Computers and Mathematics with Applications 59 (2010) 2636–2646

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Decentralized robust set-valued state estimation in networked multiple sensor systems[☆]

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ARTICLE INFO

Article history:
Received 6 September 2009
Received in revised form 10 January 2010
Accepted 13 January 2010

Keywords:
Robust control
Networked control systems
State estimation under uncertainty
Sensor fusion
Decentralized estimation

ABSTRACT

This paper addresses a decentralized robust set-valued state estimation problem for a class of uncertain systems via a data-rate constrained sensor network. The uncertainties of the systems satisfy an energy-type constraint known as an integral quadratic constraint. The sensor network consists of spatially distributed sensors and a fusion center where set-valued state estimation is carried out. The communications from the sensors to the fusion center are through data-rate constrained communication channels. We propose a state estimation scheme which involves coders that are implemented in the sensors, and a decoder–estimator that is located at the fusion center. Their construction is based on the robust Kalman filtering techniques. The robust set-valued state estimation results of this paper involve the solution of a jump Riccati differential equation and the solution of a set of jump state equations.

1. Introduction

Sensor networks have been widely applied to the state estimation and control of large-scale control systems. There are a number of advantages of using sensor networks or multisensors; e.g., more information can be gathered through the use of sensor fusion techniques, geographical constraints can be overcome by using a number of spatially distributed sensors, and reliability is improved from some degree of redundancy of sensors. For instance, sensor fusion has been widely applied to vehicle and missile guidance; see e.g. [1,2].

The communications in a sensor network are often implicitly assumed to be of infinite precision or to have an infinite bit rate. Due to the enormous growth in communication technology, it is becoming more common to employ data-rate constrained communication networks for the exchange of information between system components. However, classical estimation theory cannot be applied since the measurement information is sent via data-rate constrained communication channels, hence, the estimator only observes the transmitted sequence of finite-valued symbols. As a matter of fact, in recent years, there has been a significant interest in the problems of state estimation and control with communication; for state estimation, see, e.g., [3–15]; for both control and estimation, see, e.g, [16–25].

In relation to the state estimation problem, estimation of stochastic systems via communication channels were studied by using: recursive linear minimum variance estimator [5,10]; H_∞ filter [11]; dynamic Markov jump filter [12]; and least-squares filters [8,13,14]. For instance, the approach taken in [8,13,14] is based on the covariance functions of the process that is included in the observation equation, and hence the knowledge of the signal state-space model is not required that is in contrast to [5,10–12]. State estimation using quantized measurements with logarithmic quantizers was studied in [15].

☆ This work was supported by the Australian Research Council.
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doi:10.1016/j.camwa.2010.01.032

As a matter of fact, the above-mentioned papers mainly considered state estimation via communication channels with effects, such as delay, noises, quantization, and packet dropouts during transmission, in a *stochastic* setting. In Ref. [4], it is shown that if Shannon capacity bound is met, a stable observer with limited computation complexity and memory per unit time can be constructed for estimating discrete-time partially observed linear systems via limited-capacity noisy communication channels. Necessary and sufficient conditions for reconstruction and stability of linear stochastic partially observed systems via additive white Gaussian noise channels were derived in [16]. Using the conditions, an encoder, decoder, and controller of mean-square reconstruction and stability were also designed. In order to reduce sensor data traffic, a modified Kalman filter, that operates when the sensor values change more than a specified value, was proposed in [9].

When the statistics of the external noises and uncertain parameters in the systems are unknown, an H_∞ filter for linear systems with uncertainties that belong to $L_2[0, \infty)$ was developed in [7] in a deterministic setting with communication channels subject to bounded delays and data-packet dropouts. For a similar class of systems and also in a deterministic setup, but in a different approach, a constructive coder-decoder scheme for a robust *set-valued* state estimation problem of continuous-time uncertain linear systems via limited capacity communication channels was proposed in [3]. The proposed coder–decoder was developed using the idea of the recursive set-valued state estimation technique [26] that originated from the deterministic interpretation of the Kalman filter [27]. When the Kalman filter is viewed from this deterministic view point, the set-valued state estimation problem turns out to be a linear quadratic (LQ) optimal control problem in which time is reversed.

Using the approach taken by [26], but with a different coding scheme, a constructive algorithm for robust estimation was proposed in [6] for a wider class of uncertain systems. The uncertainties of the systems considered in [6] are defined by a certain Integral Quadratic Constraint (IQC) (see, e.g., [28]), and this class of uncertainties allows for nonlinear, time-varying, dynamic uncertainties. However, only point state estimates can be obtained by the algorithm proposed in [6] as compared to the set-valued state estimate. Indeed, besides the work [3], the problem of *robust set-valued* state estimation via limited communication channels is seldom reported in the literature to the best of the authors' knowledge.

Even though the works of [3,6] provide constructive algorithms that allow one to reliably estimate the state of an uncertain system through communication networks, a drawback of their proposed coding schemes is that they are centralized, requiring that *all* the measurement information is available to a single centralized coder. The coder uses the information to obtain a state estimate that is then encoded and sent to a decoder. However, this scheme may not be practical in a sensor network or multisensor setting since the sensors may be spatially distributed or geographically separated. The transmission of all the measurement information to a centralized coder, and the transmission of the full state estimate to a decoder, will take up a significant amount of bandwidth, as bandwidth is always a constraint in a communication network. Therefore, it is more realistic to transmit each sensor measurement to the decoder directly, rather than collecting all the measurements from the sensors and processing them at a centralized coder.

In this paper, we consider a decentralized robust *set-valued* state estimation via a data-rate constrained, or limited capacity, sensor network. The sensors in the network are spatially distributed. Instead of transmitting all the measurements to a centralized coder as in [3,6], we employ a decentralized scheme and design a coder for each individual sensor. Each encoded measurement is sent to a remotely-located fusion center where a decoder and a robust state estimator are embedded. The fusion center combines all the received codewords from the remote sensors and produces a set-valued state estimate, which is an ellipsoid, that over-bounds the true set of possible states of the uncertain system.

The systems considered in this paper consist of uncertainties that satisfy the IQC and, as mentioned before, this class of uncertainties allows for nonlinear, time-varying, dynamic uncertainties. Our proposed algorithm computes a *set-valued* state estimate at the fusion center by using the codewords generated at the remote sensors. A major benefit of having a set-valued estimate over a point estimate is that not only is the point estimate available, but also the upper and lower bounds of the true system states can be computed at the fusion center or decoder. In other words, the range of possible states of the uncertain system can also be obtained at the fusion center. The robust set-valued state estimation results of this paper involve the solution of a jump Riccati differential equation and the solution of a set of jump state equations. The jump Riccati differential equation has discontinuous right-hand sides and it behaves like a standard Riccati differential equation between sampling instants, but its solution exhibits finite jumps at the sample times.

The main advantages of this paper over the works of [3,6] are: (1) a *decentralized* scheme is used instead of a centralized scheme; (2) the communication overhead is *reduced*; (3) a set-valued state estimate is obtained for a larger class of uncertain systems; and (4) a *continuous* set-valued state estimate is obtained at the decoder rather than at discrete sampling times. As for the communication overhead, the schemes proposed in [3,6] require sensors-to-coder *and* coder-to-decoder communication links; whereas in this paper, we only need communication links from sensors to a decoder and hence communication overhead can be significantly reduced. Another advantage of this paper when compared with the work [6] is that here we obtain a set-valued state estimate, rather than a point-valued state estimate, of an uncertain system at the remotely-located decoder. From the set-valued state estimate, upper and lower bounds of the true system states can be computed. In contrast, the scheme proposed in [6] only allows us to retrieve a point-valued state estimate and the bounds of the true system states are not available at the decoder. Furthermore, the state estimate obtained by the algorithms in [3, 6] is available only at discrete times rather than continuously available as in our scheme. In other words, the inter-sampling behavior was not studied and considered in [3,6], but it is considered in this paper.

Furthermore, the improved feature of this decentralized scheme as compared to other schemes previously proposed in the area of state estimation (or control) over communication networks is that here we only require a simple quantization at the coders within the sensors, and the use of the central processing unit (CPU) is *not* necessary in the coders. We only need a CPU at the fusion center to perform state estimation. In contrast, the coding schemes in the previous works (e.g., Refs. [18,3,19,6,22,23]) require that the centralized coder is equipped with a state estimator, and hence the use of a CPU in the coder is necessary. Therefore, our proposed decentralized scheme is more applicable than the previous scheme, since we only need simple and low computational-cost coders in the sensors.

The paper is organized as follows. In Section 2, we formulate the problem of decentralized robust state estimation via a data-rate constrained sensor network. To solve the problem, some preliminary results are presented in Section 3. In Section 4, a design of coders and decoder–estimator that solves the proposed problem is introduced. Finally, a practical example is presented in Section 5 to demonstrate the effectiveness of the proposed algorithms.

2. Problem statement

Consider the time-varying uncertain system defined over the finite time interval $[0, NT]$:

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)w(t)\\ z(t) &= K(t)x(t)\end{aligned} \tag{1}$$

where $N > 0$ is an integer, $T > 0$ is a given constant, $x \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^p$ is an *uncertainty input*, $z(t) \in \mathbb{R}^q$ is the *uncertainty output* and $A(\cdot)$, $B(\cdot)$, and $K(\cdot)$ are bounded piecewise continuous matrix functions defined on $[0, NT]$. The uncertainty input $w(\cdot)$ depends dynamically on the uncertainty output $z(\cdot)$, and it may describe some dynamic uncertainties such as those that arise from unmodeled dynamics.

The decentralized estimation problem studied in this paper is to robustly estimate the state of the uncertain system (1) by a sensor network consisting of l low-cost and low-power sensors, namely $\{\Omega_1, \Omega_2, \dots, \Omega_l\}$, that are spatially distributed. For each sensor Ω_i, the observation or measurement $y_i(\cdot) \in \mathbb{R}^{m_i}$ is corrupted by a noise $v_i(\cdot) \in \mathbb{R}^{m_i}$ and it is given by

$$y_i(t) = C_i(t)x(t) + v_i(t), \quad i = 1, 2, \dots, l, \tag{2}$$

where the measurement matrix $C_i(\cdot) \in \mathbb{R}^{m_i \times n}$ is bounded and piecewise continuous over the time interval $[0, NT]$.

The information of the measurement $y_i(\cdot)$ from each sensor Ω_i is passed on to a *fusion center* that is remotely located from the sensors. The only way of communicating information from the sensors to the fusion center is via digital communication channels. In other words, each sensor Ω_i not only observes the measurement $y_i(\cdot)$, but also converts it into a finite-length codeword for transmitting the information to the fusion center. To convert the measurement $y_i(\cdot)$ into a finite-length codeword, each sensor Ω_i is equipped with a *coder* $\mathcal{F}_i$ that takes the measurement $y_i(\cdot)$ and encodes this measurement into a codeword $h_i(\cdot)$.

The channel connecting the sensor Ω_i to the fusion center carries one discrete-valued symbol $h_i(kT)$ at time kT, selected from a coding alphabet $\mathcal{H}_i$ of size ν_i. Here $T > 0$ is a given period and $k = 0, 1, 2, \dots, N$. This restricted number ν_i of codewords $h_i(kT)$ is determined by the transmission data rate of the channel. We assume that the channel is a perfect noiseless channel and there is no time delay.

Using this communication channel, the codeword $h_i(kT)$ produced by the coder $\mathcal{F}_i$ is transmitted to the fusion center. A decoder and a robust state estimator are embedded in the fusion center. The job of the fusion center is to combine all the received codewords $h_1(kT), h_2(kT), \dots, h_l(kT)$ from the remote sensors and to produce a set-valued state estimate $\mathcal{X}_t$, for all $t \in [kT, (k+1)T)$, that over-bounds the true set of possible state $x(t)$ of system (1) over the time interval $[kT, (k+1)T)$. The decoder and the state estimator within the fusion center are called *decoder–estimator* $\mathcal{G}$.

We define the total number of measurements from all the sensors as $\bar{m} := m_1 + m_2 + \cdots + m_l$. Let $h(\cdot) = [h_1(\cdot) \;\; h_2(\cdot) \;\; \dots \;\; h_l(\cdot)]' \in \mathbb{R}^{\bar{m}}$ be the vector of codewords produced by the sensors. Then the coders and the decoder–estimator are in the form: for $k = 0, 1, 2, \dots, N$, **Coders** $(i = 1, 2, \dots, l)$: $h_i(kT) = \mathcal{F}_i\left(y_i(\cdot)\,|_0^{kT}\right)$; **Decoder-estimator:** $\mathcal{X}_t = \mathcal{G}\,(h(T), h(2T), \dots, h(kT)), \forall t \in [kT, (k+1)T)$. A schematic of the proposed robust state estimation via a data-rate constrained sensor network is illustrated in Fig. 1.

Notation 2.1. *Let $x = [x_1 \; x_2 \; \cdots \; x_n]'$ be a vector from $\mathbb{R}^n$. Then $\|x\|_\infty := \max_{j=1,\dots,n} |x_j|$. Furthermore, $\|\cdot\|$ denotes the standard Euclidean vector norm: $\|x\| := \sqrt{\sum_{j=1}^n x_j^2}$.*

Notation 2.2. *The set $\mathcal{F} = \{\mathcal{F}_1, \; \mathcal{F}_2, \; \dots \; , \mathcal{F}_l\}$ denotes the collection of coders in the sensors. The vectors $y = [y_1 \; y_2 \dots y_l]' \in \mathbb{R}^{\bar{m}}$ and $v = [v_1 \;\; v_2 \;\; \dots \;\; v_l]' \in \mathbb{R}^{\bar{m}}$ denote the augmented measurement vector from all the sensors and the augmented measurement noise vector, respectively. The measurement matrix $C(\cdot) \in \mathbb{R}^{\bar{m} \times n}$ is defined as $C(\cdot) := \left[C_1'(\cdot) \;\vdots\; C_2'(\cdot) \;\vdots\; \dots \;\vdots\; C_l'(\cdot)\right]'$.*

To solve our proposed estimation problem, we make the following assumption on the uncertain system (1) and the measurement noise in (2).

Assumption 2.1. The uncertainty $w(t)$ vector in system (1) and the augmented measurement noise vector $v(t)$ satisfy the following integral quadratic constraint (IQC). Let $Y_0 = Y_0' > 0$ be a given matrix, $x_0 \in \mathbb{R}^n$ be a given vector, $d > 0$ be a given constant, $Q(\cdot) = Q(\cdot)'$ and $R(\cdot) = R(\cdot)'$ be given bounded piecewise continuous matrix weighting functions satisfying the following condition. There exists a constant $\delta > 0$ such that $Q(t) \geq \delta I$, $R(t) \geq \delta I$ for all t. Then for a given time interval

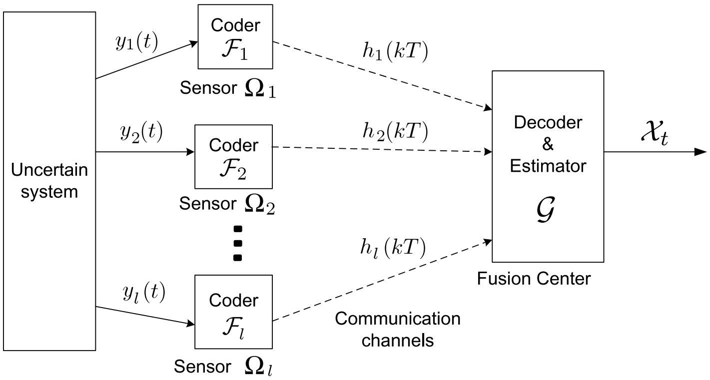

Fig. 1. Robust state estimation via a data-rate constrained sensor network.

$[0,\ s]$, $s \le NT$, we will consider the uncertainty input $w(\cdot)$, the measurement noise $v(\cdot)$ and initial condition $x(0)$ such that

$$(x(0)-x_0)'Y_0(x(0)-x_0)+\int_0^s (w(t)'Q(t)w(t)+v(t)'R(t)v(t))\mathrm{d}t \le d+\int_0^s \|z(t)\|^2\mathrm{d}t. \tag{3}$$

The description (3) in Assumption 2.1 allows the uncertainty inputs $w(\cdot)$ and $v(\cdot)$ to be dynamically dependent upon the uncertainty output $z(\cdot)$. In fact, the IQC condition (3) can be satisfied by many classes of commonly known uncertain systems [28]. For instance, consider the norm-bounded uncertain system:

$$\dot{x}=[A(t)+B(t)\Delta(t)K_1]x(t), \qquad y(t)=[C(t)+\Delta(t)K_2(t)]x(t), \tag{4}$$

where $\Delta(t)$ is an uncertainty matrix and the initial value satisfies $(x(0)-x_0)'Y_0(x(0)-x_0) \le d$. If we let $w(t)=\Delta(t)K_1(t)x(t)$, $v(t)=\Delta(t)K_2(t)x(t)$, $z(t)'=[K_1(t)x(t)\ \ K_2(t)x(t)]'$ and $\|\Delta(t)\| \le 1$ for all $t \in [0,NT)$, then the IQC condition (3) is satisfied with $Q(\cdot)\equiv R(\cdot)\equiv I$. Also, it is not hard to see that systems with sector-bounded nonlinearities (a class of nonlinear systems arose from the celebrated theory of absolute stability) can be considered as a special case of the norm-bounded uncertain systems. System (1) with uncertainties satisfying (3) is more general than the ones considered in [3] where $K(t)\equiv 0$. In other words, in this paper, we allow a much larger class of uncertainties in the systems than that of [3]; for example, the norm-bounded uncertain systems (4) described above cannot be handled by the technique proposed by [3].

Notation 2.3. *Let $y(t)=y_0(t)$ be a fixed measured output of the uncertain system (1) and let the finite-time interval $[0,s]$ be given. Furthermore, let $\mathcal{F}$ and $\mathcal{G}$ be given coders and decoder–estimator, respectively. Then, $\mathcal{X}_s[x_0, y_0(\cdot)\,|_0^s, d, \mathcal{F}, \mathcal{G}]$ denotes the set produced by the coders/decoder–estimator pair $(\mathcal{F}, \mathcal{G})$ that captures all possible state $x(s)$ at time s for the uncertain system (1) with uncertainty input $w(t)$ and measurement noise $v(t)$ satisfying the constraint (3).*

The problem of decentralized robust state estimation via a sensor network with data-rate constrained communication channels considered in this paper is the problem of constructing the coders/decoder–estimator pair $(\mathcal{F}, \mathcal{G})$ and the set $\mathcal{X}_s[x_0, y_0(\cdot)\,|_0^s, d, \mathcal{F}, \mathcal{G}]$.

Definition 2.1. The coders/decoder–estimator pair $(\mathcal{F}, \mathcal{G})$ is said to *detect* the state of system (1) via a data-rate constrained sensor network if for any vector $x_0 \in \mathbb{R}^n$, any time $s \in [0, NT]$, any constant $d > 0$, and any sampling period $T > 0$, and any fixed output $y(t)=y_0(t)$, the set $\mathcal{X}_s[x_0, y_0(\cdot)\,|_0^s, d, \mathcal{F}, \mathcal{G}]$ is bounded.

3. Preliminaries

This section presents two useful preliminary results that are important for the development of the main result in this paper. The first result concerns the robust prediction that will be applied to determining the size of a quantization region in designing coders $\mathcal{F}$ and decoder–estimator $\mathcal{G}$. The second result concerns the set-valued state estimation of uncertain continuous systems with discrete measurements.

3.1. Robustly predictable systems

We consider system (1) that satisfies the following IQC condition. Let $\hat{d} > 0$ be a given constant, $S_0 = S_0' > 0$ be a given matrix, and $Q(\cdot)=Q(\cdot)'$ be a given bounded piecewise continuous matrix weighting function satisfying the following condition. There exists a constant $\delta > 0$ such that $Q(t) \geq \delta I$ for all t. Then for a given time interval $[0, s]$, $s \leq NT$, we will consider the uncertainty input $w(\cdot)$ and initial condition $x(0)$ such that

$$x(0)'S_0x(0) + \int_0^s w(t)'Q(t)w(t)\mathrm{d}t \leq \hat{d} + \int_0^s \|z(t)\|^2\mathrm{d}t. \tag{5}$$

Next, consider the following Riccati differential equation: for all $t \in [0, NT]$,

$$-\dot{S}(t) = S(t)A(t) + A(t)'S(t) + S(t)B(t)Q(t)^{-1}B(t)'S(t) + K(t)'K(t), \qquad S(0) = S_0. \tag{6}$$

Definition 3.1. Uncertain system (1) and (5) is said to be *robustly predictable* on $[0, NT]$ if for any time $s \in [0, NT]$ and any constant $\hat{d} > 0$, the set $\mathcal{X}_s[\hat{d}]$ is bounded where $\mathcal{X}_s[\hat{d}]$ denotes the set of all possible state $x(s)$ at time s for the uncertain system (1) with uncertainty input $w(t)$ and initial condition $x(0)$ satisfying the constraint (5).

Theorem 3.1. *Consider system (1). Let $S_0 = S_0' > 0$ be a given matrix, and $Q(\cdot) = Q(\cdot)' > 0$ be a given matrix function such that condition (5) holds over time interval $[0, NT]$. Then, for a given constant $\hat{d} > 0$ and any time $s \in [0, NT]$, the system (1) and (5) is robustly predictable on $[0, NT]$ if and only if the Riccati equation (6) has a solution over $[0, NT]$ such that $S(\cdot) = S(\cdot)' > 0$. Furthermore, the set $\mathcal{X}_s[\hat{d}]$ is given by $\mathcal{X}_s[\hat{d}] = \{x_s \in \mathbb{R}^n : x_s'S(s)x_s \leq \hat{d}\}$.*

Proof. The proof is similar to the proof of Theorem 2.1 in [26]. Here, we set $x_0 = 0$, and the cost function is defined by $J[x_s, w(\cdot)] := x(0)'S_0x(0) + \int_0^s w(t)'Q(t)w(t) - x(t)'K(t)'K(t)x(t)\mathrm{d}t \leq \hat{d}$. (See Appendix for details.) □

Theorem 3.1 shows that the state $x(\cdot)$ of the uncertain system (1) belongs to the ellipsoid $\mathcal{X}_s[\hat{d}]$ if its hypothesis is satisfied. In other words, the bounds on $x(\cdot)$ can be evaluated from $\mathcal{X}_s[\hat{d}]$ for $s \leq NT$. In fact, Theorem 3.1 is markedly different from [6, Lemma 5.3.2] that was introduced to compute the ellipsoid that bounds the *state estimate* $\hat{x}(\cdot)$ rather than the actual state $x(\cdot)$. Since in [6], their coder encodes the state estimate $\hat{x}(\cdot)$ and the knowledge on the bounds of $\hat{x}(\cdot)$ is required to quantize it. On the other hand, our proposed coders will not encode $\hat{x}(\cdot)$ instead they will encode the measurements $y_i(\cdot), i = 1, \ldots, l$, that are related to $x(\cdot)$ not $\hat{x}(\cdot)$. When comparing Theorem 3.1 with [26, Theorem 2.1], Theorem 3.1 is a prediction result whereas [26, Theorem 2.1] is a filtering result.

Corollary 3.1. *Suppose that Assumption 2.1 holds and the Riccati equation (6) has a solution $S(\cdot) = S(\cdot)' > 0$ over $[0, NT]$ with initial condition $S(0) = Y_0$. Then system (1) is robustly predictable on $[0, NT]$.*

Proof. Using d, x_0 and Y_0 in (3), we define a constant $\hat{d} := \lambda_{\max}(Y_0)\left(\|x_0\| + \sqrt{d/\lambda_{\min}(Y_0)}\right)^2$, where $\lambda_{\max}(Y_0)$ and $\lambda_{\min}(Y_0)$ are the largest and smallest eigenvalues of the matrix Y_0 respectively. Inequality (3) then implies inequality (5) with $S_0 = Y_0$. Using Theorem 3.1, system (1) is robustly predictable over $[0, NT]$. □

3.2. Robust state estimation with discrete quantized measurements

In this subsection, we again consider the continuous-time system (1), but assume that the remote sensors observe the system at discrete times and also that these sensors quantize their measurements. We define the augmented discrete quantized measurement equation as:

$$\bar{y}(kT) = C(kT)x(kT) + \bar{v}(kT) \tag{7}$$

where

$$\bar{y}(\cdot) = \begin{bmatrix}\bar{y}_1(\cdot) & \bar{y}_2(\cdot) & \ldots & \bar{y}_l(\cdot)\end{bmatrix}' \in \mathbb{R}^{\bar{m}},$$
$$\bar{v}(\cdot) = \begin{bmatrix}\bar{v}_1(\cdot) & \bar{v}_2(\cdot) & \ldots & \bar{v}_l(\cdot)\end{bmatrix}' \in \mathbb{R}^{\bar{m}}$$

are the discrete quantized measurement vector and measurement noise vector, respectively. In (7), the measurement $\bar{y}(kT)$ is the sampled and quantized version of $y(t)$ in (2), and it is only available at discrete time instances $T, 2T, \ldots, NT$. In this case, the measurement noise $\bar{v}(kT)$ in (7) resulted from two sources, namely $v(kT)$ (i.e., the noise $v(t)$ in (2) when $t = kT$) and the quantization noise. Thus, the measurement noise of the sampled and quantized measurement can be written as

$$\bar{v}(kT) = v(kT) + \bar{y}(kT) - y(kT)$$

where $(\bar{y}(kT) - y(kT))$ is defined as the quantization noise at kT.

Suppose that the uncertainty $w(\cdot)$ and the measurement noise $\bar{v}(\cdot)$ satisfy a Sum Integral Quadratic Constraint (SIQC) such that

$$(x(0) - x_0)'P_0^{-1}(x(0) - x_0) + \int_0^s w(t)'\bar{Q}(t)w(t)\mathrm{d}t + \sum_{kT \leq s} \bar{v}(kT)'\bar{R}\bar{v}(kT) \leq \bar{d} + \int_0^s \|z(t)\|^2\mathrm{d}t, \tag{8}$$

where P_0, $\bar{Q}(\cdot)$ and $\bar{R}(\cdot)$ are given symmetric positive definite weighting matrices of suitable dimensions, and $\bar{d} > 0$ is a given constant. Note that $\bar{Q}(\cdot)$, $\bar{R}(\cdot)$ and $\bar{d}$ can be different from $Q(\cdot)$, $R(\cdot)$ and d in (2) and (3).

Notation 3.1. *Let $\bar{y}(kT) = \bar{y}_0(kT)$ be a given fixed discrete measurement of system (1). The set $\bar{\mathcal{X}}_s[x_0, \bar{y}_0(\cdot)\,|_0^s, \bar{d}]$ denotes a set containing all the possible states of system (1) at time $s \in [0, NT]$ with uncertainty and measurement noise satisfying SIQC (8).*

The following set-valued state estimation of an uncertain continuous system with discrete measurements is a special case of Theorem 6.3.1 in [29]. Before we state this result, we introduce a notation: the term $\nu(t^-)$ denotes the limit of the function $\nu(\cdot)$ at the point t from the left; i.e., $\nu(t^-) := \lim_{\epsilon>0,\epsilon\to 0} \nu(t-\epsilon)$.

Theorem 3.2. *Let $P_0 = P_0' > 0$ be a given matrix, $\bar{Q}(\cdot) = \bar{Q}(\cdot)' > 0$ and $\bar{R}(\cdot) = \bar{R}(\cdot)' > 0$ be given matrix functions. Consider uncertain system (1) and constraint (8) with discrete measurement (7). Then, the set $\bar{\mathcal{X}}_s[x_0, \bar{y}_0(\cdot)\,|_0^s, \bar{d}]$ is bounded over $[0, NT]$ if and only if the following jump Riccati equation*

$$\begin{aligned} \dot{P}(t) &= A(t)P(t) + P(t)A(t)' + B(t)Q^{-1}(t)B(t)' + P(t)K(t)'K(t)P(t), \quad \text{for } t \neq kT \\ P(kT) &= [P^{-1}(kT^-) + C(kT)'\bar{R}(kT)C(kT)]^{-1}, \quad \text{for } k = 1, 2, \ldots, N \end{aligned} \tag{9}$$

has a solution over $[0, NT]$ such that $P(\cdot) = P(\cdot)' > 0$ and $P(0) = P_0$. Furthermore, the set $\bar{\mathcal{X}}_s[x_0, \bar{y}_0(\cdot)\,|_0^s, \bar{d}] = \{x_s \in \mathbb{R}^n : (x_s - \hat{x}(s))'P(s)^{-1}(x_s - \hat{x}(s)) \leq \bar{d} + \bar{\rho}(s)\}$ for any $s \in [0, NT]$, where $\hat{x}(\cdot)$ is the solution to the following jump state equation:

$$\begin{aligned} \dot{\hat{x}}(t) &= [A(t) + P(t)K(t)'K(t)]\hat{x}(t), \quad \text{for } t \neq kT \\ \hat{x}(kT) &= \hat{x}(kT^-) + P(kT^-)C(kT)'\bar{R}(kT)\left(\bar{y}(kT) - C(kT)\hat{x}(kT^-)\right), \quad \text{for } k = 1, 2, \ldots, N, \end{aligned} \tag{10}$$

with initial condition $\hat{x}(0) = x_0$, and the function $\bar{\rho}(s)$ is defined as $\bar{\rho}(s) := \int_0^s \|K(t)\hat{x}(t)\|^2 \mathrm{d}t - \sum_{kT\leq s} \|\bar{R}(kT)^{1/2}(C(kT)\hat{x}(kT) - \bar{y}_0(kT))\|^2$.

Proof. See Theorem 6.3.1 in [29]. □

Remark 3.1. The jump Riccati differential equation (9) behaves like a standard Riccati differential equation between sampling instants, but its solution exhibits finite jumps at the sample times.

4. Coders and Decoder-estimator

In this section, we design coders $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_l\}$ and a decoder–estimator $\mathcal{G}$ that solve the state estimation problem proposed in Section 2. For each sensor Ω_i, its coder $\mathcal{F}_i$ measures $y_i(t)$ and converts it into a finite-length codeword $h_i(kT)$ through sampling and quantization. The coder $\mathcal{F}_i$ is static and does not have any memory, and therefore, computations at the coder can be kept at minimal.

To construct each $\mathcal{F}_i$, we first need to know the possible range of $y_i(\cdot)$ for quantization *a priori*. By using Corollary 3.1(Robust predictability), if system (1) with uncertainty $w(\cdot)$ and initial condition $x(0)$ satisfying the IQC (5), and the Riccati equation (6) has a solution over $[0, NT]$ such that $S(\cdot) = S(\cdot)' > 0$, then for all $s \in [0, NT]$,

$$\|x(s)\|_\infty \leq \beta\sqrt{\hat{d}}, \quad \beta := \max_{k=0,1,2,\ldots,N}\left(\max_{j=1,2,\ldots,n}\sqrt{[S(kT)^{-1}]_{j,j}}\right), \tag{11}$$

where $[S(kT)^{-1}]_{j,j}$ denotes the (j, j) element of the matrix $S(kT)^{-1}$.

Since each measurement matrix $C_i(\cdot)$ in (2) is a bounded piecewise continuous matrix function, there exists a constant $\gamma_i > 0$ such that $\max_{k=0,1,2,\ldots,N} \|C_i(kT)\|_\infty = \gamma_i$, where $\|C_i(\cdot)\|_\infty$ denotes the maximum row sum matrix norm of the matrix $C_i(\cdot)$, i.e., $\|C_i(\cdot)\|_\infty := \max_i \sum_{j=1}^n |[C_i(\cdot)]_{i,j}|$. To get a bound for $y_i(\cdot)$, we impose the following assumption on the measurement noise (2).

Assumption 4.1. The measurement noise $v_i(\cdot)$ in (2) from each sensor Ω_i is bounded and there exists a known bound $\alpha_i > 0$ such that $\|v_i(s)\| \leq \alpha_i$ for all $s \leq NT$.

Then a bound L_i for the measurement $y_i(\cdot)$ over the time interval $[0, NT]$ can be defined as follows:

$$L_i := \gamma_i\beta\sqrt{\hat{d}} + \alpha_i \geq \|y_i(s)\|_\infty \tag{12}$$

for all $s \in [0, NT]$. The bound L_i (12) can be pre-computed without the knowledge of the actual output $y_i(\cdot)$. This bound is then used to define a quantization region for the output measurement $y_i(kT)$, for $k = 0, 1, 2, \ldots, N$.

Each coder $\mathcal{F}_i$, $i = 1, 2, \ldots, l$ uses uniform quantization of the measurement $y_i(\cdot)$. Let the set $\mathcal{B}_{L_i} := \{y_i \in \mathbb{R}^{m_i} : \|y_i\|_\infty \leq L_i\}$ be the quantization region. The measurement $y_i(\cdot)$ is quantized by simply dividing the quantization region $\mathcal{B}_{L_i}$ uniformly into $q_i^{m_i}$ hypercubes where q_i is a specified integer. For each $j \in \{1, 2, \ldots, m_i\}$, we divide the corresponding component of the vector $y_i = [y_{i,1}\ y_{i,2} \ldots\ y_{i,m_i}]'$ into q_i intervals as follows:

$$I_1^j(L_i) := \left\{ y_{i,j} : y_{i,j} \in \left[-L_i,\ -L_i + \frac{2L_i}{q_i} \right) \right\};$$
$$I_2^j(L_i) := \left\{ y_{i,j} : y_{i,j} \in \left[-L_i + \frac{2L_i}{q_i}, -L_i + \frac{4L_i}{q_i} \right) \right\};$$
$$\vdots$$
$$I_{q_i}^j(L_i) := \left\{ y_{i,j} : y_{i,j} \in \left[L_i - \frac{2L_i}{q_i}, L_i \right] \right\}.$$

Then for any $y_i \in \mathcal{B}_{L_i}$, y_i belongs to one of the hypercubes in $\mathcal{B}_{L_i}$. In other words, there exist unique m_i integers $\theta_{i,1}, \theta_{i,2}, \ldots, \theta_{i,m_i} \in \{1, 2, \ldots, q_i\}$ such that $y_i \in I^1_{\theta_{i,1}}(L_i) \times I^2_{\theta_{i,2}}(L_i) \times \cdots \times I^{m_i}_{\theta_{i,m_i}}(L_i)$, where $I^1_{\theta_{i,1}}(L_i) \times I^2_{\theta_{i,2}}(L_i) \times \cdots \times I^{m_i}_{\theta_{i,m_i}}(L_i)$ is one of the $q_i^{m_i}$ hypercubes containing y_i. Corresponding to the integers $\theta_{i,1}, \theta_{i,2}, \ldots, \theta_{i,m_i}$, we define the vector η_i as follows:

$$\eta_i(\Theta_i) := -L_i + \left[\frac{L_i(2\theta_{i,1} - 1)}{q_i}\ \frac{L_i(2\theta_{i,2} - 1)}{q_i}\ \ldots\ \frac{L_i(2\theta_{i,m_i} - 1)}{q_i} \right]'$$

where $\Theta_i := [\theta_{i,1}\ \theta_{i,2}\ \ldots\ \theta_{i,m_i}]'$. The vector $\eta_i(\cdot)$ is the center of the hypercube $I^1_{\theta_{i,1}}(L_i) \times I^2_{\theta_{i,2}}(L_i) \times \cdots \times I^{m_i}_{\theta_{i,m_i}}(L_i)$ containing the original point y_i. Each one of the hypercubes in the quantization region $\mathcal{B}_{L_i}$ will be assigned a codeword $h_i(kT) = \Theta_i$ and the coder $\mathcal{F}_i$ will transmit the codeword $h_i(kT)$ corresponding to the current measurement vector $y_i(kT)$. By defining $\bar{y}_i(kT) := \eta_i(\Theta_i)$, for a given $\epsilon > 0$, we can choose $q_i > 0$ such that

$$\|y_i(kT) - \bar{y}_i(kT)\|_\infty \le L_i/q_i \le \epsilon, \tag{13}$$

for all $k = 0, 1, 2, \ldots, N$. In other words, ϵ gives the quantization error and it can be controlled by varying the parameter q_i. However, the allowable quantization parameter q_i is limited by the capacity of the communication channel between the sensor Ω_i and the fusion center. If q_i is unbounded, it means the measurement $y_i(kT)$ can be transmitted with an infinite precision.

Now we are in a position to introduce our proposed coders and decoder–estimator:

Coder $\mathcal{F}_i$ $(i = 1, 2, \ldots, l)$: for $k = 0, 1, 2, \ldots, N$,

$$h_i(kT) = \Theta_i, \quad \text{for } y_i \in I^1_{\theta_{i,1}}(L_i) \times I^2_{\theta_{i,2}}(L_i) \times \cdots \times I^{m_i}_{\theta_{i,m_i}}(L_i). \tag{14}$$

Decoder-estimator $\mathcal{G}$:

$$\begin{aligned} &\text{Consists of jump state Eq. (10) and Riccati equation (9) with } \hat{x}(0) = x_0, \quad P(0) = Y_0^{-1}, \\ &\text{and } \bar{y}(kT) = [\eta_1(\Theta_1)\ \eta_2(\Theta_2)\ \ldots\ \eta_l(\Theta_l)]', \quad \text{for } h(kT) = [\Theta_1\ \Theta_2\ \ldots\ \Theta_l]'. \end{aligned} \tag{15}$$

The main result of this paper is then stated as follows:

Theorem 4.1. *Consider the uncertain system (1) and (2). Let $\bar{R} = \operatorname{diag}\{r_1^{-1}, r_2^{-1}, \ldots, r_{\bar{m}}^{-1}\}$ be a given diagonal constant matrix with $r_i > 0$, and let $T > 0$ and $\epsilon > 0$ be given constants, and $s \in (0, NT]$ be given. Suppose that Assumptions 2.1 and 4.1 hold and also that the solution $S(\cdot)$ to the Riccati equation (6) with initial condition $S(0) = Y_0$ and the solution $P(\cdot)$ to the jump Riccati equation (9) with initial condition $P(0) = Y_0^{-1}$ are both defined and positive-definite on the interval $[0, NT]$. Furthermore, suppose that the quantization parameter q_i satisfies*

$$q_i \ge L_i/\epsilon, \quad i = 1, 2, \ldots, l, \tag{16}$$

where L_i is defined in (12). Then the coders/decoder–estimator pair $(\mathcal{F}, \mathcal{G})$ (14), (15) detects the state of system (1), (2) via a data-rate constrained sensor network and the set $\mathcal{X}_s[x_0, y_0(\cdot)\mid_0^s, d, \mathcal{F}, \mathcal{G}]$ is given by

$$\mathcal{X}_s[x_0, y_0(\cdot)\mid_0^s, d, \mathcal{F}, \mathcal{G}] = \{x_s \in \mathbb{R}^n : (x_s - \hat{x}(s))' P(s)^{-1} (x_s - \hat{x}(s)) \le d + \rho(s)\} \tag{17}$$

where $\rho(s) := \int_0^s \|K(t)\hat{x}(t)\|^2 \mathrm{d}t + N(\|\alpha\| + \epsilon\sqrt{\bar{m}})^2/r - \sum_{kT \le s} \|\bar{R}^{1/2}(C(kT)\hat{x}(kT) - \bar{y}_0(kT))\|^2$, $\alpha := [\alpha_1\ \alpha_2\ \ldots \alpha_l]'$ and $r := \min_{i \le \bar{m}}\{r_i\}$. The state $\hat{x}(\cdot)$ is defined by (15) with initial condition x_0, $\bar{y}_0(\cdot)$ is the sampled and quantized signal of the fixed measurement vector $y_0(\cdot)$.

Proof. By using Assumption 2.1 and the solution $S(\cdot)$ to the Riccati equation (6), Corollary 3.1 and Assumption 4.1 allow us to determine a bound L_i (12) for the each measurement $y_i(\cdot)$ for $i = 1, 2, \ldots, l$. Next, let $y(\cdot) = [y_1(\cdot)\ y_2(\cdot)\ \ldots\ y_l(\cdot)]'$ be a fixed measurement vector of the uncertain system (1), (3) taken by sensors $\Omega_1, \Omega_2, \ldots, \Omega_l$. At time $t = kT$, $k = 0, 1, 2, \ldots, N$, the decoder–estimator $\mathcal{G}$ receives the codeword vector $h(kT) = [h_1(\cdot)\ h_2(\cdot)\ \ldots\ h_l(\cdot)]'$ corresponding to the measurement vector $y(\cdot)$. Then the decoder–estimator $\mathcal{G}$ decodes $h(kT)$ into $\bar{y}(kT) = [\bar{y}_1(kT)\ \bar{y}_2(kT)\ \ldots\ \bar{y}_l(kT)]'$. The vector $\bar{y}(kT)$ is the sampled and quantized version of the measurement vector $y(t)$. Since the quantization parameter q_i satisfies (16) for all $i = 1, 2, \ldots, l$, we obtain $\|y(kT) - \bar{y}(kT)\|_\infty \le \epsilon$ by using (12) and (13). Therefore, the decoder–estimator observes

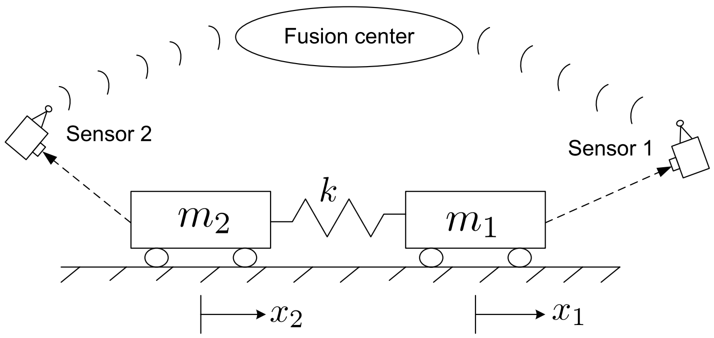

Fig. 2. Estimation of a two-mass-spring system via a data-rate constrained sensor network.

the continuous system (1) with discrete measurements $\bar{y}(\cdot)$ at $t = 0, T, 2T, \dots, NT$ where $\bar{y}(kT) = C(kT)x(kT) + \bar{v}(kT)$ and $\bar{v}(kT) := v(kT) + \bar{y}(kT) - y(kT)$. Moreover, the uncertainty input vector $[w(\cdot)\ \ \bar{v}(\cdot)]'$ satisfies the SIQC (8) with $\bar{Q}(\cdot) = Q(\cdot)$, $\bar{R} = \text{diag}\{r_1^{-1}, r_2^{-1}, \dots, r_{\bar{m}}^{-1}\}$ and $\bar{d} := d + N(\|\alpha\| + \epsilon\sqrt{\bar{m}})^2/r$. Also, the jump Riccati differential equation (9) has a solution $P(\cdot) = P(\cdot)' > 0$ with initial value $P(0) = Y_0^{-1}$ over the time interval $[0\ \ NT]$. Then using the state estimator (9), (10) and (15) for the decoder–estimator $\mathcal{G}$, the result of the theorem can be obtained from Theorem 3.2, and $\mathcal{X}_s[x_0, y_0(\cdot)\,|_0^s, d, \mathcal{F}, \mathcal{G}] := \bar{\mathcal{X}}_s[x_0, \bar{y}_0(\cdot)\,|_0^s, \bar{d}]$. □

Remark 4.1. The set $\mathcal{X}_s[x_0, y_0(\cdot)\,|_0^s, d, \mathcal{F}, \mathcal{G}]$ is an ellipsoid and the centroid of this ellipsoid $\hat{x}(s)$ can be used to provide a point-valued state estimate at time $t = s$. The set $\mathcal{X}_s$ captures all the possible state $x(s)$ of system (1). However, it may be conservative and hence the tightness of this set deserves further investigation.

When comparing Theorem 4.1 with the results given in [6], Theorem 4.1 is distinctly different, for our decoder not only provides $\hat{x}(\cdot)$ (i.e., a point estimate of $x(\cdot)$) at any time, but it also gives the upper and lower limits of the estimation error $(x(s) - \hat{x}(s))$. These limits are indeed determined from the ellipsoid $\mathcal{X}_s$ that is available at the decoder. On the other hand, the results given in [6] only guarantee that the error $\|\tilde{x}(kT) - \hat{x}(kT)\|$ is arbitrarily small where $\tilde{x}(\cdot)$ is some state estimate of $x(\cdot)$ obtained at the coder, and the *actual* state estimation error $\|x(kT) - \hat{x}(kT)\|$ is unknown at the decoder side. When the algorithm proposed by [6] is adopted, to find $\|x(kT) - \tilde{x}(kT)\|$, one needs to use the information of the output $y(\cdot)$, but it is unavailable to the decoder of [6]. Furthermore, the state estimate $\hat{x}(\cdot)$ obtained at the decoder of [6] is only a discrete-time sequence, namely $\hat{x}(kT)$, $k = 1, 2, \dots, NT$, that is in sharp contrast with our results presented in this paper.

5. Illustrative example

We consider a state estimation problem of an uncertain two-mass-spring system via a data-rate constrained sensor network. The system to be estimated is commonly used in a well-known benchmark example in robust control (see, e.g. [28]). It consists of two masses connected by a spring as shown in Fig. 2. The masses are assumed to be $m_1 = 1$ and $m_2 = 1$, whereas the spring constant k of the spring is uncertain. The spring constant k has a nominal value of $k_0 = 1.25$, but can vary up to 15% of its nominal value. Based on these parameters, a model of the dynamics of the mass-spring system can be obtained from [28] and is described by the equation

$$\dot{x}(t) = Ax(t) + Bw(t), \qquad z(t) = Kx(t) \tag{18}$$

where $x := [x_1\ x_2\ \dot{x}_1\ \dot{x}_2]' \in \mathbb{R}^4$,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1.25 & 1.25 & 0 & 0 \\ 1.25 & -1.25 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ -0.1875 \\ 0.1875 \end{bmatrix}, \qquad K = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix},$$

and the uncertainty input $w(t)$ is given by $w(t) = \Delta(t)z(t)$, and $\Delta(t)$ is an uncertain function satisfying $|\Delta(t)| \le 1$ for all $t \ge 0$.

In this example, we employ two sensors to estimate the state of the system. Sensors 1 and 2 measure the positions of Mass 1 and Mass 2. The encoded measurements from Sensors 1 and 2 are sent to the fusion center via data-rate constrained communication channels. A schematic of the state estimation of the uncertain mass-spring system via two sensors is shown in Fig. 2. We let $y_1(t)$ and $y_2(t)$ be the measurements taken by Sensors 1 and 2, respectively. The measurement equations are given by $y_1(t) = C_1x(t) + v_1(t) = [1\ 0\ 0\ 0]x(t) + v_1(t)$ and $y_2(t) = C_2x(t) + v_2(t) = [0\ 1\ 0\ 0]x(t) + v_2(t)$, where $v_1(t) = \alpha_1\Delta(t)$ and $v_2(t) = \alpha_2\Delta(t)$ are some measurement noises.

Using Sensors 1 and 2, we are interested in estimating the state of system (18) over the time interval $[0,\ \ 10]$ seconds with a sampling period of $T = 0.4$ s in the coders. Assuming that: the initial condition of system (18) is $x(0) = [1\ -1\ 1\ -0.5]'$;

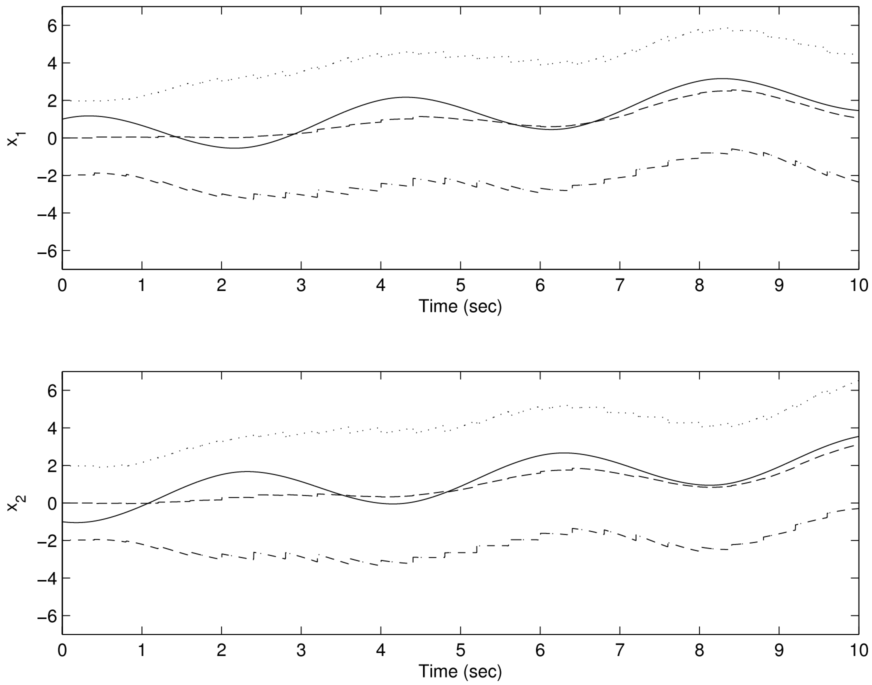

Fig. 3. Estimation of states x_1 (Top) and x_2 (Bottom). True value $x(t)$ (—), estimate $\hat{x}(t)$ (– –), upper bound (···), lower bound (·—).

the time-varying uncertain function $\Delta(t)$ is given by $\Delta(t) = \sin 2\pi t$; and the bounds of the measurement noises are $\alpha_1 = \alpha_2 = 0.3$. We choose the vector $x_0 = [0\ 0\ 0\ 0]'$ and the matrices Y_0, S_0, $\bar{R}$ and scalar Q as follows: $Y_0 = S_0 = \text{diag}\{25, 25, 25, 25\}$, $\bar{R} = \text{diag}\{1, 1\}$ and $Q = 1$. Then the parameters d and $\hat{d}$ can be chosen as $d = \hat{d} = 90$ so that both conditions (3) and (5) hold. Using the initial conditions $P(0) = Y_0^{-1}$, $S(0) = Y_0$, the solutions of both the Riccati differential equation (6) and the jump Riccati differential equation (9) are defined and positive definite over the time interval $[0,\ 10]$.

The solution $S(t)$ of the Riccati differential equation (6) allows us to estimate the bounds L_1 and L_2 of the measurements $y_1(\cdot)$ and $y_2(\cdot)$, respectively via Eqs. (11) and (12). The bounds are found to be $L_1 = L_2 = 19.1$. Therefore, given a quantization error bound as $\epsilon = 0.1$, we pick the quantization parameters q_1 and q_2 as $q_1 = q_2 = 386$ so that condition (16) holds. Finally, using the proposed coders $\mathcal{F}_1$ and $\mathcal{F}_2$ (14) for Sensors 1 and 2, and the decoder–estimator $\mathcal{G}$ (15) for the fusion center together with the above-mentioned design parameters, we obtain simulation results for the state estimation of system (18) and they are shown in Figs. 3 and 4. The upper and lower bounds of the state estimates can then be computed from (17) as we are dealing with set-valued state estimate rather that point estimation. We point out that such upper and lower bounds *cannot* be obtained at the decoder side by the algorithm proposed in [6].

This simulation example was implemented in MATLAB/Simulink package using the jump differential equations (9) and (10). These equations consist of the terms $\hat{x}(kT^-)$, $P(kT^-)$ that are the left-handed limits of these functions. For instance, the jump differential equation (10) for the state estimate $\hat{x}(\cdot)$ was simulated using the integrator block in Simulink. To capture the jump behavior, the initial value of the integrator was reset at discrete time instances, $T, 2T, 3T, \dots, NT$ with values $\hat{x}(T), \hat{x}(2T), \hat{x}(3T), \dots, \hat{x}(NT)$, respectively. These values were calculated using the second equation of (10). As for the left-handed limit $\hat{x}(kT^-)$, it is the integrator output right before the integrator was reset at kT. Following the same steps, the jump Riccati differential equation (9) of $P(\cdot)$ was also simulated in a similar manner, except that $P(\cdot)$ is a matrix whereas $\hat{x}(\cdot)$ is a vector. In this case, the integration was performed in an element-by-element manner, i.e., $[\dot{P}(\cdot)]_{i,j}$, $i, j = 1, 2, \dots, n$. Again the left-handed limit $P(kT^-)$ is the integrator output right before the integrator was reset at kT. Hence, the matrix $P^{-1}(kT^-)$, i.e, the inverse of $P(kT^-)$, can be evaluated since $P(\cdot) = P'(\cdot) > 0$.

To calculate the upper and lower bounds of the estimate, Eq. (17) was used, since the set $\mathcal{X}_s[x_0, y_0(\cdot)\,|_0^s, d, \mathcal{F}, \mathcal{G}]$ is an ellipsoid (with centroid at $\hat{x}(s)$) that captures all the possible state $x(s)$ of system (1). Using (17), at $s \in (0, NT]$, one can obtain the following inequality

$$|x_{s,i} - \hat{x}_i(s)| \le \sqrt{[P(s)]_{i,i}(d + \rho(s))} =: \mathcal{D}_i(s), \tag{19}$$

for $i = 1, 2, \dots, n$, where $x_{s,i}$ and $\hat{x}_i(s)$ are the ith-component of the n-dimensional vectors x_s and $\hat{x}(s)$, respectively, and $[P(kT)]_{i,i}$ denotes the (i, i) element of the matrix $P(kT)$. The vector x_s is the actual state of the system at time s, and the upper and lower bounds of the estimate at $s \in (0, NT]$ are

$$\hat{x}_i(s) - \mathcal{D}_i(s) \le x_{s,i} \le \hat{x}_i(s) + \mathcal{D}_i(s), \tag{20}$$

for $i = 1, 2, \dots, n$.

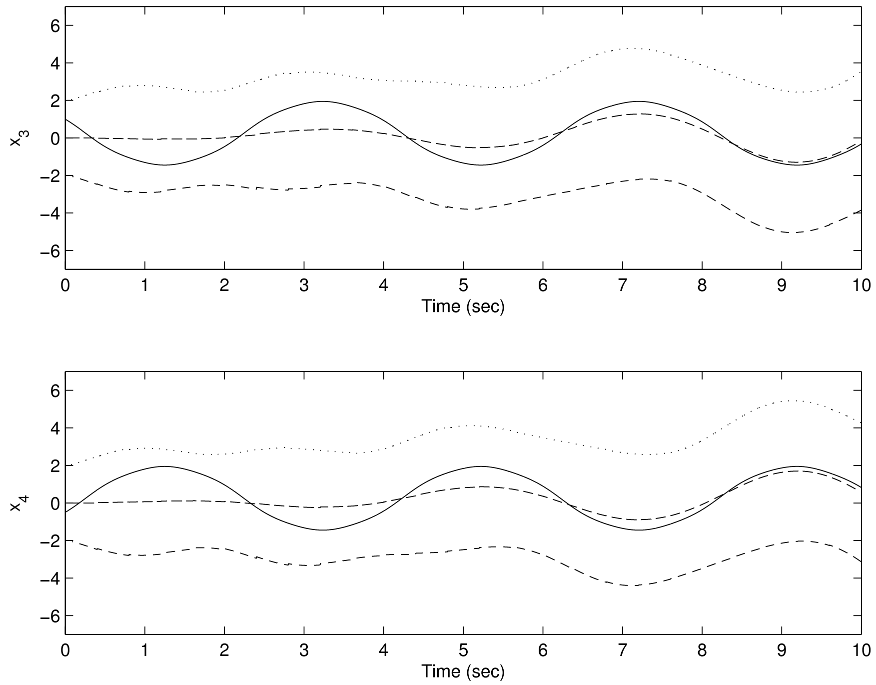

Fig. 4. Estimation of states x_3 (Top) and x_4 (Bottom). True value $x(t)$ (—), estimate $\hat{x}(t)$ (– –), upper bound (· · ·), lower bound (·—).

6. Conclusions

In this paper, we have studied a decentralized robust set-valued state estimation problem for uncertain systems via a data-rate constrained or limited capacity sensor network. The considered sensor network consists of spatially distributed low-power and computationally-limited sensors, and a fusion center where the robust set-valued state estimation is carried out. The communications from the sensors to the fusion center are through data-rate constrained communication channels. We have proposed a state estimation scheme which involves coders and a decoder–estimator, and they were developed based on the robust Kalman filtering techniques. The proposed scheme was developed in a deterministic setting and it only requires simple coders that can be implemented by low-power sensors. To compute the robust set-valued state estimate, the solution of a jump Riccati differential equation and the solution of a set of jump state equations are utilized in the decoder–estimator. In the future, the results of this paper will be extended to networks that have noisy communication channels such as data packet dropouts, and delay during the transmission.

Appendix

The proof provided here is for the completeness of the paper. *Necessity:* Let $s \in [0, NT]$ be given and consider the uncertain system (1) and (5) defines over $[0, s]$. By definition of $\mathcal{X}_s[\hat{d}]$, $x_s \in \mathcal{X}_s[\hat{d}]$ if and only if there exist vector functions $x(\cdot)$ and $w(\cdot)$ satisfying (1) such that $x(s) = x_s$ and the constraint (5) holds for all $t \in [0, s]$. It implies that $x_s \in \mathcal{X}_s[\hat{d}]$ if and only if there exists an input $w(\cdot) \in \mathbf{L}_2[0, s]$ such that $J[x_s, w(\cdot)] \leq \hat{d}$, where $J[x_s, w(\cdot)] := x(0)'S_0x(0) + \int_0^s w(t)'Q(t)w(t) - x(t)'K(t)'K(t)x(t)\mathrm{d}t \leq \hat{d}$, and $x(\cdot)$ is the solution to system (1) with input $w(\cdot)$ and boundary condition $x(s) = x_s$. The functional $J[\cdot]$ is a homogeneous quadratic functional with a terminal cost term, namely $x(0)'S_0x(0)$. Consider the set $\mathcal{X}_s[1]$ corresponding to $\hat{d} = 1$. Since $\mathcal{X}_s[1]$ is bounded, there exists a constant $h_s > 0$ such that all vectors $x_s \in \mathbb{R}^n$ with $\|x_s\| = h_s$ do not belong to the set $\mathcal{X}_s[1]$. Hence $J[x_s, w(\cdot)] > 1$ for all $x_s \in \mathbb{R}^n$ such that $\|x_s\| = h_s$ for all $w(\cdot) \in \mathbf{L}_2[0, s]$. Since J is a homogeneous quadratic functional, we have $J[ax_s, aw(\cdot)] = a^2J[x_s, w(\cdot)]$ and hence $\inf_{w(\cdot)\in\mathbf{L}[0,s]} J[x_s, w(\cdot)] > 0$ for all $s \in [0, NT]$ and $x_s \neq 0$. This optimization problem subject to the constraint defined by system (1) is a linear quadratic (LQ) optimal regulation problem in which time is reversed (i.e., from s to 0)and a sign indefinite quadratic cost function is being considered. By using results from LQ control theory, there exists a solution $S(\cdot)$ to the Riccati equation (6) for all $s \in [0, NT]$ with initial condition $S(0) = S_0$.

Sufficiency: For a given time interval $[0, s]$, we have already shown that $x_s \in \mathcal{X}_s[\hat{d}]$ if and only if there exists $w(\cdot) \in \mathbf{L}[0, s]$ such that $J[x_s, w(\cdot)] \leq \hat{d}$ is satisfied. Next, consider the optimization problem $\min_{w(\cdot)\in\mathbf{L}_2[0,s]} J[x_s, w(\cdot)]$, where the minimum is taken over all $x(\cdot)$ and $w(\cdot)$ connected by (1) with the boundary condition $x(s) = x_s$. This optimal control problem is the standard LQ control problem with a sign indefinite cost function. Using the results from the theory of LQ optimal control, we have $\min_{w(\cdot)\in\mathbf{L}_2[0,s]} J[x_s, w(\cdot)] = x_s'S(s)x_s$. Finally, we have that $x_s'S(s)x_s \leq \hat{d}$. This completes the proof of Theorem 3.1.

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