{N-3} + 4g_{N-2} + g_{N-1}].
\end{align*}
\]
(b) To obtain the fourth-order boundary formula we define:
\[
y_1 - 2y_2 + y_3 = \frac{h^2}{12} (y'' + 10y'' + y''), \quad i = 1,
\]
\[
y_{N-3} - 2y_{N-2} + y_{N-1} = \frac{h^2}{12} (y''_{N-3} + 10y''_{N-2} + y''_{N-1}), \quad i = N - 1,
\]
using Eq. (1) we have
\[
\begin{align*}
&\left(-1 - \frac{h^2}{12} f_1\right) y_1 + \left(2 - \frac{10h^2}{12} f_2\right) y_2 + \left(-1 - \frac{h^2}{12} f_3\right) y_3 = -\frac{h^2}{12} [g_1 + 10g_2 + g_3], \\
&\left(-1 - \frac{h^2}{12} f_{N-1}\right) y_{N-1} + \left(2 - \frac{10h^2}{12} f_{N-2}\right) y_{N-2} + \left(-1 - \frac{h^2}{12} f_{N-1}\right) y_{N-1} = -\frac{h^2}{12} [g_{N-3} + 10g_{N-2} + g_{N-1}].
\end{align*}
\]
By expanding (13) in Taylor series about \( x_i \), we obtain the following local truncation error:
\[
T_i = \left[ \frac{1}{6} (7\alpha + \beta) - (4p + q) \right] h^4 y_i^{(4)} + \left[ \frac{1}{180} (31\alpha + \beta) - \frac{1}{12} (16p + q) \right] h^8 y_i^{(8)} + \left[ \frac{1}{131040} (1611\alpha + 31\beta) - \frac{1}{360} (4p + q) \right] h^{10} y_i^{(10)} + O(h^{10}).
\]
(13) and (16) hold for any choice of \( \alpha \) and \( \beta \), provided that \( \alpha + \beta = \frac{1}{4} \).

**Remark 1.** Second-order method

For \( \alpha = \frac{1}{4}, \quad \beta = \frac{1}{4} \),
and \( p = 0.04063489941134321703 \),
\( q = 0.25412730690212937985 \)
and \( s = 0.41047570631347259688 \)
gives \( T_i = O(h^4) \).

**Remark 2.** Fourth-order method

For \( \alpha = \frac{1}{6}, \quad \beta = \frac{1}{3} \),
\( p = \frac{1}{120}, \quad q = \frac{26}{120} \)
and \( s = \frac{66}{120} \),
gives \( T_i = O(h^6) \).

6. Numerical example

We now consider two numerical examples illustrating the comparative performance of the nonpolynomial quintic algorithm over the nonpolynomial quadratic spline method.

**Example 1.** Consider the boundary value problem
\[
y''(x) + y = -1.
\]
Subject to
\[
y^{(1)}(0) = \frac{1 - \cos(1)}{\sin(1)} = -y^{(1)}(1).
\]
The analytic solution of (17) is
\[ y(x) = \cos(x) + \frac{1 - \cos(1)}{\sin(1)} \sin(x) - 1. \]

Example 2. Consider the boundary value problem

\[ y'' + xy = (3 - x - x^2 + x^3) \sin x + 4x \cos x. \]  \hspace{1cm} (19)

Subject to

\[ y'(0) = -1, \quad y'(1) = 2 \sin(1). \] \hspace{1cm} (20)

The analytic solution of (19) is
\[ y(x) = (x^2 - 1) \sin x. \]

Nonpolynomial quintic spline solution of Example 1

We compare our approximate solutions (for second order) approximate solutions by the method of Ramadan et al. and the exact solutions in Table 1. We also compute the maximum observed errors in absolute values for Example 1.

Next, we compare our approximate solutions (for fourth order) approximate solutions of the method of Ramadan et al. and exact solutions in Table 2. We also calculate the maximum observed errors in absolute values for Example 1.

Nonpolynomial quintic spline solution of Example 2

Next, we repeat the same things for Example 2 and mention the results in Table 3.

We next use our approximate solutions for (fourth order) and repeat Example 2 (see Table 4).

7. Graphical representation of results

The difference between the approximate values obtained by our algorithms, by Ramadan et al. method and the exact values for Examples 1 and 2 are graphically represented by Figs. 1 and 2 respectively.
In this paper we used a nonpolynomial quintic spline function to develop a numerical algorithm for solving second order boundary value problems associated with heat transfer. The result obtained by our algorithm is better than that
Table 4

<table>
<thead>
<tr>
<th>N</th>
<th>Exact value</th>
<th>$y_i$ (approximate) (by Ramadan et al. [10])</th>
<th>$y_i$ (approximate) (our method)</th>
<th>$E$ (maximum absolute error)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.31476545468332</td>
<td>0.29235298746385</td>
<td>0.30897112889976</td>
<td>5.7943E−3</td>
</tr>
<tr>
<td>8</td>
<td>0.27741522214089</td>
<td>0.27474575964644</td>
<td>0.27712336777804</td>
<td>2.9185E−4</td>
</tr>
<tr>
<td>16</td>
<td>0.20662533356214</td>
<td>0.20630125764185</td>
<td>0.20660395401995</td>
<td>2.1379E−5</td>
</tr>
<tr>
<td>32</td>
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<td>0.13738851911992</td>
<td>1.6947E−6</td>
</tr>
<tr>
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<td>0.10033723274673</td>
<td>0.10034202346574</td>
<td>1.5332E−7</td>
</tr>
</tbody>
</table>

obtained by a nonpolynomial quadratic spline as compared in Tables 1–4. The present method outlined here is competitive with other methods like collocation, finite difference and polynomial spline methods. Not only does algorithm yield better approximating solutions, but it also reduces the cost of computation in comparison to other procedures.

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References