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Boolean-Rank-Preserving Operators and Boolean-Rank-1 Spaces

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ABSTRACT

We study the extent to which certain theorems on linear operators on field-valued matrices carry over to linear operators on Boolean matrices. We obtain analogues and near analogues of several such theorems. One of these leads us to consider linear spaces of $m \times n$ Boolean matrices whose nonzero members all have Boolean rank 1. We obtain a structure theorem for such spaces that enables us to determine the maximum Boolean dimension of such spaces and their maximum cardinality.

1. INTRODUCTION

Partly because of their association with nonnegative real matrices, Boolean matrices [(0, 1)-matrices with the usual arithmetic except $1+1=1$] have been the subject of research by many authors. Recently Kim [5] has published a compendium of results on the theory and applications of Boolean matrices.

Often, parallels are sought for results known for field-valued matrices or other rings. See e.g. de Caen and Gregory [3], Rao and Rao [11, 12], and Richman and Schneider [13].

We first study the extent to which known properties of linear operators preserving the ranks of field-valued matrices carry over to operators on Boolean matrices.

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Suppose T is a linear operator on $\mathcal{M}$, the $m \times n$ matrices over an algebraically closed field **F**. Say that T is a

(i) *(U, V)-operator* if there exist invertible matrices U and V such that $T(A) = UAV$ for all A in $\mathcal{M}$, or $m = n$ and $T(A) = UA^tV$ for all A in $\mathcal{M}$.

(ii) *rank preserver* if $\mathrm{rank}(T(A)) = \mathrm{rank}(A)$ for all A in $\mathcal{M}$.

(iii) *rank* 1 *preserver* if $\mathrm{rank}(T(A)) = 1$ whenever $\mathrm{rank}(A) = 1$ for all A in $\mathcal{M}$.

The fact that (i), (ii), and (iii) are equivalent was established in the work of Marcus and Moyls [7, 8], Westwick [15], and Lautemann [6].

We obtain nearly analogous results for "linear" operators on Boolean matrices (operators that fix 0 and preserve sums), employing a particular definition of Boolean rank. Section 2 contains all definitions and other preliminaries.

Boolean-rank-1 preservers are discussed in Section 3. Marcus and Moyls [8] and Westwick [15] have shown that

(1.1) *Over* **F**, *T preserves rank* 1 *if and only if T is a (U,V)-operator.*

We show that over **B**, although all (U,V)-operators are rank-1 preservers, the converse is false. In analogy with terminology introduced in [8], we call a family of Boolean matrices consisting of 0 and some Boolean-rank-1 matrices a "Boolean-rank-1 space" if it is closed under addition. Using a particular definition of Boolean dimension, we characterize Boolean (U,V)-operators in a partial analogue of (1.1).

THEOREM 3.1. *T is a Boolean (U,V)-operator if and only if T is an invertible rank-*1 *preserver if and only if T is a rank-*1 *and rank-*2 *preserver which preserves the dimension of every Boolean-rank-*1 *space.*

Section 4 concerns Boolean-rank preservers. Marcus and Moyls [7] and Lautemann [6] have shown that

(1.2) *Over* **F**, *T is a rank preserver if and only if T is a (U,V)-operator.*

We obtain an exact analogue for $m \times n$ Boolean matrix operators:

THEOREM 4.1. *If* $\min(m, n) > 1$, *then T is a Boolean-rank preserver if and only if T is a Boolean (U,V)-operator.*

It follows from (1.1) that

(1.3) *Over* **F**, *T is a rank preserver if T is a rank*-1 *preserver.*

We obtain a near analogue:

THEOREM 4.2. *T is a Boolean-rank preserver if T preserves Boolean ranks* 1 *and* 2.

We present a structure theorem (Theorem 5.1) for Boolean-rank-1 spaces in Section 5. This is used in Section 6 to determine the maximum cardinality $c(m,n)$ and maximum dimension $d(m,n)$ of Boolean-rank-1 spaces of $m \times n$ Boolean matrices:

THEOREM 6.1. *For every* $1 \leqslant m \leqslant n$, $c(m,n) = 2^n + 2^{m-1} - 1$ *and* $d(m,n) = d(1,n) + d(1,m-1)$.

The definition of Boolean dimension chosen for this paper, while providing most of the analogues sought, has some amusing consequences. For example, a k-dimensional (Boolean) space can have a subspace of dimension exceeding k; it is known that $d(1,k)$ is asymptotically equal to $2^k\sqrt{2/k\pi}$. In Section 2 we discuss this and other ways that consequences of such Boolean concepts differ from (or imitate) their field counterparts.

2. NOTATION, DEFINITIONS, AND OTHER PRELIMINARIES

We let $\mathscr{M}_{m,n}(\mathbf{B})$ denote the set of all $m \times n$ matrices with entries in $\mathbf{B} = \{0,1\}$, the two-element Boolean algebra. Arithmetic in $\mathbf{B}$ follows the usual rules except that $1+1=1$. The usual definitions for adding and multiplying matrices apply to Boolean matrices as well. Throughout this paper we shall adopt the convention that $m \leqslant n$, unless otherwise specified. Also lowercase, boldface letters will represent vectors, all vectors $\mathbf{u}$ are column vectors ($\mathbf{u}^t$ is a row vector), and $J_{m,n}$ denotes the matrix in $\mathscr{M}_{m,n}(\mathbf{B})$ all of whose entries are 1.

2.1. *Rank*

There are several notions of rank for Boolean matrices. We have found the following definition useful for our purposes. It appears in [5], where it is ascribed to B. M. Schein [14]. It also appears in graph-theoretic form in J. Orlin's paper [9], and in [4].

If A is a nonzero $m \times n$ Boolean matrix, its *Boolean rank*, $b(A)$, is the least integer k for which there exist $m \times k$ and $k \times n$ Boolean matrices B and C with $A = BC$. The Boolean rank of 0 is 0.

It is well known (see e.g. [9] or [5]) that $b(A)$ is the least k such that A is the sum of k matrices of Boolean rank 1.

Although Boolean rank enjoys many properties of the rank of field-valued matrices [e.g. $b(A)=b(A^t)$, $b(AB)\leqslant \min(b(A), b(B))$], there are others which it fails to enjoy. For example, even though $b(A)=r$, A may contain no $r\times r$ submatrix of Boolean rank r (see [4]).

2.2. *Singularity and Invertibility of Matrices*

We say that a Boolean matrix A is *singular* if $A\mathbf{x}=\mathbf{0}$ for some vector $\mathbf{x}\neq\mathbf{0}$ [$\mathbf{x}\in\mathcal{M}_{n,1}(\mathbf{B})$].

Note that having full Boolean rank [i.e. $b(A)=m$] is a sufficient, but not a necessary condition for nonsingularity when A is $m\times m$, and that the nonsingularity of a square matrix does not guarantee the nonsingularity of its transpose, A^t. For any A in $\mathcal{M}_{m,n}(\mathbf{B})$, A is nonsingular if and only if A has no zero column.

An $n\times n$ Boolean matrix A is said to be *invertible* if for some X, $AX=XA=I_n$, where I_n is the $n\times n$ identity matrix. This matrix X is necessarily unique when it exists. It is then denoted A^{-1}. It is well known that the permutation matrices are the only invertible Boolean matrices (see e.g. [13] or [3] or Lemma 2.5.1(d) below), and therefore $A^{-1}=A^t$ when A is invertible. The characterization of nonsingularity given above shows that nonsingularity does not imply invertibility. Of course, invertible matrices are nonsingular.

2.3. *Subspaces, Bases, and Dimension*

For our purposes, we can define a *Boolean vector space* to be any subset of $\mathbf{B}^m$ [$=\mathcal{M}_{m,1}(\mathbf{B})$] containing $\mathbf{0}$ which is closed under addition.

If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbf{B}^m$, we say $\mathbf{x}$ *absorbs* $\mathbf{y}$, written $\mathbf{x}\geqslant\mathbf{y}$, if $x_i=0$ only when $y_i=0$, for all $1\leqslant i\leqslant m$.

If $\mathbf{V},\mathbf{W}$ are vector spaces with $\mathbf{V}\subseteq\mathbf{W}$, then $\mathbf{V}$ is called a *subspace* of $\mathbf{W}$. We identify $\mathcal{M}_{m,n}(\mathbf{B})$ with $\mathbf{B}^{mn}$ in the usual way when we discuss it as a Boolean vector space and consider its subspaces.

Let $\mathbf{V}$ be a Boolean vector space. If S is a subset of $\mathbf{V}$, then $\langle S\rangle$ denotes the intersection of all subspaces of $\mathbf{V}$ containing S. This is a subspace of $\mathbf{V}$ too, called the subspace *generated by* S. If $S=\{\mathbf{s}_1,\mathbf{s}_2,\dots,\mathbf{s}_p\}$, then $\langle S\rangle=\{\sum_{i=1}^p x_i\mathbf{s}_i : x_i\in\mathbf{B}\}$, the set of *linear combinations of* S. Note that $\langle\varnothing\rangle=\{\mathbf{0}\}$. Define the *dimension* of $\mathbf{V}$, written $\dim(\mathbf{V})$, to be the minimum of the cardinalities of all subsets S of $\mathbf{V}$ generating $\mathbf{V}$. We call a generating set of cardinality equal to $\dim(\mathbf{V})$ a *basis* of $\mathbf{V}$. It is a curious fact that every Boolean vector space has only one basis. This was noted at least as far back as 1967 in [10], where it was stated without proof.

A subset of **V** is called *independent* if none of its members is a linear combination of the others. Evidently every basis is independent. The following proves the uniqueness of the basis and establishes the fact that every independent set is the basis for the space it generates.

LEMMA 2.3.1. *If S is an independent subset of the Boolean vector space* **V**, *then S is contained in every subset of* **V** *generating* $\langle S\rangle$.

Proof. We may assume that $S \neq \varnothing$. Suppose $\langle T\rangle = \langle S\rangle$ and $T \subseteq \mathbf{V}$. Let $\mathbf{c} \in S$; then $\mathbf{c}$ is a linear combination of members of T, each of which is a linear combination of members of S. But S is independent, so $\mathbf{c} \geqslant \mathbf{b} \geqslant \mathbf{c}$ for some $\mathbf{b} \in T$. Therefore $\mathbf{c} \in T$. ■

In contrast with vector spaces over fields, a Boolean vector space **V** may have several subspaces with the same dimension as **V**. For example, let $\mathbf{x} = [0,1,1]$, $\mathbf{v} = [1,1,0]$, and $\mathbf{z} = [1,1,1]$. Let $\mathbf{V} = \langle \mathbf{x},\mathbf{y}\rangle$; then $\langle \mathbf{x},\mathbf{z}\rangle$ and $\langle \mathbf{y},\mathbf{z}\rangle$ are two-dimensional subspaces of **V**, neither of which equals **V**.

Even more disconcerting, **V** can have subspaces whose dimensions *exceed* dim(**V**). For example, let **V** be the subspace of $\mathbf{B}^4$ generated by the set S of six vectors **x** having exactly two entries equal to 0. Then dim(**V**) = 6 because S is independent, even though **V** is subspace of a 4-dimensional space.

Just how big can the dimension of a subspace of $\mathbf{B}^n$ be? For all $n \geqslant 1$, let $\beta(n)$ denote the maximum dimension of all subspaces of $\mathbf{B}^n$. It is shown in [5, §1.4] that

$$\left(1+\frac{1}{n}\right)\binom{n}{[n/2]} \leqslant \beta(n) \leqslant \left(1+\frac{1}{\sqrt{n}}\right)\binom{n}{[n/2]} \qquad \text{for all} \quad n \geqslant 1. \tag{2.3.1}$$

Therefore

$$\frac{\beta(n)}{\binom{n}{[n/2]}} \to 1 \text{ as } n \to \infty.$$

Using Wallis's product representation of π (see e.g. [2, p. 225]), it can be shown that $\lim_{n\to\infty}\beta(n)/2^n\sqrt{2/n\pi} = 1$. That is, $\beta(n)$ is asymptotic to $2^n\sqrt{2/n\pi}$.

As with vector spaces over a field, the intersection of two subspaces **U**, **W** of a Boolean vector space is always a subspace, but their union seldom is.

However, if $\mathbf{W}$ *absorbs* $\mathbf{U}$—that is, $\mathbf{w} \geqslant \mathbf{u}$ for all nonzero $\mathbf{w}$ in $\mathbf{W}$ and all $\mathbf{u}$ in $\mathbf{U}$—then it's easy to verify that $\mathbf{U} \cup \mathbf{W}$ is a Boolean vector space.

In Section 6 we shall need the following lemma.

LEMMA 2.3.2. *If* $\mathbf{U}, \mathbf{W}$ *are subspaces of the same Boolean vector space,* $\mathbf{W} \geqslant \mathbf{U}$, *and* $\mathbf{U} \cap \mathbf{W} = \{0\}$, *then*

$$\dim(\mathbf{U} \cup \mathbf{W}) = \dim(\mathbf{U}) + \dim(\mathbf{W}).$$

Proof. Let $\mathscr{C}$, $\mathscr{D}$ be the bases of $\mathbf{U}$ and $\mathbf{W}$ respectively, and $\mathscr{B} = \mathscr{C} \cup \mathscr{D}$. Then $\mathscr{B}$ is independent and generates $\mathbf{U} \cup \mathbf{W}$, so $\mathscr{B}$ is a basis for $\mathbf{U} \cup \mathbf{W}$ by Lemma 2.3.1. ∎

2.4. Linear Transformations, Operators, and Matrix Representation

If $\mathbf{V}, \mathbf{W}$ are Boolean vector spaces, a mapping $T: \mathbf{V} \to \mathbf{W}$ which preserves sums and $\mathbf{0}$ is said to be a (Boolean) *linear transformation.* If $\mathbf{V} = \mathbf{W}$, the word *operator* is used instead of "transformation." Evidently, when T is linear its behavior on $\mathbf{V}$'s basis determines its behavior completely. As with transformations of vector spaces over fields, by ordering the bases of $\mathbf{V}$ and $\mathbf{W}$ we can represent T by an $m \times n$ matrix $[t_{ij}]$ in an analogous way. But the t_{ij} are not usually uniquely defined by Boolean T, so T may have several matrix representations for the same bases orderings.

A matrix $A \in \mathscr{M}_{m,n}(\mathbf{B})$ determines a linear transformation T_A of $\mathbf{B}^n$ into $\mathbf{B}^m$ by

$$T_A(\mathbf{x}) = A\mathbf{x} \qquad \text{for all} \quad \mathbf{x} \in \mathbf{B}^n. \tag{2.4.1}$$

The *image* of $\mathbf{V}$ in W, $T(\mathbf{V})$, is generated by the image, $T(\mathscr{B})$, of the basis $\mathscr{B}$ of $\mathbf{V}$. This proves:

LEMMA 2.4.1. *For every linear Boolean transformation* T,

$$\dim(T(\mathbf{V})) \leqslant \dim(\mathbf{V}).$$

LEMMA 2.4.2. *If the Boolean linear transformation* $T: \mathbf{V} \to \mathbf{W}$ *is injective, then* $\dim(T(\mathbf{V})) = \dim(\mathbf{V})$ *and* T *maps the basis of* $\mathbf{V}$ *onto the basis of* $T(\mathbf{V})$.

Proof. Follows from Lemma 2.4.1. ∎

2.5. *Invertibility of Transformations*

A transformation $T: \mathbf{V} \to \mathbf{W}$ is *invertible* if and only if T is injective and $T(\mathbf{V}) = \mathbf{W}$.

As with vector spaces over fields, the inverse, T^{-1}, of a Boolean linear transformation T is also linear. Here is another familiar-sounding proposition:

LEMMA 2.5.1. *If $T: \mathbf{V} \to \mathbf{W}$ is a surjective Boolean linear transformation, then T is invertible if and only if T preserves the dimension of every subspace of $\mathbf{V}$.*

Proof. If T is not injective, then, for some $\mathbf{x} \neq \mathbf{y}$, T reduces the dimension of $\langle \mathbf{x}, \mathbf{y} \rangle$. Conversely, if T is invertible, then the conclusion follows by Lemma 2.4.2. ∎

The finiteness of $|\mathbf{V}|$ and the previous lemma give us:

COROLLARY 2.5.1. *If T is a linear Boolean operator on $\mathbf{V}$, then the following statements are equivalent:*

(a) *T is invertible;*
(b) *T is injective;*
(c) *T is surjective;*
(d) *T permutes the basis of $\mathbf{V}$;*
(e) *T preserves the dimension of every subspace of $\mathbf{V}$.*

We note that T_A is invertible if and only if A is invertible, so that Corollary 2.5.1(d) is equivalent to the fact mentioned in Section 2.2, that the invertible Boolean matrices are the permutation matrices.

The main application of the ideas in Section 2.5 in this paper is to the linear operators on the space $\mathcal{M}_{m,n}(\mathbf{B})$ of all $m \times n$ Boolean matrices. Let $\Delta_{m,n} = \{(i,j): 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$, $E_{i,j}^{m,n}$ be the $m \times n$ matrix whose (i,j)th entry is 1 and whose other entries are all 0, and $\mathcal{E}_{m,n} = \{E_{i,j}^{m,n} : (i,j) \in \Delta_{m,n}\}$.

Fix m and n, and suppress them as sub- and superscripts. It follows directly from Corollary 2.5.1 that

COROLLARY 2.5.2. *The linear operator T on $\mathcal{M}(\mathbf{B})$ is invertible if and only if T permutes $\mathcal{E}$ if and only if T preserves the dimension of every subspace of $\mathcal{M}(\mathbf{B})$.*

We can describe any operator T on $\mathcal{M}(\mathbf{B})$ by expressing $(T(X))_{ij}$ as a scalar-valued function of X for all $(i,j) \in \Delta$. The operator T will be linear if and only if each component function t_{ij}: $X \to (T(X))_{ij}$ is a linear transformation of $\mathcal{M}(\mathbf{B})$ into $\mathbf{B}$.

Applying Corollary 2.5.2, we see that the operator T on $\mathcal{M}(\mathbf{B})$ is invertible if and only if there exists a permutation τ of Δ such that $T([x_{ij}])=[x_{\tau(i,j)}]$ for all X in $\mathcal{M}(\mathbf{B})$.

2.6 Boolean Rank-1 Matrices and Rank-1 Spaces

It is easy to verify that (just as with field-valued matrices) the Boolean rank of A is 1 if and only if there exist nonzero (Boolean) vectors $\mathbf{x}$ and $\mathbf{y}$ $[\mathbf{x}\in\mathcal{M}_{m,1}(\mathbf{B})$ and $\mathbf{y}\in\mathcal{M}_{n,1}(\mathbf{B})]$ such that $A=\mathbf{x}\mathbf{y}^t$. Unlike the corresponding situation for field-valued matrices, these vectors $\mathbf{x}$ and $\mathbf{y}$ are uniquely determined by A. Therefore there are exactly $(2^m-1)(2^n-1)$ rank-1 $m\times n$ Boolean matrices.

We use the notation $A\leqslant B$ to mean $b_{ij}=0$ only if $a_{ij}=0$. Equivalently, $A\leqslant B$ if and only if $A+B=B$.

For any vector $\mathbf{x}$, let $|\mathbf{x}|$ be the number of nonzero entries in $\mathbf{x}$, and when $A=\mathbf{a}\mathbf{b}^t$ is not zero, define the *perimeter of* A, $p(A)$, as $|\mathbf{a}|+|\mathbf{b}|$.

The following is a useful lemma whose straightforward proof we omit.

LEMMA 2.6.1. *If $A\leqslant B$ and $b(A)=b(B)=1$, then $p(A)<p(B)$ unless $A=B$.*

Analogously with [8, 15], we define a subspace of $\mathcal{M}_{m,n}(\mathbf{B})$ whose nonzero members have Boolean rank 1 as a *rank-1 space.*

If $A=\mathbf{a}\mathbf{x}^t$ is a rank-1 matrix, then $\mathbf{a}$ and $\mathbf{x}$ are uniquely determined by A. We call $\mathbf{a}$ the *left factor*, and $\mathbf{x}$ the *right factor* of A.

LEMMA 2.6.2. *If A, B, and $A+B$ are rank-1 matrices and neither $A\leqslant B$ nor $B\leqslant A$, then A, B, and $A+B$ have a common factor.*

Proof. Let $A=\mathbf{a}\mathbf{x}^t$, $B=\mathbf{b}\mathbf{y}^t$, and $C=A+B=\mathbf{c}\mathbf{z}^t$ be the "factorizations" of A, B, and C. We have for all i, j

$$a_i\mathbf{x}+b_i\mathbf{y}=c_i\mathbf{z} \quad\text{and}\quad x_j\mathbf{a}+y_j\mathbf{b}=z_j\mathbf{c}.$$

If $\mathbf{a}\nleqslant\mathbf{b}$ and $\mathbf{b}\nleqslant\mathbf{a}$, then for some i, j, $\mathbf{x}=c_i\mathbf{z}$ and $\mathbf{y}=c_j\mathbf{z}$. But $\mathbf{x}\neq\mathbf{0}$ and $\mathbf{y}\neq\mathbf{0}$, so $\mathbf{x}=\mathbf{y}=\mathbf{z}$. Thus A,B,C have a common right factor. If $\mathbf{a}\leqslant\mathbf{b}$, then $\mathbf{x}\nleqslant\mathbf{y}$ (as $A\nleqslant B$). Therefore $\mathbf{a}=\mathbf{b}$ and A, B, and C have a common left factor. A parallel argument holds if $\mathbf{b}\leqslant\mathbf{a}$. ∎

CONVENTION. Since we can write 0 as $\mathbf{0}\mathbf{x}^t$ or $\mathbf{a}\mathbf{0}^t$ for all $\mathbf{a}$ and $\mathbf{x}$, let us agree to say that 0 and A have a common left factor and a common right factor for any rank-1 matrix A.

3. RANK-1-PRESERVING OPERATORS

As was mentioned in Section 1, Marcus, Moyls [8], and Westwick [15] showed that if T is a linear operator on $\mathcal{M}_{m,n}(\mathbf{F})$ ($\mathbf{F}$ algebraically closed) and T maps rank-1 matrices to rank-1 matrices (i.e., T *preserves* rank-1 matrices) then (and only then) T is a (U,V)-operator. This result does not hold in the Boolean case.

The following example shows that not all rank-1-preserving operators T are of the form $T(X)=UXV$ for some nonsingular U,V^t, contrary to the situation for algebraically closed fields. Since invertible Boolean matrices are nonsingular, it also shows that not all rank-1-preserving operators T are (U,V)-operators.

EXAMPLE 3.1. Let

$$T\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right)=(b+e+c+f)\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}+\begin{bmatrix} a & 0 & d \\ 0 & 0 & 0 \end{bmatrix}.$$

Here, T is a linear operator and $b(T(X))=1$ whenever $b(X)=1$ (in fact whenever $X\neq 0$). If there existed nonsingular U and V^t such that $T(X)=UXV$ for all $X\in\mathcal{M}_{23}(\mathbf{B})$, then for $j=1,2,3$, we have $T(E_{1j})=\mathbf{u}\mathbf{v}_j^t$ where $\mathbf{u}$ is the first column of U and $\mathbf{v}_j$ is the jth column of V. But

$$T(E_{11})=\begin{bmatrix} 1 \\ 0 \end{bmatrix}[1,0,0] \quad\text{and}\quad T(E_{12})=\begin{bmatrix} 1 \\ 1 \end{bmatrix}[1,1,1],$$

and hence

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\mathbf{u}=\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

a contradiction.

Suppose U and V^t are nonsingular members of $\mathcal{M}_{m,m}(\mathbf{B})$ and $\mathcal{M}_{n,n}(\mathbf{B})$ respectively, and T is the operator on $\mathcal{M}_{m,n}(\mathbf{B})$ defined by $T(X)=UXV$ for all X. Clearly T is linear. Moreover $T(X)$ has rank 1 whenever X has rank 1: Suppose X has rank 1, so that $X=\mathbf{a}\mathbf{b}^t$ where $\mathbf{a}\neq\mathbf{0}$, $\mathbf{b}\neq\mathbf{0}$. Then, $T(X)=U\mathbf{a}\mathbf{b}^tV=(U\mathbf{a})(V^t\mathbf{b})^t$, and since U and V^t are nonsingular, neither $U\mathbf{a}$ or $V^t\mathbf{b}$ is $\mathbf{0}$, so $T(X)$ has rank 1. It follows that all Boolean (U,V)-operators are rank-1 preservers.

EXAMPLE 3.2. Suppose C is a fixed rank-1 member of $\mathscr{M}_{m,n}(\mathbf{B})$, and T is the operator defined by $T(X)=C$ if $X \neq 0$ and $T(0)=0$.

Example 3.2 shows that for each k ($1 \leqslant k \leqslant n$) there is a linear operator T_k that preserves the Boolean rank of every rank-k $m \times n$ matrix but is not a (U,V)-operator when $k > 1$. [Just take C to be a fixed rank-k matrix; recall that (U,V)-operators preserve rank 1.]

Beasley [1] showed that for most $k \leqslant n$, each operator on field-valued matrices preserves the rank of rank-k matrices if and only if it is a (U,V)-operator.

We were unable to find a condition necessary and sufficient for a Boolean operator to preserve the rank of all rank-1 matrices, so the Boolean analogue of the work of Marcus, Moyls, and Westwick mentioned in Section 1, characterizing the rank-1 preservers, remains to be discovered. We have, however, found two conditions, one necessary (but not sufficient), and the other sufficient (but not necessary), which are of some help in constructing examples. These are described in the next few paragraphs.

Suppose T is a linear operator on $\mathscr{M}_{m,n}(\mathbf{B})$. Let $\mathscr{R}_i = \{T(E_{ik}): 1 \leqslant k \leqslant n\}$ and $\mathscr{C}_j = \{T(E_{kj}): 1 \leqslant k \leqslant m\}$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.

LEMMA 3.1. *T preserves the rank of all rank-1 matrices only if there exist rank-1 spaces $\mathbf{R}_i$ and $\mathbf{C}_j$ such that $\mathscr{R}_i \subseteq \mathbf{R}_i$ and $\mathscr{C}_j \subseteq \mathbf{C}_j$ for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.*

Proof. Suppose T preserves the rank of all $m \times n$ Boolean rank-1 matrices. Then as $\{E_{ij}: 1 \leqslant j \leqslant n\}$ is in the rank-1 space $\mathbf{V}_i$ of all $A \in \mathscr{M}_{m,n}(\mathbf{B})$ whose nonzero entries all lie in its ith row, it follows that $\mathscr{R}_i \subseteq T(\mathbf{V}_i)$. But $T(\mathbf{V}_i)$ is also a rank-1 space. Therefore, $T(\mathbf{V}_i)$ will serve for $\mathbf{R}_i$ of the conclusion. Similarly for $\mathbf{C}_j$. ■

Although useful for constructing examples, Lemma 3.1's condition is not sufficient:

EXAMPLE 3.3. Let

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b+c & d \\ a & 0 \end{bmatrix};$$

then T is a linear transformation on $\mathscr{M}_{22}(\mathbf{B})$, and $\mathscr{C}_1 = \mathscr{R}_1 \subseteq \{\mathbf{a}[1,0]: \mathbf{a} \in \mathbf{B}^2\}$, which is a rank-1 space. Also $\mathscr{C}_2 = \mathscr{R}_2 \subseteq \{[1,0]^t \mathbf{b}^t: \mathbf{b} \in \mathbf{B}^2\}$, which is another rank-1 space. But

$$T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

LEMMA 3.2. *T preserves the rank of all rank-1 matrices if there is a rank-1 space* $\mathbf{V}$ *such that*

(i) $\bigcup_{i=1}^{m}\mathscr{R}_i \subseteq \mathbf{V}$ *or*
(ii) $\bigcup_{j=1}^{n}\mathscr{C}_j \subseteq \mathbf{V}$.

Proof. (i): If $b(X)=1$, then $T(X)=\sum_{i=1}^{m}\sum_{j=1}^{n}x_i y_j T(E_{ij}) = \sum_{i=1}^{m}x_i[\sum_{j=1}^{n}y_j T(E_{ij})] = \sum_{i=1}^{m}x_i M_i$, where M_i is in $\mathscr{R}_i$. Therefore $T(X)$ is a sum of members of $\mathbf{V}$ and hence has rank 1. The proof of (ii) is similar. ■

The identity operator I on $\mathscr{M}_{n,n}(\mathbf{B})$ provides an example of a rank-1-preserving operator for which neither (i) or (ii) of Lemma 3.2 holds. Thus those conditions are sufficient, but not necessary.

If we add the hypothesis that T preserves the dimension of any rank-1 space (unlike matrices over fields, for which this is always true of rank-1 preservers), then the conclusion is much more restrictive, as we shall see in Theorem 3.1.

LEMMA 3.3. *If T is a linear operator on $\mathscr{M}_{m,n}(\mathbf{B})$ that preserves the dimension of all rank-1 spaces, then the restriction of T to the rank-1 matrices is injective or T reduces the rank of some rank-2 matrix to 1.*

Proof. Let $\mathscr{M}^1 = \{A\in\mathscr{M}_{m,n}(\mathbf{B}) : b(A)=1\}$ and $\mathbf{W} \equiv \{0\}\cup\{X\in \mathscr{M}^1 : T(X)=T(B)\}$ for each $B\in\mathscr{M}^1$. If $\mathbf{W}$ is a rank-1 space then $\dim(\mathbf{W}) = \dim(T(\mathbf{W}))=1$, so $\mathbf{W}=\langle B\rangle$. Thus T is injective. Otherwise there are X, Y in $\mathbf{W}$ such that $b(X+Y)=2$. ■

COROLLARY 3.3. *If T is a linear operator on $\mathscr{M}_{m,n}(\mathbf{B})$ that*

(i) *preserves the ranks of all rank-1 and rank-2 matrices and*
(ii) *preserves the dimension of all rank-1 spaces,*

then

(a) *T is invertible and*
(b) *T^{-1} satisfies* (i) *and* (ii).

Proof. Part (a): Let $\mathscr{E}$ be the basis of $\mathscr{M}_{m,n}(\mathbf{B})$, as in Section 2.5. According to Corollary 2.5.1(d), T is invertible if it permutes $\mathscr{E}$. Lemma 3.3 implies that T permutes $\mathscr{M}^1$. But $\mathscr{M}^1 \supseteq \mathscr{E}$, so it suffices to show that $T(\mathscr{E})\subseteq\mathscr{E}$. Let $E\in\mathscr{E}$; then $E=T(C)$ for some $C\in\mathscr{M}^1$. Since $C\neq 0$, we have $C\geqslant F$ for some F in the basis $\mathscr{E}$. Therefore $E\geqslant T(F)$. Then $E=T(F)$ by Lemma 2.6.1, completing the proof of part (a). Part (b) follows directly. ■

The idea of a permutation of $\Delta \equiv \{(i, j): 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$ representing an invertible operator was introduced in Section 2.5.

LEMMA 3.4. *If T is an invertible linear operator on $\mathscr{M}_{m,n}(\mathbf{B})$ that preserves the rank of every rank-1 matrix and τ is the permutation of Δ representing T, then there exist permutations α, β of $\{1,2,\dots,m\}$ and $\{1,2,\dots,n\}$ respectively such that*

(a) *$\tau(i, j) = (\alpha(i), \beta(j))$ for all $(i, j) \in \Delta$ or*
(b) *$m = n$ and $\tau(i, j) = (\beta(j), \alpha(i))$ for all $(i, j) \in \Delta$.*

Proof. We'll denote the abscissa of $\tau(i, j)$ by u_{ij} and its ordinate by v_{ij}. So $\tau(i, j) = (u_{ij}, v_{ij})$. Let $[\tau]$ be the $m \times n$ array whose (i, j)th entry is (u_{ij}, v_{ij}).

Any two entries in the same row (or column) of $[\tau]$ have a common abscissa or ordinate. This is because $T(\langle E_{ij}, E_{ik}\rangle)$ and $T(\langle E_{ji}, E_{ki}\rangle)$ are rank-1 spaces. It follows that if $u_{i1} = u_{i2}$ (respectively $v_{i1} = v_{i2}$), then u_{i1} (respectively v_{i1}) is the abscissa (respectively ordinate) of each entry in the ith row of $[\tau]$. Let $\beta_i(j) = v_{ij}$ (respectively u_{ij}). Then for all i, β_i permutes $\{1,2,\dots,n\}$. If x were a common abscissa for one row and y a common ordinate for another, then (x, y) would belong to both rows (because $m \leqslant n$ and each β_i is a permutation), contradicting the injectivity of τ. Therefore either

(1) for all $(i, j) \in \Delta$, $u_{ij} = u_{i1}$, or
(2) for all $(i, j) \in \Delta$, $v_{ij} = v_{i1}$.

Suppose (1) holds. Define $\alpha(i) = u_{i1}$ for all i, $1 \leqslant i \leqslant m$. For some j, $v_{ij} = u_{i1}$ because β_i is a permutation. It follows that (u_{i1}, u_{i1}) occurs in the ith row of $[\tau]$ and in no other. Thus α permutes $\{1,2,\dots,m\}$. If $i \neq 1$, then $T(\langle E_{1j}, E_{ij}\rangle) = \langle E_{ux}, E_{vy}\rangle$ is a rank-1 space with $u = \alpha(1)$, $v = \alpha(i)$, $x = \beta_1(j)$, and $y = \beta_i(j)$. But $\alpha(1) \neq \alpha(i)$, so $x = y$. Therefore $\beta_i = \beta_1$ for all $i \leqslant m$. Let $\beta = \beta_1$; then $\tau(i, j) = (\alpha(i), \beta(j))$ for all $(i, j) \in \Delta$. If (2) holds, then $m = n$. Let $\tau'(i, j) = (v_{ij}, u_{ij})$ for all $(i, j) \in \Delta$, and apply (1) to τ' to complete the proof of the lemma. ■

LEMMA 3.5. *If τ satisfies the conclusion of Lemma 3.4, then T is a (U, V)-operator.*

Proof. Let π be any permutation of $\{1,2,\dots,k\}$. Let $E_{i,j}^{m,n}$ denote an $m \times n$ matrix of the form E_{ij} defined in Section 2. Let $P_k(\pi) = \sum_{l=1}^{k} E_{l,\pi(l)}^{k,k}$. Then $P_k(\pi)$ is a permutation matrix. But $E_{i,j}^{m,n} E_{u,v}^{n,r} = \delta_{u,j} E_{i,v}^{m,r}$ (where $\delta_{u,j}$ is the Kronecker delta). Thus $E_{i,j}^{m,n}P_n(\pi) = E_{i,\pi(j)}^{m,n}$, and hence $P_m(\alpha^{-1})E_{i,j}^{m,n}P_n(\beta) = E_{\alpha(i),\beta(j)}^{m,n}$.

If (a) holds in Lemma 3.4, then we define $U = P_m(\alpha^{-1})$ and $V = P_n(\beta)$. If A is any $m \times n$ Boolean matrix, we have $A = \Sigma\{E_{i,j}: a_{ij} = 1\}$ and hence $T(A) = \Sigma\{E_{\tau(i,j)}: a_{ij} = 1\} = \Sigma\{UE_{ij}V: a_{ij} = 1\} = UAV$. If (b) holds in Lemma 3.4, define $U = P_n(\beta^{-1})$ and $V = P_n(\alpha)$. Let T' be the operator on $\mathcal{M}_{m,m}(\mathbf{B})$ defined by $T'(A) = [T(A)]^t$ for all A in $\mathcal{M}_{m,m}(\mathbf{B})$. Then $T'(E_{ij}) = E_{\alpha(i),\beta(j)}$, so $T'(A) = P_n(\alpha^{-1})AP_n(\beta)$ by the result for conclusion (a). Hence $T(A) = UA^tV$. ■

THEOREM 3.1. *If T is a linear operator on $\mathcal{M}_{m,n}(\mathbf{B})$, then the following statements are equivalent*:

(a) *T is invertible and preserves the rank of all rank-1 matrices.*

(b) *T preserves the ranks of all rank-1 matrices and rank-2 matrices and preserves the dimension of all rank-1 spaces.*

(c) *T is a (U, V)-operator.*

Proof. Lemmas 3.4 and 3.5 show that (a) implies (c). Statement (b) implies (a) by Corollary 3.3. So it suffices to show that (c) implies (b). Any operator T that satisfies (c) is invertible; in fact $T^{-1}(A) = U^{-1}AV^{-1}$ or $T^{-1}(A) = U^{-1}A^tV^{-1}$. Such operators are clearly rank-1 preservers. The rest is implied by Lemma 2.4.2. ■

4. BOOLEAN RANK PRESERVERS

The following is true for operators T on $m \times n$ matrices over an algebraically closed field $\mathbf{F}$. It characterizes the *rank-preserving* operators (see Marcus and Moyls [7] or Lautemann [6]).

(4.1) *The rank of $T(A)$ equals the rank of A for all A if and only if T is a (U, V)-operator.*

In this case, the characterization carries over completely to Boolean operators (Theorem 4.1). The theorem quoted at the outset of Section 3 implies that:

(4.2) *Over* $\mathbf{F}$, *T is rank-preserving if and only if T preserves the rank of each rank-1 matrix.*

Theorem 4.2 below gives a nearly exact analogue.

LEMMA 4.1. *If A, B are in $\mathcal{M}_{m,n}(\mathbf{B})$, $A \neq B$, $p(A) \geqslant p(B)$, $m > 1$, and $b(A) = b(B) = 1$, then there exists C in $\mathcal{M}_{m,n}(\mathbf{B})$ such that $b(A + C) = 1$ and $b(B + C) = 2$.*

Proof. If $b(A + B) = 2$, then the conclusion is obtained by letting $C = A$. So we may assume that $b(A + B) = 1$. Define E_{pq} as in Section 2.5.

Factoring A, B, and E_{pq}, we have $A = \mathbf{a}\mathbf{x}^t$, $B = \mathbf{b}\mathbf{y}^t$, and $E_{pq} = \mathbf{e}_p\mathbf{f}_q^t$. By our hypotheses and Lemma 2.6.1, $A \nleqslant B$. Therefore, Lemma 2.6.2 implies that (i) $\mathbf{a} = \mathbf{b}$ and $\mathbf{x} \neq \mathbf{y}$, or (ii) $\mathbf{x} = \mathbf{y}$ and $\mathbf{a} \neq \mathbf{b}$, or (iii) $\mathbf{b} \leqslant \mathbf{a}$, $\mathbf{b} \neq \mathbf{a}$, $\mathbf{y} \leqslant \mathbf{x}$, and $\mathbf{y} \neq \mathbf{x}$.

In any case, there exist k, l such that $b_k = y_l = 1$, because $B \neq 0$.

Case (i): We have $\mathbf{x} \nleqslant \mathbf{y}$ because $\mathbf{a} = \mathbf{b}$ and $A \nleqslant B$. So we can select $j \leqslant n$ so that $x_j = 1$ and $y_j = 0$. Since $m > 1$, we can choose $i \leqslant m$ so that $i \neq k$. Now $b(E_{ij} + E_{kl}) = 2$, because $k \neq i$ and $l \neq j$. Let $C = (\mathbf{a} + \mathbf{e}_i)\mathbf{x}^t$. Then $B + C \geqslant E_{ij} + E_{kl}$. Thus $b(B + C) = 2$. On the other hand, $A \leqslant C$, so $b(A + C) = b(C) = 1$. The other cases are treated similarly. ∎

LEMMA 4.2. *If T is a linear operator on $\mathcal{M}_{m,n}(\mathbf{B})$ with $m > 1$, and T is not invertible but preserves the rank of rank-1 matrices, then T decreases the rank of some rank-2 matrix.*

Proof. By the proof of Corollary 3.3, T is not injective on $\mathcal{M}^1$ so $T(X) = T(Y)$ for some X, Y in $\mathcal{M}^1$ with $X \neq Y$. Without loss of generality we may suppose that $p(X) \geqslant p(Y)$. By Lemma 4.1, there is some matrix D such that $b(X + D) = 2$ while $b(Y + D) = 1$. However, $T(X + D) = T(X) + T(D) = T(Y + D)$. ∎

THEOREM 4.1. *Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbf{B})$ with $m > 1$. Then T is a rank preserver if and only if T is a (U, V)-operator.*

Proof. Theorem 3.1 and Lemma 4.2 prove the necessity of the condition given for rank preservation. To prove the sufficiency, we invoke the characterization of $b(A)$ quoted in Section 2.1: $b(A)$ is the least integer k for which k rank-1 matrices whose sum is A exist. Therefore $b(L(A)) \leqslant b(A)$ whenever L is a linear rank-1 preserver. Now each (U, V)-operator and its inverse are rank-1 preservers, so such operators preserve all ranks. ∎

THEOREM 4.2. *Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbf{B})$. Then T is a rank preserver if (and only if) T preserves the ranks of all rank-1 and rank-2 matrices.*

Proof. We may assume $m > 1$. If T preserves ranks 1 and 2, then T is invertible (by Lemma 4.2) and hence a rank preserver by Theorems 3.1 and 3.4. ∎

Thinking of a Boolean matrix M as a zero-one matrix over a field, the (field) rank of M is 1 if and only if $b(M) = 1$, and the field rank of M is 2 only if $b(M) = 2$. This proves the following.

COROLLARY 4.1. *Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbf{B})$. Then T preserves the Boolean rank of all matrices if it preserves the field rank of rank-1 and field rank-2 zero-one $m \times n$ matrices.*

5. THE STRUCTURE OF BOOLEAN-RANK-1 SPACES

In analogy with Marcus and Moyls [8] and Westwick [15], define a subspace of $\mathcal{M}_{m,n}(\mathbf{B})$ whose nonzero members have Boolean rank-1 as a (Boolean-) *rank-*1 *space.*

It can be shown that there are only two kinds of rank-1 spaces in $\mathcal{M}_{m,n}(\mathbf{F})$. They are of the form $\{\mathbf{a}\mathbf{x}^t : \mathbf{x} \in \mathbf{S}\}$ for some subspace $\mathbf{S}$ of $\mathbf{F}^n$ or of the form $\{\mathbf{y}\mathbf{b}^t : y \in \mathbf{T}\}$ for some subspace $\mathbf{T}$ of $\mathbf{F}^m$. We call the former "left factor spaces" and the latter "right factor spaces." Therefore, rank-1 spaces of matrices over a field are just the factor spaces.

The structure of Boolean rank-1 matrices turns out to be only slightly more complicated. We show that every rank-1 space $\mathbf{V}$ contains a unique set of $d \geqslant 1$ factor spaces $\{\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_d\}$ whose union is $\mathbf{V}$, whose pairwise intersections are $\{0\}$, and which are sequentially absorbed, each by the next: $\mathbf{W}_1 \leqslant \mathbf{W}_2 \leqslant \cdots \leqslant \mathbf{W}_d$. (Recall that $\mathbf{U} \leqslant \mathbf{W}$ means that $\mathbf{u} \leqslant \mathbf{w}$ for all $\mathbf{u}$ in $\mathbf{U}$ and all nonzero $\mathbf{w}$ in $\mathbf{W}$.)

Suppose $\mathbf{V}$ is a Boolean rank-1 space and $A = \mathbf{a}\mathbf{x}^t$ is any nonzero member of $\mathbf{V}$. Let $\mathbf{L}(A)$ [respectively, $\mathbf{R}(A)$] be the set of members of $\mathbf{V}$ having the same left factor $\mathbf{a}$ [respectively, right factor $\mathbf{x}$] as A. (*Factor* was defined in Section 2.6.) Now, $\mathbf{L}(A)$ and $\mathbf{R}(A)$ are subspaces of $\mathbf{V}$ [$0 \in \mathbf{R}(A) \cap \mathbf{L}(A)$ by the convention mentioned in Section 2.6].

We call $\mathbf{L}(A)$ and $\mathbf{R}(A)$ the *left* and *right factor spaces* of A. Evidently $\mathbf{L}(A) \cap \mathbf{R}(A) = \{0, A\} = \langle A \rangle$.

LEMMA 5.1. *If A is in a rank-*1 *space* $\mathbf{V}$, *then $A \geqslant B$ for all $B \in \mathbf{L}(A)$ or $A \geqslant C$ for all $C \in \mathbf{R}(A)$.*

Proof. If the contrary holds, then it can be shown that $b(B+C)>1$ for some $B \in \mathbf{L}(A)$ and $C \in \mathbf{R}(A)$. ∎

Applying Lemma 2.6.1, we can obtain the following corollary.

COROLLARY 5.1. *If A has the least perimeter of all X in $\mathbf{L}(A) \cup \mathbf{R}(A)$, then $\mathbf{L}(A)=\langle A\rangle$ or $\mathbf{R}(A)=\langle A\rangle$.*

COROLLARY 5.2. *If A has the least perimeter of all X in $\mathbf{L}(A) \cup \mathbf{R}(A)$ then $\mathbf{L}(A) \cup \mathbf{R}(A)=\mathbf{L}(A)$ or $\mathbf{L}(A) \cup \mathbf{R}(A)=\mathbf{R}(A)$.* ∎

Proof. Immediate from Corollary 5.1. ∎

If $B \geqslant A$ but A and B have no common factors, we write $B>A$ (or $A<B$). In other words, $\mathbf{b}\mathbf{w}^t > \mathbf{a}\mathbf{x}^t$ if and only if $\mathbf{b} \geqslant \mathbf{a}$, $\mathbf{w} \geqslant \mathbf{x}$, $\mathbf{b} \neq \mathbf{a}$, and $\mathbf{w} \neq \mathbf{x}$.

LEMMA 5.2. *If A and B are in a rank-1 space $\mathbf{V}$ and $B>A$, then $B \geqslant X$ for all X in $\mathbf{L}(A) \cup \mathbf{R}(A)$.*

Proof. Suppose $A=\mathbf{a}\mathbf{x}^t$, $B=\mathbf{b}\mathbf{w}^t$. Suppose $X=\mathbf{a}\mathbf{y}^t$ is in $\mathbf{L}(A)$. Without loss of generality we may assume $X \notin \langle A\rangle$ and $\mathbf{y} \nleqslant \mathbf{x}$. Choose j so that $y_j=1$ and $x_j=0$. For some i, $b_i=1$ and $a_i=0$ because $B>A$. For some p and q, $a_p=1$ and $x_q=1$. But $\mathbf{x} \leqslant \mathbf{w}$, so that $w_q=1$. Let $D=B+X$; then $d_{pj}=1=d_{iq}$ and $d_{ij}=w_j$. But $b(D)=1$, $p \neq i$, and $q \neq j$. Therefore $d_{pq}=d_{ij}=1$, so $w_j=1$. Thus $w_j=1$ whenever $y_j=1$ and $x_j=0$. Hence $\mathbf{w} \geqslant \mathbf{y}$. Therefore, $B \geqslant X$. If X were in $\mathbf{R}(A)$, a symmetric argument could have been used. ∎

THEOREM 5.1. *If $\mathbf{V}$ is a rank-1 space, then there exist nonzero matrices $A_1, A_2, \ldots, A_d$ in $\mathbf{V}$ and factor spaces $\mathbf{W}_i$ of the A_i such that*

(a) $\mathbf{V}=\bigcup_{1=1}^{d} \mathbf{W}_i$,

and for all $1 \leqslant i<j \leqslant d$:

(b) $\mathbf{W}_i \cap \mathbf{W}_j=\{0\}$,
(c) $\mathbf{W}_i \leqslant \mathbf{W}_j$,
(d) $p(A_i) \leqslant p(X)$ *for all nonzero X in $\mathbf{W}_i$, and*
(e) $p(X)<p(Y)$ *for all X in $\mathbf{W}_i$ and all nonzero Y in $\mathbf{W}_j$.*

Proof. (By induction on $|\mathbf{V}|$.) If $|\mathbf{V}| = 2$, then $\mathbf{V} = \{0, A\}$ and the proposition is true. Suppose it is true for all rank-1 spaces of cardinality at most c and that $\mathbf{V}$ is a rank-1 space with $|\mathbf{V}| = c + 1 \geqslant 3$. Let A_1 be a member of $\mathbf{V}$ of least perimeter and $\mathbf{W}_1 = \mathbf{L}(A_1) \cup \mathbf{R}(A_1)$. By Corollary 5.2, $\mathbf{W}_1 = \mathbf{L}(A_1)$ or $\mathbf{W}_1 = \mathbf{R}(A_1)$. Possibly $\mathbf{W}_1 = \langle A_1 \rangle$. Let $\mathbf{U} = \mathbf{V} \backslash (\mathbf{W}_1 \backslash 0)$. If $Y \in \mathbf{U} \backslash 0$, then Y has no common factor with A_1. Also $p(Y) \geqslant p(A_1)$ by the choice of A_1. Therefore, $Y \nleqslant A_1$ by Lemma 2.6.1. Thus $Y \geqslant A_1$ by Lemma 2.6.2, so $Y > A_1$. Consequently, by Lemma 5.2,

$$\mathbf{W}_1 \leqslant \mathbf{U}. \tag{5.1}$$

So if $M = \sum\{A : A \in \mathbf{W}_1\}$, then $Y \geqslant M$ and $Y \neq M$. Therefore, if $Y_1, Y_2 \in \mathbf{U} \backslash 0$, their sum $S \geqslant M$ and $S \neq M$. Thus $S \nleqslant M$ and hence $S \notin \mathbf{W}_1$. This shows that $\mathbf{U}$ is a rank-1 subspace of $\mathbf{V}$ and $|\mathbf{U}| \leqslant |\mathbf{V}| - 1 \leqslant c$. By the induction hypothesis, there exist matrices $A_2, A_3, \ldots, A_d$ with factor spaces $\mathbf{W}_2, \mathbf{W}_3, \ldots, \mathbf{W}_d$ respectively, all in $\mathbf{U}$, satisfying conclusions (a) through (e). Now we verify these conclusions for $A_1, A_2, \ldots, A_d$ and $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_d$.

The definition of $\mathbf{U}$ implies $\mathbf{V} = \mathbf{W}_1 \cup \mathbf{U}$ and $\mathbf{W}_1 \cap \mathbf{U} = \{0\}$. By the induction hypothesis, $\mathbf{U} = \cup_{i=2}^{d} \mathbf{W}_i$ and $\mathbf{W}_i \cap \mathbf{W}_j = \{0\}$ for $2 \leqslant i < j \leqslant d$. Therefore, $\mathbf{V} = \cup_{i=1}^{d} \mathbf{W}_i$ and $\mathbf{W}_i \cap \mathbf{W}_j = \{0\}$ for all $1 \leqslant i < j \leqslant d$. [(a) and (b) are established.]

Suppose $X \in \mathbf{W}_i$ and $Y \in \mathbf{W}_j \backslash 0$ for some $j > i$. We have $Y \in \mathbf{U} \backslash 0$ and hence $X \leqslant Y$ by (5.1) if $i = 1$. If $i > 1$ then $X \leqslant Y$ by the induction hypothesis [(c) is established].

We have $p(A_i) \leqslant p(X)$ for all nonzero $X \in \mathbf{W}_i$ by the definition of A_1 if $i = 1$, or by the induction hypothesis if $i > 1$. We have $p(X) < p(Y)$ for all X in $\mathbf{W}_i$ and all nonzero Y in $\mathbf{W}_j$ when $j > i$ by the induction hypothesis when $i > 1$. If $i = 1$ then $Y \geqslant X$ for all Y in $\mathbf{U} \backslash 0$, as we noted in (5.1). But $p(Y) > p(X)$ by Lemma 2.6.1 and conclusion (b) of this theorem. [(d) and (e) are established.] ■

Note that the union of any family of factor subspaces $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_d$ that satisfies (b) and (c) is a rank-1 space.

The proof of Theorem 5.1 suggests the following algorithm for determining the factor subspaces $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_d$ of any rank-1 space $\mathbf{V}$.

(1) Let $i = 0$ and $\mathbf{V}_0 = \mathbf{V}$.

(2) Let A_i be a member of $\mathbf{V}_i \backslash 0$ of least perimeter, $\mathbf{W}_i = \mathbf{L}(A_i) \cup \mathbf{R}(A_i)$, and $\mathbf{U} = \mathbf{V}_i \backslash (\mathbf{W}_i \backslash 0)$.

(3) If $\mathbf{U} = \{0\}$, stop. Otherwise, let $i = i + 1$; then let $\mathbf{V}_i = \mathbf{U}$ and go to step (2).

THEOREM 5.2. *The factor spaces* $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_d$ *of Theorem* 5.1 *are uniquely determined by* $\mathbf{V}$.

Proof. Suppose $\{\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_e\}$ is another set of factor spaces satisfying the conclusion of Theorem 5.1, and $\mathbf{Z}_j$ is the factor space of B_j for $j = 1, 2, \dots, e$ just as $\mathbf{W}_i$ is the factor space of A_i for $i = 1, 2, \dots, d$. Part (d) of Theorem 5.1 implies that A_1 and B_1 both have the least perimeter of all nonzero members of $\mathbf{V}$. Therefore $p(B_1) = p(A_1)$. Part (e) implies that $p(A_1) < p(X)$ for all nonzero X in $\mathbf{V} \setminus \mathbf{W}_1$. Therefore $B_1 \in \mathbf{W}_1$ and $A_1 \in \mathbf{Z}_1$. Corollary 5.2 implies that $\mathbf{Z}_1 = \mathbf{L}(B_1) \cup \mathbf{R}(B_1)$. Now $\mathbf{W}_1$ must be either $\mathbf{L}(A_1)$ or $\mathbf{R}(A_1)$. If $\mathbf{W}_1 = \mathbf{L}(A_1)$, then $\mathbf{L}(B_1) = \mathbf{L}(A_1)$ as $B_1 \in \mathbf{W}_1$. If $\mathbf{W}_1 = \mathbf{R}(A_1)$, then $\mathbf{R}(B_1) = \mathbf{R}(A_1)$ similarly. Therefore $\mathbf{Z}_1 = \mathbf{W}_1 \cup \mathbf{R}(B_1)$ or $\mathbf{Z}_1 = \mathbf{L}(B_1) \cup \mathbf{W}_1$. Consequently $\mathbf{W}_1 \subseteq \mathbf{Z}_1$. By a symmetric argument, $\mathbf{Z}_1 \subseteq \mathbf{W}_1$ and hence $\mathbf{W}_1 = \mathbf{Z}_1$. Let $\mathbf{V}_2 = \mathbf{V} \setminus \mathbf{W}_1 \setminus 0$; then we can show that $\mathbf{W}_2 = \mathbf{Z}_2$ in a similar way. Continuing inductively, we see that $\mathbf{W}_i = \mathbf{Z}_i$ for $i = 1, 2, \dots, d$. Therefore $\{0\} = \mathbf{V} \setminus \bigcup_{i \leqslant d} \mathbf{Z}_j$. Consequently $d = e$ and $\mathbf{Z}_j = \mathbf{W}_j$ for $1 \leqslant j \leqslant d$. ■

6. THE MAXIMUM CARDINALITY AND DIMENSION OF BOOLEAN RANK-1 SPACES

Let $c(m, n)$ and $d(m, n)$ denote the maximum cardinality and dimension respectively of rank-1 spaces of $m \times n$ Boolean matrices. We shall show that for all $1 \leqslant m \leqslant n$,

$$c(m, n) = 2^n + 2^{m-1} - 1 \quad \text{and} \quad d(m, n) = \beta(n) + \beta(m-1),$$

where $\beta(k)$ is (as in Section 2.3) the maximum dimension of all subspaces of $\mathbf{B}^k$ if $k \geqslant 1$. We define $\beta(0) = 0$ for convenience.

In our previous discussion of β, we pointed out that $\beta(k)/2^k\sqrt{2/k\pi}$ tends to 1 as $k \to \infty$.

We'll prove our results inductively using the structure theorem of Section 5. First, some notation.

If $\mathbf{V}$ is a rank-1 space, let $M(\mathbf{V})$ be the sum of its members. Let $\mathbf{l}(\mathbf{V})$ and $\mathbf{r}(\mathbf{V})$ denote the left and right factors respectively of $M(\mathbf{V})$. Let $l(\mathbf{V}) = |\mathbf{l}(\mathbf{V})|$ and $r(\mathbf{V}) = |\mathbf{r}(\mathbf{V})|$.

LEMMA 6.1.

(a) *If* $\mathbf{V}$ *is a left factor space, then* $|\mathbf{V}| \leqslant 2^{r(\mathbf{V})}$ *and* $\dim(\mathbf{V}) \leqslant \beta(r(\mathbf{V}))$.

(b) *If* $\mathbf{V}$ *is a right factor space, then* $|\mathbf{V}| \leqslant 2^{l(\mathbf{V})}$ *and* $\dim(\mathbf{V}) \leqslant \beta(l(\mathbf{V}))$.

Proof. Part (a): Let $\hat{\mathbf{V}} = \{\mathbf{y} \in \mathbf{B}^n : \mathbf{l}(\mathbf{V})\mathbf{y}^t \in \mathbf{V}\}$; then $\hat{\mathbf{V}}$ is a subspace of $\mathbf{B}^n$ isomorphic to $\mathbf{V}$. Let $r = r(\mathbf{V})$. Now $\mathbf{y} \leqslant \mathbf{r}(\mathbf{V})$ for all $\mathbf{y}$ in $\hat{\mathbf{V}}$, so $\hat{\mathbf{V}}$ is isomorphic to a subspace of $\mathbf{B}^r$. Therefore $|\mathbf{V}| \leqslant 2^r$ and $\dim(\mathbf{V}) \leqslant \beta(r)$. This establishes (a). Part (b) is proven symmetrically. ∎

LEMMA 6.2. *Suppose* $\mathbf{V} = \mathbf{U} \cup \mathbf{W}$, $\mathbf{U} \cap \mathbf{W} = \{0\}$, *and* $\mathbf{U} \leqslant \mathbf{W}$. *If* $\mathbf{U}$ *and* $\mathbf{W}$ *are rank*-1 *subspaces of* $M_{m,n}(\mathbf{B})$, $\mathbf{W}$ *is a left factor space, and* $\delta = r(\mathbf{V}) - r(\mathbf{U})$, *then*

(i) $|\mathbf{W}| \leqslant 1 + 2^{\delta}$ *and* $\dim(\mathbf{W}) \leqslant 1 + \beta(\delta)$, *and*
(ii) $|\mathbf{W}| \leqslant 2^{\delta}$ *and* $\dim(\mathbf{W}) \leqslant \beta(\delta)$ *when* $l(\mathbf{V}) = l(\mathbf{U})$.

Proof. Let $\mathbf{ab}^t = M(\mathbf{U})$, $\mathbf{cd}^t = M(\mathbf{W})$, and $\hat{\mathbf{W}} = \{\mathbf{y} \in \mathbf{B}^n : \mathbf{cy}^t \in \mathbf{W}\}$. Then $\mathbf{b} \leqslant \mathbf{y} \leqslant \mathbf{d}$ for all nonzero $\mathbf{y}$ in $\hat{\mathbf{W}}$. Therefore $\hat{\mathbf{W}} \backslash \mathbf{b}$ is a subspace of $\mathbf{B}^n$, isomorphic to a subspace of $\mathbf{B}^{\delta}$. Thus

$$|\hat{\mathbf{W}} \backslash \mathbf{b}| \leqslant 2^{\delta} \quad \text{and} \quad \dim(\hat{\mathbf{W}} \backslash \mathbf{b}) \leqslant \beta(\delta). \tag{6.1}$$

Note that $l(\mathbf{V}) = l(\mathbf{W})$, $r(\mathbf{V}) = r(\mathbf{W})$, and $\delta = r(\mathbf{W}) - r(\mathbf{U})$ because $M(\mathbf{V}) = M(\mathbf{W})$. If $l(\mathbf{V}) = l(\mathbf{U})$, then $\mathbf{a} = \mathbf{c}$ because $M(\mathbf{U}) \leqslant M(\mathbf{V})$. Therefore $\mathbf{cb}^t \in \mathbf{U} \backslash 0$, so $\mathbf{b} \notin \hat{\mathbf{W}}$. Consequently $\hat{\mathbf{W}} = \hat{\mathbf{W}} \backslash \mathbf{b}$ and (ii) is implied by (6.1). If $\mathbf{b} \in \hat{\mathbf{W}}$, then $\langle \mathbf{b} \rangle$ and $\hat{\mathbf{W}} \backslash \mathbf{b}$ satisfy the hypotheses of Lemma 2.3.2. Therefore whether $\mathbf{b} \in \hat{\mathbf{W}}$ or not, $\dim(\hat{\mathbf{W}}) = \dim(\langle \mathbf{b} \rangle) + \dim(\hat{\mathbf{W}} \backslash \mathbf{b})$. But $|\hat{\mathbf{W}}| = 1 + |\hat{\mathbf{W}} \backslash \mathbf{b}|$, so (i) follows from (6.1). ∎

LEMMA 6.3. *Suppose* $\mathbf{V}$ *is any rank*-1 *space. If* g *and* h *denote the maximum and minimum of* $\{r(\mathbf{V}), l(\mathbf{V})\}$ *respectively, then*

$$|\mathbf{V}| \leqslant 2^g + 2^{h-1} - 1 \quad \text{and} \quad \dim(\mathbf{V}) \leqslant \beta(g) + \beta(h-1). \tag{6.2}$$

Proof. By induction on $p(\mathbf{V})$, the perimeter of $M(\mathbf{V})$. Lemma 6.1 implies that (6.2) holds for all factor spaces. In particular, if $p(\mathbf{V}) \leqslant 3$, then (6.2) holds for $\mathbf{V}$. Suppose that (6.2) holds for all rank-1 spaces $\mathbf{Y}$ with $3 \leqslant p(\mathbf{Y}) < p(\mathbf{V})$. Without loss of generality, we may assume that $\mathbf{V}$ is not a factor space. Therefore $d > 1$ in the factor-space decomposition of $\mathbf{V}$ given in Section 5: $\mathbf{V} = \bigcup_{i \leqslant d} \mathbf{W}_i$.

Let $\mathbf{U} = \bigcup_{i < d} \mathbf{W}_i$ and $\mathbf{W} = \mathbf{W}_d$. Parts (c) and (d) of Theorem 5.1 imply that $p(\mathbf{U}) < p(\mathbf{W})$. But $M(\mathbf{V}) = M(\mathbf{W})$ because $\mathbf{W} \geqslant \mathbf{U}$. Therefore $p(\mathbf{U}) < p(\mathbf{V})$. Let $j = \max(r(\mathbf{U}), l(\mathbf{U}))$ and $k = \min(r(\mathbf{U}), l(\mathbf{U}))$. Then

$$|\mathbf{U}| \leqslant 2^j + 2^{k-1} - 1 \quad \text{and} \quad \dim(\mathbf{U}) \leqslant \beta(j) + \beta(k-1)$$

by the inductive hypotheses. But $\mathbf{U}\cap\mathbf{W}=\{0\}$ by Theorem 5.1(b). Also $\beta(\mathbf{V})=\beta(\mathbf{U})+\dim(\mathbf{W})$ by Lemma 2.3.2. Therefore

$$\begin{aligned}|\mathbf{V}|&\leqslant 2^{j}+2^{k-1}+|\mathbf{W}|-2\\ \dim(\mathbf{V})&\leqslant\beta(j)+\beta(k-1)+\dim(\mathbf{W}).\end{aligned}\tag{6.3}$$

We note for future reference that

$$l(\mathbf{V})\geqslant l(\mathbf{U})\quad\text{and}\quad r(\mathbf{V})\geqslant r(\mathbf{U}),\tag{6.4}$$

with equality holding in at most one of these expressions. This is because $M(\mathbf{V})\geqslant M(\mathbf{U})$ but $p(\mathbf{V})>p(\mathbf{U})$.

Case 1: $g=r(\mathbf{V})$. Suppose that $\mathbf{W}$ is a left factor space. If $j=r(\mathbf{U})$, then by (6.3) and Lemma (6.2),

$$\begin{aligned}|\mathbf{V}|&\leqslant 2^{j}+2^{g-j}+2^{k-1}+\epsilon-2,\\ \dim(\mathbf{V})&\leqslant\beta(j)+\beta(k-1)+\beta(g-j)+\epsilon,\end{aligned}\tag{6.5}$$

where $\epsilon=1$ if $l(\mathbf{V})>l(\mathbf{U})$ and $\epsilon=0$ if $l(\mathbf{V})=l(\mathbf{U})$. If $g>j$, then $|\mathbf{V}|\leqslant 2^{g}+2^{h-1}-1$ by (6.4) and (6.5). If $g=j$, then $h>k$ by (6.4), so $|\mathbf{V}|\leqslant 2^{g}+2^{h-2}$ by (6.5). Thus $|\mathbf{V}|$ satisfies (6.2). If $h>k$, then $\epsilon=1$ and $\dim(\mathbf{V})\leqslant\beta(j)+\beta(g-j)+\beta(k-1)+1$. It follows easily from the definition of β that

$$\beta(x)+\beta(y)\leqslant\beta(x+y)\tag{6.6}$$

for all nonnegative integers x and y. Therefore $\beta(j)+\beta(g-j)\leqslant\beta(g)$ and $\beta(k-1)+1\leqslant\beta(k)\leqslant\beta(h-1)$. Thus $\dim(\mathbf{V})$ satisfies (6.2) when $h>k$. If $h=k$ then $g>j$ by (6.4) and $\epsilon=0$, so (6.5) and (6.6) imply that $\dim(\mathbf{V})$ satisfies (6.2). Thus (6.2) is satisfied when $j=r(\mathbf{U})$. If $j=l(\mathbf{U})$, then by (6.3) and Lemma 6.2,

$$\begin{aligned}|\mathbf{V}|&\leqslant 2^{j}+2^{g-k}+2^{k-1}+\epsilon-2,\\ \dim(\mathbf{V})&\leqslant\beta(j)+\beta(g-k)+\beta(k-1)+\epsilon,\end{aligned}\tag{6.7}$$

where $\epsilon=1$ if $h>j$ and $\epsilon=0$ if $h=j$. Now $g>k$. Otherwise we would have $g=k$, contradicting (6.4). Thus

$$|\mathbf{V}|\leqslant 2^{j}+2^{g-1}+\epsilon-2\leqslant 2^{h}+2^{g-1}-1.$$

Also

$$\dim(\mathbf{V}) \leqslant \beta(j)+\beta(g-1)+\epsilon \leqslant \beta(h)+\beta(g-1).$$

Since the sequences $\{2^q - 2^{q-1}\}$ and $\{\beta(q)-\beta(q-1)\}$ are monotonically increasing, it follows that (6.2) is satisfied when $j = l(\mathbf{U})$.

Suppose $\mathbf{W}$ is a right factor space and $j = r(\mathbf{U})$; then $|\mathbf{W}| \leqslant 2^{h-k}+\epsilon$ and $\dim(\mathbf{W}) \leqslant \beta(h-k)+\epsilon$, where $\epsilon = 1$ if $g > j$ and $\epsilon = 0$ if $g = j$ (by a symmetric version of Lemma 6.2 for right factor spaces). We have $|\mathbf{V}| \leqslant (2^j + 2^{k-1} - 1) + (2^{h-k}+\epsilon) - 1$. If $h = k$, then $g > j$, so $|\mathbf{V}| < 2^g + 2^{h-1} - 1$. If $h > k$, then $|\mathbf{V}| \leqslant 2^j + 2^{h-1} - 1 < 2^g + 2^{h-1} - 1$. We also have $\dim(\mathbf{V}) \leqslant \beta(j)+\beta(k-1)+\beta(h-k)+\epsilon \leqslant \beta(j)+\beta(h-1)+\epsilon \leqslant \beta(g)+\beta(h-1)$. If $j = l(\mathbf{U})$, then $|\mathbf{W}| \leqslant 2^{h-j}+\epsilon$ and $\dim(\mathbf{W}) \leqslant \beta(h-j)+\epsilon$ where $\epsilon = 1$ if $g > k$ and $\epsilon = 0$ if $g = k$. But we've seen that $g \neq k$. Therefore

$$|\mathbf{V}| \leqslant (2^j + 2^{h-j}) + 2^{k-1} - 1,$$

$$\dim(\mathbf{V}) \leqslant \beta(j)+\beta(h-j)+\beta(k-1)+1.$$

Consequently

$$|\mathbf{V}| \leqslant 2^h + 2^{k-1} - 1 + \epsilon,$$

$$\dim(\mathbf{V}) \leqslant \beta(h)+\beta(k-1)+1,$$

where $\epsilon = 1$ if $h = j$ and $\epsilon = 0$ otherwise (i.e. when $h > j$). But $g > k$, as we've seen, so $2^{k-1} \leqslant 2^{g-1} - 1$ and $\beta(k-1) \leqslant \beta(g) - 1$. Therefore $|\mathbf{V}| \leqslant 2^h + 2^{g-1} - 1$ and $\dim(\mathbf{V}) \leqslant \beta(h)+\beta(g-1)$. We noted above that these inequalities imply (6.2).

Case 2: $g = l(\mathbf{V})$. Let $\mathbf{X}^t = \{X^t : X \in \mathbf{X}\}$; then $\mathbf{X}^t$ is isomorphic to $\mathbf{X}$, $l(\mathbf{X}) = r(\mathbf{X}^t)$, and $r(\mathbf{X}) = l(\mathbf{X}^t)$. We have $r(\mathbf{V}^t) \geqslant l(\mathbf{U}^t)$ in this case, so by case 1,

$$|\mathbf{V}^t| \leqslant 2^{r(\mathbf{V}^t)} + 2^{l(\mathbf{V}^t)-1} - 1$$

$$\dim(\mathbf{V}^t) \leqslant \beta\big(r(\mathbf{V}^t)\big)+\beta\big(l(\mathbf{V}^t)-1\big).$$

Thus

$$|\mathbf{V}| \leqslant 2^g + 2^{h-1} - 1 \quad \text{and} \quad \dim(\mathbf{V}) \leqslant \beta(g)+\beta(h-1). \qquad \blacksquare$$

THEOREM 6.1. *For every* $1 \leqslant m \leqslant n$, $c(m,n) = 2^n + 2^{m-1} - 1$ *and* $d(m,n) = \beta(n)+\beta(m-1)$.

Proof. In view of Lemma 6.1 it is sufficient to exhibit rank-1 spaces of $m \times n$ matrices achieving the bounds.

Let $\mathbf{e}$ be the first column of the $m \times m$ identity matrix and $\mathbf{X} = \{\mathbf{y} \in \mathbf{B}^m \setminus \mathbf{e} : \mathbf{y} \geqslant \mathbf{e}\} \cup \{0\}$.

Let $\mathbf{U} = \{\mathbf{e}\mathbf{z}^t : \mathbf{z} \in \mathbf{B}^n\}$, $\mathbf{W} = \{\mathbf{x}J_{1n} : \mathbf{x} \in \mathbf{X}\}$, and $\mathbf{G} = \mathbf{U} \cup \mathbf{W}$. Then $\mathbf{G}$ is a rank-1 space because $\mathbf{U} \leqslant \mathbf{W}$, and $|\mathbf{G}| = 2^n + 2^{m-1} - 1$ because $\mathbf{U} \cap \mathbf{W} = \{0\}$.

Let $\mathbf{P}$ be a subspace of $\mathbf{B}^n$ of dimension $\beta(n)$, and $\mathbf{Y}$ a subspace of $\mathbf{X}$ of dimension $\beta(m-1)$. Let $\mathbf{R} = \{\mathbf{e}\mathbf{z}^t : \mathbf{z} \in \mathbf{P}\}$, $\mathbf{S} = \{\mathbf{y}J_{1n} : \mathbf{y} \in \mathbf{Y}\}$, and $\mathbf{H} = \mathbf{R} \cup \mathbf{S}$. Then $\mathbf{H}$ is a rank-1 space because $\mathbf{R} \leqslant \mathbf{S}$ and $\dim(\mathbf{H}) = \beta(n) + \beta(m-1)$ by Lemma 2.3.2 because $\mathbf{R} \cap \mathbf{S} = \{0\}$. ■

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