A Hermitian Canonical Form for Complex Matrices under Consimilarity

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ABSTRACT

We produce an explicit Hermitian canonical form for complex square matrices under consimilarity. We apply a simple algorithmic procedure to a concanonical form for complex matrices to construct a form that is not only canonical but also Hermitian. We also show that a similar algorithmic procedure can be used to produce an explicit real canonical form for complex matrices under consimilarity.

1. PRELIMINARIES

We denote the set of all $m$-by-$n$ complex matrices by $M_{m,n}$; $M_n \equiv M_{n,n}$. For $A \in M_n$, we denote the transpose by $A^T$, the complex conjugate by $\overline{A}$, and the Hermitian adjoint by $A^* = \overline{A^T}$. We say that two matrices $A, B \in M_n$ are consimilar if there is a nonsingular $P \in M_n$ such that $P^{-1}AP = B$. We write $A \prec B$ if $A$ is similar to $B$, and $A \sim B$ if $A$ is consimilar to $B$. We say that $A \in M_n$ is condiagnoalizable if $A$ is diagonalizable by consimilarity, i.e., there is a nonsingular $P \in M_n$ such that $P^{-1}AP = \Lambda$, where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$. There is no loss of generality in assuming that $\Lambda = \text{diag}(|\lambda_1|, \ldots, |\lambda_n|)$; the nonnegative diagonal entries of $\Lambda$ are called coneigenvalues of $A$ [4]. Note that a unitary consimilarity is a unitary congruence. Clearly, unitarily condiagnoalizable matrices are symmetric matrices. Conversely, any symmetric matrix is unitarily condiagnoalizable by Takagi’s theorem [9].

A mapping $T : V \rightarrow W$ between complex vector spaces $V$ and $W$ is called an antilinear transformation if $T(\alpha x + \beta y) = \overline{\alpha}T(x) + \overline{\beta}T(y)$ for all $\alpha, \beta \in \mathbb{C}$.
and all \( x, y \in V \). Just as similar matrices are matrix representations of a single linear transformation in different bases, consimilar matrices are matrix representations of a single antilinear transformation in different bases [3].

A nonsingular matrix \( Q \in M_n \) is called orthogonal if \( QQ^T = I \). The following result is well known.

**Theorem 1.1.** Let \( A, B \in M_n \) be symmetric. Then \( A \) is similar to \( B \) if and only if \( A \) is orthogonally similar to \( B \).

A nonsingular \( E \in M_n \) is called involutory if \( EE = I \) and is called coninvolutory if \( EE = 1 \). The coninvolutory matrices can be characterized by the following lemma [6, Lemma (4.1)].

**Lemma 1.2.** A given \( E \in M_n \) is coninvolutory if and only if there is a nonsingular \( P \in M_n \) such that \( E = PP^{-1} \).

The following theorem shows that the consimilarity class of \( A \in M_n \) can be determined by the similarity class of \( AA \) and a rank condition [4, Theorem 4.1].

**Theorem 1.3.** Let \( A, B \in M_n \). Then \( A \sim B \) if and only if \( AA \sim BB \) and the following alternating-product rank condition is satisfied:

\[
\text{rank}\left[(AA)^kA\right] = \text{rank}\left[(BB)^kB\right], \quad k = 0, 1, \ldots, \lfloor n/2 \rfloor.
\]

Theorem 1.3 can be used to deduce a canonical form under consimilarity that we call the **concanonical form**; actually, there are several useful variants of the canonical form, all equivalent under consimilarity. For a given \( A \in M_n \), suppose \( P^{-1}AAP = J(AA) \), where \( J(AA) \) is a Jordan canonical form of \( AA \). If \( J_c \) is any matrix in \( M_n \) that satisfies the alternating-product rank condition and is such that \( J_cJ_c \sim J(AA) \), then by Theorem 1.3 \( J_c \) is consimilar to \( A \). See [4] for a proof of Theorem 1.3 and a discussion of several other concanonical forms of a matrix.

Let

\[
J_n(\lambda) = \begin{bmatrix}
\lambda & 1 & & \\
& \ddots & \ddots & \\
& & \ddots & 1 \\
0 & & & \lambda
\end{bmatrix} \in M_{n_1},
\]
a Jordan block of size $n_i$, and let the Jordan canonical form of $AA^\top$ be

$$J(AA^\top) = J_{pos}(AA^\top) \oplus \tilde{J}_{nek}(AA^\top) \oplus \tilde{J}_{com}(AA^\top),$$

where the respective direct summands are Jordan matrices with all nonnegative, negative, and complex nonreal eigenvalues, respectively (if any). Then the concanonical form $J_c(A)$ of $A \in M_n$ is such that

$$J_c(A) = J_P \oplus Q_N \oplus Q_C,$$

in which

$$J_P \equiv I_{m_1}(\lambda_1) \oplus \cdots \oplus I_{m_r}(\lambda_r),$$

where all $\lambda_i \geq 0$, and $\lambda_i^2$ are the nonnegative eigenvalues of $AA^\top$ such that

$$J_P J_P = J_{m_1}^2(\lambda_1) \oplus \cdots \oplus J_{m_r}^2(\lambda_r) \sim J_{pos}(AA^\top),$$

and the blocks $J_{m_i}(\lambda_i)$ with $\lambda_i = 0$ are determined by the alternating product rank condition;

$$Q_N \equiv N_{2n_1}(\mu_1) \oplus \cdots \oplus N_{2n_r}(\mu_r),$$

where all $\mu_i > 0$, and $-\mu_i^2 < 0$ are the negative eigenvalues of $AA^\top$,

$$N_{2n_1}(\mu_i) \equiv \begin{bmatrix} 0 & J_{n_1}(\mu_i) \\ -J_{n_1}(\mu_i) & 0 \end{bmatrix} \in M_{2n_1},$$

and

$$N_{2n_1}(\mu_i) N_{2n_1}(\mu_i) = \begin{bmatrix} J_{n_1}^2(\mu_i) & 0 \\ 0 & -J_{n_1}^2(\mu_i) \end{bmatrix} \in M_{2n_1},$$

such that $Q_N \tilde{Q}_N \sim J_{nek}(AA^\top)$; and

$$Q_C \equiv C_{2k_1}(\xi_1) \oplus \cdots \oplus C_{2k_s}(\xi_s),$$
where all $\xi_i \in \mathbb{R}$, $\xi_i^2$ are the complex nonreal eigenvalues of $A\bar{A}$,

$$C_{2k}(\xi_i) = \begin{bmatrix} 0 & J_k(\xi_i) \\ J_k(\xi_i) & 0 \end{bmatrix},$$

and

$$C_{2k}(\xi_i)\overline{C_{2k}(\xi_i)} = \begin{bmatrix} J_k(\xi_i)^2 & 0 \\ 0 & J_k(\xi_i)^2 \end{bmatrix} \in M_{2k},$$

such that $Q_C \overline{Q_C} \sim J_{\text{com}}(A\bar{A})$.

$N_{2k}(\mu)$ and $C_{2k}(\xi)$ are called the quasi-Jordan blocks.

Note that $J_\mu(A)J_\mu(A)$ is a direct sum of terms of the forms $J_\mu(\lambda)$, $-J_\mu(\mu) \oplus -J_\mu(\mu)$, or $J_\mu(\xi) \oplus J^2(\xi)$ with $\lambda > 0$, $\mu > 0$, and $\xi \notin \mathbb{R}$, and is similar to a Jordan canonical form of $A\bar{A}$.

The proposition below follows easily from the above observations.

**Proposition 1.8.** Let $B \in M_n$ be a given nonsingular matrix. Then there is a nonsingular $A \in M_n$ such that $B = A\bar{A}$ if and only if

(a) each negative eigenvalue of $B$ has even algebraic multiplicity, and

(b) any complex eigenvalues of $B$ occur only in complex conjugate pairs.

Proposition (1.8) completely classifies the set of nonsingular matrices of the form $A\bar{A}$.

We say $A \in M_n$ is positive definite if $x^*Ax > 0$ for all $x \in \mathbb{C}^n$ and is positive semidefinite if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$. A positive definite matrix is necessarily nonsingular. The following is easily verified from the definition of a positive semidefinite matrix.

**Lemma 1.9.** Let $A = [a_{ij}] \in M_n$ be positive semidefinite, and suppose that $a_{kk} = 0$ for some $1 < k < n$. Then $a_{kj} = 0$ and $a_{ik} = 0$ for all $i, j = 1, \ldots, n$.

It is well known that the product of a positive definite $A \in M_n$ and a Hermitian $B \in M_n$ is diagonalizable and has real eigenvalues [7, Theorem (7.6.3)]. It is also known [5] that the product of any two positive semidefinite matrices is diagonalizable and has nonnegative eigenvalues. In particular, we have

**Proposition 1.10.** If $A \in M_n$ is positive semidefinite, then $A\bar{A}$ is diagonalizable and has nonnegative eigenvalues.
The following is immediate from Proposition 1.10.

**Corollary 1.11.** Let \( A \in M_n \) be positive semidefinite, and suppose that \( \text{rank}(A) - \text{rank}(AA^*) = r \). Then \( 0 \leq r \leq [n/2] \), and the concanonical form \( J_c(A) \) is the direct sum of a nonnegative diagonal matrix and \( r \) 2-by-2 nilpotent Jordan blocks, i.e., \( A \) is consimilar to a matrix of the form \( \Lambda \oplus N \), in which

\[
\Lambda = \begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2 \\
& \ddots \\
0 & \cdots & \cdots & \lambda_{n-2r}
\end{bmatrix} \in M_{n-2r},
\]

where each \( \lambda_i \geq 0 \) and \( \lambda_i^2 \) is a nonnegative eigenvalue of \( AA^* \), and

\[
N = \begin{bmatrix}
J_2(0) & 0 \\
0 & J_2(0) \\
& \ddots \\
0 & \cdots & 0 & J_2(0)
\end{bmatrix} \in M_{2r}.
\]

**Proof.** See [5].

**Corollary 1.12.** Let \( A \in M_n \) be positive semidefinite. Then \( A \) is condiaogonalizable if and only if \( \text{rank}(A) = \text{rank}(AA^*) \).

**Proof.** If \( A \) is condiaogonalizable, then \( \text{rank}(A) = \text{rank}(AA^*) \) by Theorem 1.3. The converse is also immediate by Corollary 1.11, since \( r = 0 \).

2. **Hermitian Canonical Form of \( A \in M_n \)**

**Under Consimilarity**

It is well known that any \( A \in M_n \) is similar to a symmetric matrix. In fact, \( A \) is similar to a symmetric Jordan canonical form [7, Chapter 4], denoted by \( J_s(A) \):

\[
A \sim J_s(A) = S_n(\lambda_1) \oplus \cdots \oplus S_n(\lambda_k),
\]
where

\[
S_m(\lambda) = \frac{1}{2} \begin{bmatrix}
2\alpha & 1 & 0 \\
1 & \ddots & \ddots \\
\vdots & \ddots & 1 \\
0 & 1 & 2\alpha
\end{bmatrix}
\]

\[
\begin{bmatrix}
2\beta & 0 & -1 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
2\beta & -1 & \ddots & \ddots \\
0 & 2\beta - 1 & 0 & \ddots \\
i0 & -1 & 2\beta & 1 \\
\vdots & \ddots & \ddots & \ddots \\
-1 & \ddots & \ddots & 1 \\
0 & 1 & 0 & 2\beta
\end{bmatrix} \in M_{m},
\]

if \( \lambda = \alpha + i\beta, \alpha, \beta \in \mathbb{R} \), where \( \lambda \) is an eigenvalue of \( A \).

We have analogous results for consimilarity. Any \( A \in M_n \) is consimilar to a Hermitian matrix [4, Theorem 4.14]. The proof of this result relies on direct application of a theorem of Hua [8] in which he shows that a certain class of symmetric matrices can be factored into the product of a Hermitian matrix and its complex conjugate. We will use an elementary algorithmic procedure to derive a form that is not only Hermitian but also canonical. For the derivation, we just need to observe the following fact.

**Lemma 2.1.** Let \( A \in M_{m} \). If \( A \) is consimilar to a Hermitian matrix, then there is a nonsingular symmetric \( S \in M_{m} \) such that \( S^{-1}AS = A^* \). Conversely, if
there is a nonsingular symmetric $S \in M_n$ such that $S^{-1}A \bar{S} = A^*$, then for any nonsingular $P$ such that $S = PP^T$, $P^{-1}AP$ is Hermitian.

Proof. Suppose there is a Hermitian matrix $H$ such that $P^{-1}AP = H$. Then $P^TAP = H^* = H$. Thus, $P^{-1}AP = P^TAP = (PP^T)^{-1}A(PP^T) = A^*$, where $PP^T$ is symmetric.

Conversely, suppose there is a nonsingular symmetric matrix $S$ such that $S^{-1}AS = A^*$. Let $P$ be any nonsingular matrix such that $PP^T = S$. Then $(PP^T)^{-1}A(PP^T) = A^*$, or $P^{-1}AP = P^TAP = (P^{-1}AP)^*$. Therefore, $P^{-1}AP$ is Hermitian.

From Lemma 2.1 we note that for a given $A \in M_n$, if we can find a symmetric $S \in M_n$ such that $S^{-1}AS = A^*$, then for any nonsingular $P \in M_n$ such that $PP^T = S$, $P^{-1}AP$ is Hermitian. Bevis, Hall, and Hartwig have shown that every $A \in M_n$ is consimilar to $A^*$ via a symmetric matrix [2]. Thus, there always exists a nonsingular symmetric $S$ that transforms $A$ into $A^*$ by consimilarity. Therefore, the following steps can be taken to obtain a Hermitian canonical form under consimilarity:

**Step 1.** Reduce the given $A \in M_n$ to a concanonical form $J_c(A)$ by consimilarity.

**Step 2.** Find a simple nonsingular symmetric $S \in M_n$ that transforms each of the Jordan or quasi-Jordan blocks in $J_c(A)$ to its Hermitian adjoint by consimilarity.

**Step 3.** Find a nonsingular $P$ such that $PP^T = S$.

**Step 4.** Compute $P^{-1}J_c(A)P$ to obtain a Hermitian canonical form.

Now, let $A \in M_n$ be given.

**Step 1:** Let $R^{-1}AR = J_c(A)$, where $J_c(A) = J_F \oplus Q_N \oplus Q_C$ in (1.4).

**Step 2:** Let

\[
I_k = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \in M_k, \quad B_k = \begin{bmatrix} 0 & & & 1 \\ & 0 & & \\ & & \ddots & \\ 1 & & & 0 \end{bmatrix} \in M_k,
\]

\[
D_{2k} = B_k \oplus B_k = \begin{bmatrix} B_k & 0 \\ 0 & B_k \end{bmatrix} \in M_{2k}.
\]

Notice that $I_k$, $B_k$, and $D_{2k}$ are nonsingular, real, symmetric, and also
involutory matrices. We note the following:

(a) \( B_k^{-1} J_k(\lambda) B_k = B_k J_k(\lambda) B_k - J_k^T(\lambda) \), and \( J_k^T(\lambda) = J_k^*(\lambda) \) if \( \lambda \in \mathbb{R} \).

(b) We have

\[
(-iD_{2k})^{-1}N_{2k}(\mu)(-iD_{2k}) = iD_{2k}N_{2k}(\mu)iD_{2k}
\]

\[
= \begin{bmatrix}
0 & -B_k J_k(\mu) B_k \\
B_k J_k(\mu) B_k & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & -J_k^T(\mu) \\
J_k^T(\mu) & 0
\end{bmatrix} = N_{2k}^T(\mu)
\]

and \( N_{2k}^T(\mu) = N_{2k}^*(\mu) \) for \( \mu > 0 \).

(c) We have

\[
D_{2k}^{-1}C_{2k}(\xi)D_{2k} = D_{2k}C_{2k}(\xi)D_{2k}
\]

\[
= \begin{bmatrix}
0 & B_k J_k(\xi) B_k \\
B_k J_k(\xi) B_k & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & J_k^T(\xi) \\
J_k^T(\xi) & 0
\end{bmatrix} = C_{2k}^*(\xi).
\]

Therefore, \( B_k \) is the basic symmetric matrix that satisfies the statement of step 2.

Step 3: Let

\[
S_k = \frac{1}{\sqrt{2}} (I_k + iB_k) \in M_k,
\]

and let

\[
P_k = e^{-i(\pi/4)S_k}.
\]
Then, $S_k$ and $P_k$ are coninvolutory and symmetric matrices such that

$$S_k S_k^T = S_k^2 = iB_k, \quad S_k^{-1} = \bar{S}_k,$$

$$P_k P_k^T = P_k^2 = -iS_k^2 = B_k, \quad P_k^{-1} = \bar{P}_k.$$ 

$$(S_k \oplus S_k)^2 = iD_{2k}, \quad \text{and} \quad (P_k \oplus P_k)^2 = D_{2k}.$$ 

It is easy to see that $B_k$, $S_k$, and $P_k$ commute with each other.

Step 4: We compute consimilarity for each of the Jordan or quasi-Jordan blocks of $J_c(A)$ to obtain explicit Hermitian blocks.

(i) For

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \in M_k, \quad \lambda \in \mathbb{R},$$

from (a) in step 2 and by $P_k P_k^T = P_k^2 = B_k$ in step 3, Lemma 2.1 guarantees that the consimilarity $P_k J_k(\lambda) P_k = P_k J_k(\lambda) P_k$ yields a Hermitian matrix: $P_k J_k(\lambda) P_k = H_k(\lambda)$, where

$$H_k(\lambda) = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} + \frac{1}{i} \begin{bmatrix} 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$= H_k(\lambda)^* \in M_k, \quad \lambda \in \mathbb{R}.$$ 

(ii) For

$$N_{2k}(\mu) = \begin{bmatrix} 0 & J_k(\mu) \\ -J_k(\mu) & 0 \end{bmatrix} \in M_{2k}, \quad \mu > 0,$$

from (b) in step 2 and by $(S_k \oplus S_k)^2 = iD_{2k}$ in step 3, Lemma 2.1 guarantees that the consimilarity $(S_k \oplus S_k)^{-1} N_{2k}(\mu)(S_k \oplus S_k) = (S_k \oplus S_k) N_{2k}(\mu)(S_k \oplus S_k)$
yields a Hermitian matrix

\[
\begin{pmatrix}
\mathbb{S}_k & \mathbb{S}_k
\end{pmatrix} N_{2k}(\mu) \begin{pmatrix}
\mathbb{S}_k & \mathbb{S}_k
\end{pmatrix} = K_{2k}(\mu),
\]

where

\[
K_{2k}(\mu) = \begin{bmatrix}
0 & -iH_k(\mu) \\
iH_k(\mu) & 0
\end{bmatrix} \in M_{2k}, \quad \mu > 0.
\]

(iii) For

\[
C_{2k}(\xi) = \begin{bmatrix}
0 & J_k(\xi) \\
J_k(\xi) & 0
\end{bmatrix} \in M_{2k},
\]

where \(\xi \in \mathbb{C}\) is nonreal, from (c) of step 2 and by \((P_k \oplus P_k)^2 = D_{2k}\) in step 3, Lemma 2.1 guarantees that the consimilarity \(C_{2k}(\xi)(P_k \oplus P_k) = (P_k \oplus P_k)C_{2k}(\xi)(P_k \oplus P_k)\) yields a Hermitian matrix.

\[
(P_k \oplus P_k)C_{2k}(\xi)(P_k \oplus P_k) = L_{2k}(\xi),
\]

where

\[
L_{2k}(\xi) = \begin{bmatrix}
0 & H_k(\xi) \\
H_k(\xi)^* & 0
\end{bmatrix} \in M_{2k},
\]

in which

\[
H_k(\xi) = \begin{bmatrix}
\frac{1}{2} & \xi & \frac{1}{2} \\
\cdot & \cdot & \cdot \\
\frac{1}{2} & \cdot & 0
\end{bmatrix} + \frac{i}{2} \begin{bmatrix}
0 & 1 & \cdot & \cdot & \cdot & 0 \\
-1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & 1 & 0
\end{bmatrix} \in M_k,
\]

with \(\xi \in \mathbb{C}\) nonreal.

We summarize the above procedure as a theorem.

**Theorem 2.2.** Let \(A \in M_n\) be given. The Hermitian canonical form of \(A\) under consimilarity, denoted by \(J_H(A)\), is a direct sum of three Hermitian
matrices:

\[ J_H(A) = H_F \oplus K_N \oplus K_C, \]

with

\[ H_F = H_{m_1}(\lambda_1) \oplus \cdots \oplus H_{m_p}(\lambda_p), \tag{2.3} \]

where all \( \lambda_i \geq 0 \), and \( \lambda_i^2 \) are the nonnegative eigenvalues of \( \Lambda \bar{\Lambda} \);

\[ K_N = K_{2n_1}(\mu_1) \oplus \cdots \oplus K_{2n_r}(\mu_r), \tag{2.4} \]

where all \( \mu_i > 0 \), \( -\mu_i^2 \) are the negative eigenvalues of \( \Lambda \bar{\Lambda} \), and

\[ K_{2n_i}(\mu_i) = \begin{bmatrix} 0 & -iH_{n_i}(\mu_i) \\ iH_{n_i}(\mu_i) & 0 \end{bmatrix} \in M_{2k}; \]

and

\[ K_C = L_{2k_1}(\xi_1) \oplus \cdots \oplus L_{2k_s}(\xi_s), \tag{2.5} \]

where all \( \xi_i \in \mathbb{C} \) are nonreal, \( \xi_i^2 \) are the nonreal eigenvalues of \( \Lambda \bar{\Lambda} \), and

\[ L_{2k_1}(\xi_i) = \begin{bmatrix} 0 & H_{k_1}(\xi_i) \\ H_{k_1}(\xi_i)^* & 0 \end{bmatrix} \in M_{2k_i}, \]

in which

\[ H_m(\lambda) = \frac{1}{2} \begin{bmatrix} 1 & 2\lambda \\ & \ddots & \ddots \\ & \ddots & 1 \\ 2\lambda & 1 \end{bmatrix} \]

\[ + i \begin{bmatrix} 0 & 1 & & \cdots & 0 \\ -1 & & \ddots & \ddots & \cdots \\ & \cdots & \ddots \ddots \ddots \cdots \cdots \\ 0 & \cdots & \cdots & 1 & 0 \end{bmatrix} \in M_m, \quad \lambda \in \mathbb{C}. \]
Then $A$ is consimilar to $J_H(A)$. The direct summands (2.3), (2.4), and (2.5) in $J_H(A)$ correspond exactly to the direct summands (1.5), (1.6), and (1.7) in $J_c(A)$, respectively. The Hermitian canonical form $J_H(A)$ is unique up to a permutation of the Hermitian blocks that compose $H_F$, $K_N$, and $K_C$.

Note that $H_m(\lambda)$ is Hermitian if $\lambda$ is real. Since $J_H(A)$ is derived from a concanonical form $J_c(A)$ of $A$, the uniqueness of $J_H(A)$ is the same as the uniqueness of $J_c(A)$.

**Corollary 2.6.** Two $n$-by-$n$ complex matrices are consimilar if and only if they have the same Hermitian canonical form.

It is known that a positive definite matrix is always con diagonalizable [6, Theorem (4.3)], and we can see that its Hermitian canonical form must be a diagonal matrix. Therefore, the Hermitian canonical form $J_H(A)$ of $A \in M_n$ cannot contain a positive definite block of dimension greater than one. Indeed, Hermitian blocks $K_{2n}(\mu_j)$ or $L_{2n}(\xi_j)$ that correspond to negative or complex nonreal eigenvalues of $AA^*$, respectively, cannot be positive definite or positive semidefinite. Since $H_m(\lambda) \in M_n$ is nonsingular for $\lambda \neq 0$, if we let $x \in \mathbb{C}^n$ be such that $H_m(\lambda)x = (-1, -1, \ldots, -1)^T \in \mathbb{C}^m$, then for $y = (1, \ldots, 1)^T \in \mathbb{C}^m$,

\[
\begin{bmatrix} y^* & x^* \end{bmatrix} \begin{bmatrix} 0 & H_m(\lambda) \\ H_m^*(\lambda) & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = (y^*H_m(\lambda)x)^* + [y^*H_m(\lambda)x] = -2n < 0.
\]

We note further that $H_m(\lambda) \in M_n$ is positive definite only if $m = 1$ when $\lambda > 0$, and is positive semidefinite only if $m = 1$ or 2 when $\lambda = 0$.

For $m = 1$, $H_1(\lambda) = \frac{1}{2}[(2\lambda + i(0)]$ is clearly positive definite when $\lambda > 0$, and is positive semidefinite when $\lambda = 0$. For $m = 2$,

\[
H_2(\lambda) = \frac{1}{2} \begin{bmatrix} 1 & 2\lambda - i \\ 2\lambda + i & 1 \end{bmatrix},
\]

and $\det H_2(\lambda) = \frac{1}{2}(1 - 4\lambda^2 - 1) = -2\lambda^2$. Thus, $H_2(\lambda)$ cannot be positive definite when $\lambda > 0$, and is positive semidefinite when $\lambda = 0$. Note that

\[
H_2(0)H_2(0)^* = 0 \in M_2.
\]
For $m > 2$, 

$$H_m(\lambda) = \frac{1}{2} \begin{bmatrix} 0 & -i & 1 & 2\lambda \\
 i & 0 & -i & 1 & 2\lambda \\
i & 0 & \ddots & \ddots & 1 \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
 1 & 2\lambda & \cdots & 0 & -i \\
 2\lambda & 1 & \cdots & i & 0 \end{bmatrix} \in M_m.$$ 

Since the $1,1$ entry of $H_m(\lambda)$ is zero but the corresponding off-diagonal entries are not all zero, $H_m(\lambda)$ cannot be positive semidefinite for all $\lambda > 0$, by Lemma (1.9).

**Corollary 2.7.** Let $A \in M_n$ be positive semidefinite, and suppose that $\text{rank}(A) - \text{rank}(AA) - r$. Then $0 \leq r \leq \lfloor n/2 \rfloor$, and the Hermitian canonical form $J_H(A)$ is of the form $J_H(A) = \Lambda \oplus \Gamma$, in which 

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\
 \lambda_2 & \ddots \\
 0 & \ddots & \lambda_{n-2r} \end{bmatrix} \in M_{n-2r},$$

where each $\lambda_i \geq 0$, and $\lambda_i^2$ is a nonnegative eigenvalue of $AA$; and 

$$\Gamma = \begin{bmatrix} H_2(0) & \ddots & 0 \\
 H_2(0) & \ddots & \ddots \\
 0 & \ddots & H_2(0) \end{bmatrix} \in M_{2r},$$

where

$$H_2(0) = \frac{1}{2} \begin{bmatrix} 1 & i \\
 i & 1 \end{bmatrix}.$$ 

**Proof.** The result follows immediately from Corollary 1.11 and Theorem 2.2. \[\blacksquare\]
Let

\[ F_{2k} = \begin{bmatrix} 0 & B_k \\ B_k & 0 \end{bmatrix} \in M_{2k}. \]

Then it is simple to check that a real, symmetric, and involutory matrix \( F_{2k} \) commutes with \( S_k \oplus S_k \) and \( P_k \oplus P_k \).

Now, note from (i), (ii), and (iii) of step 4 that

\[ B_k H_k(\lambda) B_k = B_k \bar{P}_k J_k(\lambda) \bar{P}_k B_k = \bar{P}_k B_k J_k(\lambda) B_k \bar{P}_k \]
\[ = \bar{P}_k J_k^T(\lambda) \bar{P}_k = H_k^T(\lambda) = \overline{H_k(\lambda)}, \]

\[ F_{2k} K_{2k}(\mu) F_{2k} = F_{2k} \left( \frac{S_k \oplus S_k}{N_{2k}(\mu)} \right) F_{2k} \]
\[ = \left( \frac{S_k \oplus S_k}{N_{2k}(\mu)} \right) F_{2k} \]
\[ = \left( \frac{S_k \oplus S_k}{N_{2k}(\mu)} \right) = K_{2k}(\mu) = \overline{K_{2k}(\mu)}. \]

Since \( B_k \) and \( F_{2k} \) are real, symmetric, and involutory matrices, the following is immediate.

\section*{Corollary 2.8.} \textit{Let} \( J_H(A) \) \textit{be a Hermitian canonical form of} \( A \in M_n \) \textit{under consimilarity. Then} \( J_H(A) \) \textit{is consimilar to} \( J_H(A) \) \textit{via a real, symmetric, and involutory matrix.}

An interesting consequence of Theorem 2.2 is that there is nothing special about Hermitian matrices under consimilarity, just as there is nothing special about symmetric matrices under similarity. Every consimilarity class in \( M_n \) contains at least one Hermitian matrix; in fact, it contains a Hermitian concanonical form. Thus, every antilinear transformation on \( \mathbb{C}^n \) has a Hermitian basis representation. Another consequence of Theorem 2.2 is that any matrix is real diagonalizable by a mixture of similarity and consimilarity.
COROLLARY 2.9. Let $A \in M_n$ be given. There are nonsingular $P_1, P_2 \in M_n$ and unitary $U_1, U_2 \in M_n$ such that

$$U_1^* \left( P_1^{-1} A P_1 \right) U_1 \quad \text{and} \quad U_2^* \left( P_2^{-1} A P_2 \right) U_2$$

are real diagonal matrices.

Proof. We only need to note that $A \in M_n$ is always consimilar to a Hermitian matrix, since a Hermitian matrix is real diagonalizable by unitary similarity, and that $A \in M_n$ is always similar to a symmetric matrix, since a symmetric matrix is real diagonalizable by unitary congruence.

If $J_c(A)$ is a concanonical form of $A \in M_n$ then $J_c(A)J_c(A)$ is always similar to a Jordan canonical form of $AA$. Between $J_H(A)$ and the symmetric Jordan canonical form $J_s(AA)$, the following holds.

COROLLARY 2.10. Let $J_H(A)$ be a Hermitian canonical form of $A \in M_n$ under consimilarity. Let $J_s(AA)$ be the symmetric Jordan canonical form of $AA \in M_n$. Then $J_s(AA)$ is orthogonally similar to $J_H(A)J_H(A)$.

Proof. $J_H(A)J_H(A) \overset{\text{sym}}{\sim} J_c(A)J_c(A) \overset{\text{sym}}{\sim} J_s(AA)$. Thus, $J_H(A)J_H(A) \overset{\text{sym}}{\sim} J_s(AA)$. But since $J_H(A)J_H(A)$ is symmetric, $J_s(AA)$ is orthogonally similar to $J_H(A)J_H(A)$ by Theorem 1.1.

The following result is shown by Hua [8]. We give a simple proof.

COROLLARY 2.11. Let $B \in M_n$ be a given nonsingular and symmetric matrix. Then there is a Hermitian $H \in M_n$ such that $B = HH$ if and only if

(a) each negative eigenvalue of $B$ has even algebraic multiplicity, and
(b) any complex eigenvalues of $B$ occur only in complex conjugate pairs.

Proof. The forward implication is immediate by Proposition 1.8. Conversely, suppose a nonsingular symmetric $B \in M_n$ satisfies (a) and (b). By Proposition 1.8, $B = AA$ for some nonsingular $A \in M_n$. By Theorem 2.2, $B = P^{-1}J_H(A)PP^{-1}J_H(A)P = P^{-1}J_H(A)J_H(A)P$. Since $J_H(A)J_H(A)$ is symmetric, there is an orthogonal $Q \in M_n$ such that $B = Q^TJ_H(A)J_H(A)Q = \left[Q^TJ_H(A)Q\right]\left[Q^TJ_H(A)Q\right]$. The result follows if we set $H = Q^TJ_H(A)Q$. 

Let $A \in M_n$. Then $A \sim A^* \sim A^T \sim A^{-1}$ [4]. Bevis, Hall, and Hartwig [2] showed that $A$ is consimilar to $A^T$ via a Hermitian matrix and $A$ is consimilar to $A^*$ via a symmetric matrix.

**Corollary 2.12.** For $A \in M_n$,

1. $A$ is consimilar to $A^*$ via a symmetric matrix, and $A$ is consimilar to $-A^*$ via a symmetric matrix;
2. $A$ is consimilar to $A^T$ via a Hermitian matrix, and $A$ is consimilar to $-A^T$ via a skew Hermitian matrix;
3. $A$ is consimilar to a real matrix;
4. $A$ is consimilar to $A^{-1}$ via a coninvoluntary matrix.

**Proof.** By Theorem 2.2 there is a nonsingular $R \in M_n$ such that $R^{-1}AR = J_H(A) = J_H(A)^* = RTA^*R^*^{-1}$. Thus, $A = (RR^T)A^*(RR^T)^{-1} = (iRR^T) - A^*(iRR^T)^{-1}$, where both matrices $RR^T$ and $iRR^T$ are nonsingular and symmetric. Therefore, (i) holds.

By Corollary 2.8, $R^{-1}AR = J_H(A) = EJ_H(A)E - EJ_H(A)^TE$, where $E \in M_n$ is real, symmetric, and involutory. Therefore, $R^{-1}AR = EJ_H(A)^TE = ER^*A^T(R^T)^{-1}E$. Thus, $A = (RER^*)A^*(RER^*)^{-1} = (iRER^*) - A^*(iRER^*)^{-1}$, where $RER^*$ is nonsingular Hermitian and $iRER^*$ is nonsingular skew Hermitian. Thus, (ii) holds.

Since $EJ_H(A)E = J_H(A)$, where $E \in M_n$ is real involutory (thus coninvoluntary), by Lemma 1.2 there is a nonsingular $P \in M_n$ such that $P^{-1}P = E$, and hence $P^{-1}J_H(A)P = P^{-1}J_H(A)P = (P^{-1}J_H(A)P)$. Therefore, (iii) holds, since $P^{-1}J_H(A)P$ is a real matrix that is consimilar to $A$.

Assertion (iv) follows immediately from (iii). Since there is a nonsingular $R \in M_n$ such that $R^{-1}AR$ is a real matrix, $R^{-1}AR = (R^{-1}AR) = R^{-1}AR$. Thus, $RR^{-1}AR^{-1} = A$ where $(RR^{-1})(RR^{-1}) = I$.

3. **A REAL CANONICAL FORM UNDER CONSIMILARITY**

By (iii) of Corollary 2.12, any $A \in M_n$ is consimilar to a real matrix. This result seems to be due to Asano and Nakayama [1], and is also obtained in [4] using a different approach. In this section, we produce an explicit real concanonical form of $A$ using a procedure that is similar to the one we used in Section 2. Again, the procedure is algorithmic and completely elementary.
**Lemma 3.1.** Let \( A \in M_n \). If \( A \) is consimilar to a real matrix, then there is a coninvolutory \( E \in M_n \) such that \( EAE = \bar{A} \). Conversely, if there is a coninvolutory \( E \in M_n \) such that \( EAE = \bar{A} \), then for any nonsingular \( P \in M_n \) such that \( \bar{PP}^{-1} = E \), \( P^{-1}AP \) is a real matrix.

**Proof.** Suppose there is a real matrix \( R \in M_n \) such that \( P^{-1}AP = R \). Then \( \bar{P}^{-1}A\bar{P} = \bar{R} = R \). Thus, \( P^{-1}AP = \bar{P}^{-1}A\bar{P} \), and hence \( \bar{PP}^{-1} = E \) where \( \bar{P}P^{-1} \) is coninvolutory.

Conversely, suppose there is a coninvolutory \( E \in M_n \) such that \( EAE = \bar{A} \). Then let \( P \in M_n \) be nonsingular and such that \( \bar{PP}^{-1} = E \). The existence of such a \( P \) is guaranteed by Lemma 1.2. Thus, \( \bar{PP}^{-1}A\bar{PP}^{-1} = \bar{A} \), or \( P^{-1}AP = \bar{PP}^{-1}A \bar{PP}^{-1} = (P^{-1}AP) \). Therefore, \( P^{-1}AP \) is real.

Now let \( A \in M_n \) be given. By (iv) of Corollary 2.11 there always exists a coninvolutory \( E \in M_n \) such that \( EAE = A \). Therefore, if we can find a nonsingular \( P \in M_n \) such that \( \bar{PP}^{-1} = E \), then \( P^{-1}AP \) is a real matrix by Lemma 3.1. Thus, consider \( J_\sigma(A) = J_\sigma \oplus Q_N \oplus Q_C \), the concanonical form of \( A \). Since \( J_\sigma \) and \( Q_N \) are real matrices, we only need to consider \( Q_C \), a quasi-Jordan matrix corresponding to the complex nonreal eigenvalues of \( AA \): \( Q_C = C_{2k_1}(\xi_1) \oplus \cdots \oplus C_{2k_t}(\xi_t) \), where each \( \xi_i \in \mathbb{C} \) is nonreal, \( \xi_i \) is a nonreal eigenvalue of \( AA \), and

\[
C_{2k_1}(\xi_i) = \begin{bmatrix} 0 & J_{k}(\xi_i) \\ J_{k}(\xi_i) & 0 \end{bmatrix}
\]

is such that

\[
C_{2k_1}(\xi_i) \overline{C_{2k_1}(\xi_i)} = \begin{bmatrix} J^2_{k_1}(\xi_i) & 0 \\ 0 & J^2_{k_1}(\xi_i) \end{bmatrix} \in M_{2k_1}.
\]

Now let

\[
E_{2k} = \begin{bmatrix} 0 & I_k \\ I_k & 0 \end{bmatrix} \in M_{2k} \quad \text{and} \quad Z_{2k} = \begin{bmatrix} I_k & iI_k \\ I_k & -iI_k \end{bmatrix} \in M_{2k}.
\]

Then \( \overline{Z_{2k}}Z_{2k}^{-1} = E_{2k} \), where \( E_{2k} \) is a real, symmetric, and involutory matrix such that \( E_{2k}C_{2k}(\xi_i)E_{2k} = \overline{C_{2k}(\xi_i)} \). Therefore, the matrix \( Z_{2k}^{-1}C_{2k}(\xi_i)\overline{Z_{2k}} \) must be a real matrix by Lemma 3.1. Indeed, if \( \xi_i = a_i + ib_i \) for \( a_i, b_i \in \mathbb{R} \),
then consider the consimilarity

\[ Z_{2k}^{-1} C_{2k}(\xi_i) \overline{Z}_{2k} = \frac{1}{2} \begin{bmatrix} J_k(\xi_i) + J_k(\xi_i) & i(J_k(\xi_i) - J_k(\xi_i)) \\ i(J_k(\xi_i) - J_k(\xi_i)) & J_k(\xi_i) + J_k(\xi_i) \end{bmatrix} \]

\[ = \begin{bmatrix} J_k(a_i) & -b_iI_k \\ b_iI_k & J_k(a_i) \end{bmatrix} \in M_{2k}(\mathbb{R}). \]

**Theorem 3.2.** Let \( A \in M_n \), and let \( J_c(A) = J_P \oplus Q_N \oplus Q_C \) be a concanonical form of \( A \). Then \( A \) is consimilar to a real canonical form \( J_R(A) = J_P \oplus Q_N \oplus R_C \), where \( J_P \) and \( Q_N \) are Jordan and quasi-Jordan matrices corresponding to the nonnegative and negative eigenvalues of \( AA^* \), respectively, and \( R_C \) is a real matrix such that

\[ R_C = R_{2k_1}(\xi_1) \oplus \cdots \oplus R_{2k_s}(\xi_s), \quad \xi_i = a_i + ib_i, \quad a_i, b_i \in \mathbb{R} \text{ and } b_i \neq 0, \]

\[ R_{2k_i}(\xi_i) = \begin{bmatrix} J_k(a_i) & -b_iI_k \\ b_iI_k & J_k(a_i) \end{bmatrix} \in M_{2k_i}(\mathbb{R}). \]

Let \( R_{2m}(\xi) \in M_{2m} \) be given. It is interesting to note that if we construct a permutation matrix \( P \in M_{2m} \) whose first \( m \) rows are \( e_1, e_3, e_5, \ldots \) and whose last \( m \) rows are \( e_2, e_4, e_6, \ldots \), then the consimilarity \( P^T R_{2m}(\xi) P \) yields a form that is identical to the blocks of a real Jordan canonical form of a real matrix under an ordinary similarity, i.e.,

\[ P^T R_{2m}(\xi) P = \begin{bmatrix} C(a, b) & I_2 & & & \\ & C(a, b) & I_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & I_2 \\ & & & & C(a, b) \end{bmatrix}, \]

where

\[ C(a, b) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in M_2(\mathbb{R}) \]
and $\xi = a + ib$, $a, b \in \mathbb{R}$. Thus, if $\mathbf{A}\mathbf{\bar{A}}$ has no negative eigenvalues, then $\mathbf{A}$ is consimilar to a real Jordan canonical form.

REFERENCES

5 Y. P. Hong and R. A. Horn, The Jordan canonical form for the product of a Hermitian matrix and a positive semidefinite matrix, to appear.

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