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Support-type properties of convex functions of higher order and Hadamard-type inequalities

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Abstract

It is well known that every convex function $f : I \to \mathbb{R}$ (where $I \subset \mathbb{R}$ is an interval) admits an affine support at every interior point of I (i.e. for any $x_0 \in \operatorname{Int} I$ there exists an affine function $a : I \to \mathbb{R}$ such that $a(x_0) = f(x_0)$ and $a \leqslant f$ on I). Convex functions of higher order (precisely of an odd order) have a similar property: they are supported by the polynomials of degree no greater than the order of convexity. In this paper the attaching method is developed. It is applied to obtain the general result—Theorem 2, from which the mentioned above support theorem and some related properties of convex functions of higher (both odd and even) order are derived. They are applied to obtain some known and new Hadamard-type inequalities between the quadrature operators and the integral approximated by them. It is also shown that the error bounds of quadrature rules follow by inequalities of this kind.

Keywords: Approximate integration; Error bounds; Hadamard inequality; Higher-order convexity; Quadrature rules; Support theorems

1. Introduction

Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. For distinct points of I the *divided differences* of f are defined recursively as follows: $[x_1, f] := f(x_1)$ and

$$[x_1, \dots, x_{n+1}; f] := \frac{[x_2, \dots, x_{n+1}; f] - [x_1, \dots, x_n; f]}{x_{n+1} - x_1}, \quad n \in \mathbb{N},\ n \geqslant 2. \tag{1}$$

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For n distinct points $x_1, \dots, x_n \in I$ ($n \geqslant 2$) the following formula holds true:

$$[x_1, \dots, x_n; f] = \frac{D(x_1, \dots, x_n; f)}{V(x_1, \dots, x_n)}, \tag{2}$$

where

$$D(x_1, \dots, x_n; f) := \begin{vmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_n \\ \vdots & & \vdots \\ x_1^{n-2} & \dots & x_n^{n-2} \\ f(x_1) & \dots & f(x_n) \end{vmatrix}$$

and $V(x_1, \dots, x_n)$ stands for the Vandermonde determinant of the terms involved. By (2) we can immediately see that the divided differences are symmetric.

Let Π_n be the family of all polynomials of degree at most n. For k distinct points $x_1, \dots, x_k \in I$ denote by $P(x_1, \dots, x_k; f)$ the (unique) interpolation polynomial $p \in \Pi_{k-1}$ such that $p(x_i) = f(x_i)$, $i = 1, \dots, k$. Then for any $n+1$ distinct points $x_1, \dots, x_{n+1} \in I$ and for any $x \in I \setminus \{x_1, \dots, x_{n+1}\}$ we have

$$f(x) - P(x_1, \dots, x_{n+1}; f)(x) = [x_1, \dots, x_{n+1}, x; f] \prod_{i=1}^{n+1} (x - x_i). \tag{3}$$

For the definition and properties of divided differences the reader is referred to [7,8,10].

If $x_1, \dots, x_{n+1} \in I$ are distinct then Newton's Interpolation Formula holds:

$$\begin{aligned} P(x_1, \dots, x_{n+1}; f)(x) = f(x_1) &+ [x_1, x_2; f](x - x_1) + \cdots \\ &+ [x_1, \dots, x_{n+1}; f](x - x_1) \cdots (x - x_n). \end{aligned} \tag{4}$$

Next we recall the notion of convex functions of higher order. Hopf's thesis [6] from 1926 seems to be the first work devoted to this topic (the functions with nonnegative divided differences were considered but the name "convex functions of higher order" was not used). Eight years later higher-order convexity was extensively studied by Popoviciu [8] (cf. also [7,10]). Let $n \in \mathbb{N}$. A function $f : I \to \mathbb{R}$ is called *n-convex* if $[x_1, \dots, x_{n+2}; f] \geqslant 0$ for any $n+2$ distinct points $x_1, \dots, x_{n+2} \in I$. It follows by (2) that f is n-convex if and only if

$$D(x_1, \dots, x_{n+2}; f) \geqslant 0 \tag{5}$$

for any $x_1, \dots, x_{n+2} \in I$ with $x_1 < \cdots < x_{n+2}$ (since $V(x_1, \dots, x_{n+2}) > 0$).

For $n = 1$ it is not difficult to observe that the n-convexity reduces to convexity in the usual sense.

By (3) we obtain the following important property of convex functions of higher order (cf. [7,8,10]): a function $f : I \to \mathbb{R}$ is n-convex if and only if for any $x_1, \dots, x_{n+1} \in I$ with $x_1 < \cdots < x_{n+1}$ the graph of an interpolation polynomial $p := P(x_1, \dots, x_{n+1}; f)$ passing through the points $(x_i, f(x_i))$, $i = 1, \dots, n+1$, changes successively the side of the graph of f (always $p(x) \leqslant f(x)$ for $x \in I$ such that $x > x_{n+1}$, if such points do exist). More precisely,

$$\begin{aligned} &(-1)^{n+1}\big(f(x) - p(x)\big) \geqslant 0, \quad x < x_1,\ x \in I, \\ &(-1)^{n+1-i}\big(f(x) - p(x)\big) \geqslant 0, \quad x_i < x < x_{i+1},\ i = 1, \dots, n, \\ &f(x) - p(x) \geqslant 0, \quad x > x_{n+1},\ x \in I. \end{aligned} \tag{6}$$

The theorem below contains another property of higher-order convexity (cf. [7, p. 391, Corollary 1], [8, p. 27]).

Theorem A. *If $f : I \to \mathbb{R}$ is n-convex then for any $k \in \{1, \dots, n+1\}$ the divided differences $[x_1, \dots, x_k; f]$ are bounded on every compact interval $[a, b] \subset \operatorname{Int} I$.*

Convex functions of higher order have the following regularity property (cf. [3,7,8]):

Theorem B. *If $f : [a, b] \to \mathbb{R}$ is n-convex then f is continuous on (a, b) and bounded on $[a, b]$.*

All the integrals that appear in this paper are understood in the sense of Riemann. Then by Theorem B we obtain

Theorem C. *If $f : [a, b] \to \mathbb{R}$ is n-convex then f is integrable on $[a, b]$.*

For n-convex functions which are $(n+1)$-times differentiable the following result holds (cf. [12, Theorems A and B], [13, Theorems 1.2 and 1.3], cf. also [7,8,10]):

Theorem D. *Assume that $f : [a, b] \to \mathbb{R}$ is $(n+1)$-times differentiable on (a, b) and continuous on $[a, b]$. Then f is n-convex if and only if $f^{(n+1)}(x) \geqslant 0$, $x \in (a, b)$.*

It is well known that every convex function $f : I \to \mathbb{R}$ admits an affine support at every interior point of I (i.e. for any $x_0 \in \operatorname{Int} I$ there exists an affine function $a : I \to \mathbb{R}$ such that $a(x_0) = f(x_0)$ and $a \leqslant f$ on I). Convex functions of higher order (precisely of an odd order) have a similar property: they are supported by the polynomials of degree no greater than the order of convexity. Such a result was obtained by Ger [5], who assumed that the supported n-convex function, defined on an open and convex subset of a normed space, was of the class $\mathcal{C}^{n+1}$. In this paper we develop the attaching method and we use it to prove in Theorem 2 a support-type result of a general nature. As almost immediate consequences we obtain the result improving Ger's theorem (we remove the differentiability assumption) for functions defined on a real interval and more support-type properties of convex functions of higher (both odd and even) order.

In the theory of convex functions an important role is played by the famous Hermite–Hadamard inequality. It states that if $f : [a, b] \to \mathbb{R}$ is convex then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}. \tag{7}$$

The interesting study of this inequality and lots of related inequalities was given by Dragomir and Pearce [4]. In this paper we apply the above mentioned support-type properties of convex functions of higher order to obtain both known and new Hadamard-type inequalities between the quadrature operators and the integral approximated by them. We also show that the error bounds of quadrature rules follow by inequalities of this kind.

2. Attaching method

In this section we describe this method. Consider the n-convex function $f : I \to \mathbb{R}$ and take the polynomial $p \in \Pi_n$ interpolating f at $n+1$ distinct points of I. Figure 1 is drawn for

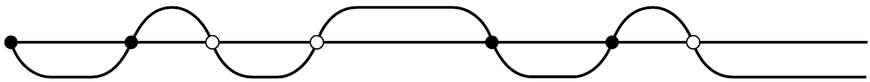

Fig. 1. The “bubbles” are attached to the “bullets.”

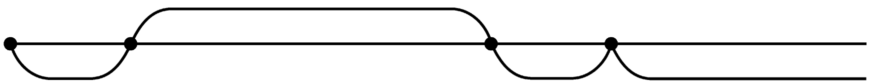

Fig. 2. The degree (its upper bound) of the polynomial is preserved.

the 6-convex function. The graph of f is represented by the horizontal straight line. Then by n-convexity the graph of p (symbolized by the curve line) meeting the graph of f changes successively its side. If the “bubbles” lying between two consecutive “bullets” tend to the nearest “bullet” situated on the left-hand side of them, then the appropriate interpolation polynomials belonging to Π_n (the graph of one of them is shown at Fig. 1) tend to some polynomial belonging to Π_n. Then we arrive at the situation that we can see at Fig. 2. Figures 1 and 2 have only an explanatory character. The curve lines are not really the graphs of polynomials and the straight line is not really the graph of f. They illustrate only the location of graphs of appropriate polynomials on a proper side of the graph of f.

3. Support-type theorem

Now we are going to prove a support-type theorem of the general nature. In the proof we would like to use the boundedness of divided differences of an n-convex function. By Theorem A this is the case when all the points involved belong to the compact subinterval of $\operatorname{Int} I$. However, we need also this property for divided differences involving additionally the boundary points of I (if they do exist). That is why we prove below the following lemma.

Lemma 1. *Let $n \in \mathbb{N}$, $A \subset \mathbb{R}$, $a \notin \operatorname{cl} A$ and $f : A \cup \{a\} \to \mathbb{R}$. If for any $k \in \{1, \dots, n\}$ the divided differences $[x_1, \dots, x_k; f]$ are bounded on A then for any $k \in \{1, \dots, n\}$ they remain bounded on $A \cup \{a\}$.*

Proof. For $k = 1$ there is nothing to prove. For $k > 1$ assume that the divided differences $[x_1, \dots, x_{k-1}; f]$ are bounded on $A \cup \{a\}$. To finish the proof it is enough to show the assertion for k-point divided differences containing a. To proceed this job take $x_1, \dots, x_{k-1} \in A$. Then by (1)

$$\big|[a, x_1, \dots, x_{k-1}; f]\big| = \frac{|[x_1, \dots, x_{k-1}; f] - [a, x_1, \dots, x_{k-2}; f]|}{|x_{k-1} - a|} \leqslant \frac{2M}{|a - A|},$$

where

$$M := \sup\big\{\big|[x_1, \dots, x_{k-1}; f]\big| \colon x_1, \dots, x_{k-1} \in A \cup \{a\}\big\},$$
$$|a - A| := \inf\big\{|a - b| \colon b \in A\big\} > 0 \quad (\text{since } a \notin \operatorname{cl} A). \qquad \square$$

Now we are ready to prove the main result of this section.

Theorem 2. *Let $n \in \mathbb{N}$ and $f: I \to \mathbb{R}$ be an n-convex function. Fix $k \in \mathbb{N}$, $k \leqslant n$, and take $x_1, \ldots, x_k \in I$ such that $x_1 < \cdots < x_k$. Assign to each point x_j $(j = 1, \ldots, k)$ the multiplicity $l_j \in \mathbb{N}$ ($l_j - 1$ stands for the number of points attached to x_j). We require $l_1 + \cdots + l_k = n + 1$ and if $x_1 = \inf I$ then $l_1 = 1$, if $x_k = \sup I$ then $l_k = 1$ (the points can be attached only to the interior points of I). Denote $I_0 = (-\infty, x_1)$, $I_j = (x_j, x_{j+1})$, $j = 1, \ldots, k-1$, and $I_k = (x_k, \infty)$. Under these assumptions there exists a polynomial $p \in \Pi_n$ such that $p(x_j) = f(x_j)$, $j = 1, \ldots, k$, and*

$$\begin{aligned} &(-1)^{n+1}\big(f(x) - p(x)\big) \geqslant 0 \quad \textit{for } x \in I_0 \cap I, \\ &(-1)^{n+1-(l_1+\cdots+l_j)}\big(f(x) - p(x)\big) \geqslant 0 \quad \textit{for } x \in I_j,\ j = 1, \ldots, k-1, \\ &f(x) - p(x) \geqslant 0 \quad \textit{for } x \in I_k \cap I. \end{aligned} \tag{8}$$

Before we start the proof let us notice that at Figs. 1 and 2 we have $n = 6$, $k = 4$ (the “bubbles” were attached to the “bullets”) and the multiplicities of the “bullets” are (from left to right) 1, 3, 1, 2, respectively.

Proof of Theorem 2. Let $m \in \mathbb{N}$. If $l_j > 1$ $(j = 1, \ldots, k)$, we take the points

$$x_j < x_j + \frac{1}{m} < \cdots < x_j + \frac{l_j - 1}{m}.$$

For m large enough all these points belong to I_j. Then the sequence

$$\left(x_1, x_1 + \frac{1}{m}, \ldots, x_1 + \frac{l_1 - 1}{m}, x_2, \ldots, x_k, x_k + \frac{1}{m}, \ldots, x_k + \frac{l_k - 1}{m}\right) \tag{9}$$

is increasingly ordered and it contains $n + 1$ points of I (because of $l_1 + \cdots + l_k = n + 1$). There exists a polynomial $p_m \in \Pi_n$ interpolating f at the points of the sequence (9). We use Newton’s Interpolation Formula (4) to write p_m. This formula contains the products of binomials of the form

$$x - x_j - \frac{s_j}{m}, \quad j = 1, \ldots, k,\ s_j = 0, \ldots, l_j - 1\ (k \leqslant n),$$

and the divided differences involving points of the sequence (9). By Theorem A and Lemma 1 the sequences of these differences containing 1 point, 2 points, $\ldots$, $n + 1$ points, respectively, are bounded and for that reason they contain the convergent subsequences. By taking (if needed) the common subsequence (α_m) of positive integers we may assume without loss of generality that all these sequences are convergent.

Let $x \in I$ and $p(x) := \lim_{m\to\infty} p_m(x)$. Then $p \in \Pi_n$ and by the construction we have $p(x_j) = f(x_j)$, $j = 1, \ldots, k$.

Let $x \in I_0 \cap I$ (if $x \in I_0 \cap I \neq \emptyset$). Then $x < x_1$ and by n-convexity and (6) $(-1)^{n+1}(f(x) - p_m(x)) \geqslant 0$ for m large enough. Tending with m to infinity we get $(-1)^{n+1}(f(x) - p(x)) \geqslant 0$.

Let $x \in I_k \cap I$ (if $x \in I_k \cap I \neq \emptyset$). Then for m large enough we have

$$x > x_k + \frac{l_k - 1}{m}.$$

We infer by (6) that $f(x) - p_m(x) \geqslant 0$, whence letting $m \to \infty$ we obtain $f(x) - p(x) \geqslant 0$.

Finally let $x \in I_j$, $j = 1, \ldots, k-1$. For m large enough we have

$$x_j + \frac{l_j - 1}{m} < x < x_{j+1}.$$

Observe that the point $x_j + \frac{l_j-1}{m}$ has in the sequence (9) the number $l_1 + \cdots + l_j$. Therefore by (6) $(-1)^{n+1-(l_1+\cdots+l_j)}(f(x) - p_m(x)) \geqslant 0$. For $m \to \infty$ we get $(-1)^{n+1-(l_1+\cdots+l_j)}(f(x) - p(x)) \geqslant 0$, which finishes the proof. □

Remark 3. In the classical setting, if $f : I \to \mathbb{R}$ admits at each point $x_0 \in \operatorname{Int} I$ an affine support, then f is convex. This is also the case for the statement of Theorem 2: it characterizes n-convexity. Indeed, to prove that f is n-convex it is enough to assume that the appropriate polynomial exists for $k = n$, $x_1, \ldots, x_n \in \operatorname{Int} I$ with $x_1 < \cdots < x_n$ and $l_1 = \cdots = l_{n-1} = 1$, $l_n = 2$. This is shown by the present author in [14, Theorem 3] in a more general setting, i.e. for convex functions with respect to Chebyshev systems (for a polynomial Chebyshev system $(1, x, \ldots, x^n)$ such a convexity reduces to n-convexity). We have formulated Theorem 2 in the form of the necessary condition since, as we can see, the sufficient condition can be weakened.

Remark 4. The polynomial obtained in Theorem 2 need not to be unique. Let $n = k = 3$. Then for a 3-convex function $f : \mathbb{R} \to \mathbb{R}$, $x_1 = -1$, $x_2 = 0$, $x_3 = 1$ and $l_1 = l_3 = 1$, $l_2 = 2$, by Theorem 2 there exists a polynomial $p \in \Pi_3$ such that $p(-1) = f(-1)$, $p(0) = f(0)$, $p(1) = f(1)$ and $p(x) \leqslant f(x)$ for $|x| > 1$, $p(x) \geqslant f(x)$ for $0 < |x| < 1$. Observe that for a 3-convex function $f(x) = x^4$, $x \in \mathbb{R}$, this assertion is fulfilled by $p_1(x) = 1$ and by $p_2(x) = x^2$.

Remark 5. The assumption $l_1 = 1$ if $x_1 = \inf I$ is essential. For $n = k = 1$ and $l_1 = 2$ Theorem 2 asserts that a convex function $f : I \to \mathbb{R}$ has an affine support at a point x_1. If x_1 is a boundary point of I it need not to be the case. Observe that a convex function $f(x) = -\sqrt{1 - x^2}$, $x \in [-1, 1]$, has no affine support both at $x_1 = -1$ and at $x_1 = 1$.

4. Some consequences of Theorem 2

We start with the support theorem for convex functions of an odd order.

Corollary 6. *Let $n \in \mathbb{N}$ be an odd number and $f : I \to \mathbb{R}$ be an n-convex function. Then for any $x_1 \in \operatorname{Int} I$ there exists a polynomial $p \in \Pi_n$ such that $p(x_1) = f(x_1)$ and $p \leqslant f$ on I.*

Proof. In Theorem 2 take $k = 1$ and $l_1 = n + 1$. □

This result needs at least two comments.

1. The support theorem for convex functions of an odd order was proved by Ger [5] with the additional assumption that the supported function is of the class $\mathcal{C}^{n+1}$. However, Ger's result holds for functions defined on an open and convex subset of a normed space. Notice at this place that if $f : I \to \mathbb{R}$ is n-convex then f is of the class $\mathcal{C}^{n-1}$ on $\operatorname{Int} I$ (cf. [7,8]). Better regularity properties must be assumed (e.g. for $n = 1$, $f(x) = |x|$ is convex, continuous and not differentiable).
2. The attaching method gives an answer to the question why convex functions of an even order need not to admit polynomial supports at every interior point of I. Namely, for an n-convex function $f : I \to \mathbb{R}$ and $x_1 \in \operatorname{Int} I$, if n is an even number then the suitable interpolation polynomial p (constructed as in the proof of Corollary 6) fulfills by (8) the inequality $p(x) \geqslant f(x)$, $x \in I$, $x < x_1$. It is easy to give an example: the 2-convex function $f(x) = x^3$ must not be supported on $\mathbb{R}$ by any quadratic polynomial (cf. [5]). However, there are some situations when convex functions of an even order do admit polynomial supports (cf. Corollary 10 below).

Remark 7. Figures 1 and 2 show that the graph of a polynomial $p \in \Pi_n$ obtained by Theorem 2 may be situated on both sides of the graph of an n-convex function f. Another possibility which may occur is that the graph of p may be situated above the graph of f, contrary to the support property (see Corollaries 9 and 11 below).

Corollary 8. *If $f:[a,b] \to \mathbb{R}$ is $(2n-1)$-convex and $x_1, \ldots, x_n \in (a,b)$, then there exists a polynomial $p \in \Pi_{2n-1}$ such that $p(x_i) = f(x_i)$, $i = 1, \ldots, n$, and $p \leqslant f$ on $[a,b]$.*

Proof. Assuming that $x_1 < \cdots < x_n$ use Theorem 2 for $2n-1$ instead of n, $k = n$ and $l_1 = \cdots = l_n = 2$. $\square$

Corollary 9. *If $f:[a,b] \to \mathbb{R}$ is $(2n-1)$-convex and $x_1 = a$, $x_2, \ldots, x_n \in (a,b)$, $x_{n+1} = b$, then there exists a polynomial $p \in \Pi_{2n-1}$ such that $p(x_i) = f(x_i)$, $i = 1, \ldots, n+1$, and $p \geqslant f$ on $[a,b]$.*

Proof. Use Theorem 2 for $2n-1$ instead of n, $k = n+1$, $l_1 = 1$, $l_2 = \cdots = l_n = 2$, $l_{n+1} = 1$. $\square$

Corollary 10. *If $f:[a,b] \to \mathbb{R}$ is $2n$-convex, $x_1 = a$, $x_2, \ldots, x_{n+1} \in (a,b)$, then there exists a polynomial $p \in \Pi_{2n}$ such that $p(x_i) = f(x_i)$, $i = 1, \ldots, n+1$, and $p \leqslant f$ on $[a,b]$.*

Proof. Use Theorem 2 for $2n$ instead of n, $k = n+1$ and $l_1 = 1$, $l_2 = \cdots = l_{n+1} = 2$. $\square$

Corollary 11. *If $f:[a,b] \to \mathbb{R}$ is $2n$-convex, $x_1, \ldots, x_n \in (a,b)$ and $x_{n+1} = b$, then there exists a polynomial $p \in \Pi_{2n}$ such that $p(x_i) = f(x_i)$, $i = 1, \ldots, n+1$, and $p \geqslant f$ on $[a,b]$.*

Proof. Use Theorem 2 for $2n$ instead of n, $k = n+1$ and $l_1 = \cdots = l_n = 2$, $l_{n+1} = 1$. $\square$

5. Hadamard-type inequalities

In this section we obtain some inequalities between the quadrature operators and the integral approximated by them. The classical inequality of this kind is the celebrated Hermite–Hadamard inequality (7).

Orthogonal polynomials

Let $w:[a,b] \to [0,\infty)$ be an integrable function such that $\int_a^b w(x)\,dx > 0$. The function w is called the *weight function*. Then $\langle f, g\rangle_w := \int_a^b f(x)g(x)w(x)\,dx$ is the inner product in the space of all integrable functions $f:[a,b] \to \mathbb{R}$. Performing for the sequence of monomials $(1, x, x^2, \ldots)$ the Gramm–Schmidt orthogonalization procedure we obtain the sequence (P_n) of polynomials orthogonal to each other on $[a,b]$ with the weight w (i.e. with respect to the above inner product). Let P_n be the member of this sequence of degree n. The well-known results from numerical analysis (cf. e.g. [9,11]) state that the polynomial P_n has n distinct zeros belonging to (a,b).

Gauss quadratures

Let (P_n) be the sequence of polynomials orthogonal to each other on $[a,b]$ with the weight function w and let $x_1, \ldots, x_n$ be the zeros of the polynomial P_n. Furthermore, let

$$w_i := \int_a^b \frac{P_n(x)w(x)}{(x-x_i)P_n'(x_i)}\,dx, \quad i=1,\dots,n,$$

$$\mathcal{G}_n(f) := \sum_{i=1}^n w_i f(x_i).$$

It is well known from numerical analysis (cf. e.g. [2,9,11,16]) that the equation

$$\int_a^b f(x)w(x)\,dx = \mathcal{G}_n(f)$$

holds for all polynomials belonging to Π_{2n-1}. If $[a,b]=[-1,1]$ and $w\equiv 1$ then $\mathcal{G}_n$ is the n-point Gauss–Legendre quadrature (cf. [9,17]).

Lobatto-type quadratures

Let (Q_n) be the sequence of polynomials orthogonal to each other on $[a,b]$ with the weight function $(x-a)(b-x)w(x)$ and let $x_1,\dots,x_{n-1}$ be the zeros of the polynomial Q_{n-1} (where Q_{n-1} is the member of this sequence of degree $n-1$). Furthermore, let

$$w_0 := \frac{1}{(b-a)Q_{n-1}^2(a)}\int_a^b Q_{n-1}^2(x)(b-x)w(x)\,dx,$$

$$w_i := \frac{1}{(b-x_i)(x_i-a)}\int_a^b \frac{Q_{n-1}(x)(x-a)(b-x)w(x)}{(x-x_i)Q_{n-1}'(x_i)}\,dx,$$

$$w_n := \frac{1}{(b-a)Q_{n-1}^2(b)}\int_a^b Q_{n-1}^2(x)(x-a)w(x)\,dx,$$

$$\mathcal{L}_{n+1}(f) := w_0 f(a) + \sum_{i=1}^{n-1} w_i f(x_i) + w_n f(b).$$

It is well known from numerical analysis (cf. e.g. [2,9]) that the equation

$$\int_a^b f(x)w(x)\,dx = \mathcal{L}_{n+1}(f)$$

holds for all polynomials belonging to Π_{2n-1}. If $[a,b]=[-1,1]$ and $w\equiv 1$ then $\mathcal{L}_{n+1}$ is the $(n+1)$-point Lobatto quadrature (cf. [9,18]).

Inequalities for Gauss quadratures and Lobatto-type quadratures

Proposition 12. *If $f:[a,b]\to\mathbb{R}$ is $(2n-1)$-convex then*

$$\mathcal{G}_n(f) \leqslant \int_a^b f(x)w(x)\,dx \leqslant \mathcal{L}_{n+1}(f).$$

Proof. By Theorem C, f is integrable on $[a,b]$. Let $x_1,\dots,x_n$ be the abscissas of the quadrature rule $\mathcal{G}_n$. By Corollary 8 there exists a polynomial $p\in\Pi_{2n-1}$ such that $p(x_i)=f(x_i)$, $i=1,\dots,n$, and $p\leqslant f$ on $[a,b]$. Then $\mathcal{G}_n(p)=\mathcal{G}_n(f)$ and by $w\geqslant 0$,

$$\int_a^b p(x)w(x)\,dx\leqslant\int_a^b f(x)w(x)\,dx.$$

Since the quadrature $\mathcal{G}_n$ is precise for polynomials belonging to Π_{2n-1}, then

$$\mathcal{G}_n(f)=\mathcal{G}_n(p)=\int_a^b p(x)w(x)\,dx\leqslant\int_a^b f(x)w(x)\,dx.$$

The second inequality we prove similarly taking as $x_1,\dots,x_{n+1}$ the abscissas of the quadrature rule $\mathcal{L}_{n+1}$ and using Corollary 9. □

Radau-type quadratures

Let (P_n) be the sequence of polynomials orthogonal to each other on $[a,b]$ with the weight function $(x-a)w(x)$ and let $x_1,\dots,x_n$ be the zeros of the polynomial P_n. Furthermore, let

$$w_0:=\frac{1}{P_n^2(a)}\int_a^b P_n^2(x)w(x)\,dx,$$

$$w_i:=\frac{1}{(x_i-a)}\int_a^b\frac{P_n(x)(x-a)w(x)}{(x-x_i)P_n'(x_i)}\,dx,\quad i=1,\dots,n,$$

$$\mathcal{R}_{n+1}^l(f):=w_0f(a)+\sum_{i=1}^n w_if(x_i).$$

If $[a,b]=[-1,1]$ and $w\equiv 1$ then $\mathcal{R}_{n+1}^l$ is the $(n+1)$-point Radau quadrature (cf. [9,19]).

Let (Q_n) be the sequence of polynomials orthogonal to each other on $[a,b]$ with the weight function $(b-x)w(x)$ and let $x_1,\dots,x_n$ be the zeros of the polynomial Q_n. Furthermore, let

$$w_i:=\frac{1}{(b-x_i)}\int_a^b\frac{Q_n(x)(b-x)w(x)}{(x-x_i)Q_n'(x_i)}\,dx,\quad i=1,\dots,n,$$

$$w_{n+1}:=\frac{1}{Q_n^2(b)}\int_a^b Q_n^2(x)w(x)\,dx,$$

$$\mathcal{R}_{n+1}^r(f):=\sum_{i=1}^n w_if(x_i)+w_{n+1}f(b).$$

It is well known from numerical analysis (cf. e.g. [2,9,19]) that the equation

$$\mathcal{R}_{n+1}^l(f)=\int_a^b f(x)w(x)\,dx=\mathcal{R}_{n+1}^r(f)$$

holds for all polynomials belonging to Π_{2n}.

Inequalities for Radau-type quadratures

Proposition 13. *If $f:[a,b]\to\mathbb{R}$ is $2n$-convex, then*

$$\mathcal{R}^l_{n+1}(f)\leqslant\int\limits_a^b f(x)w(x)\,dx\leqslant\mathcal{R}^r_{n+1}(f).$$

Proof. The proof is similar to that of Proposition 12. For the first inequality use Corollary 10 for the abscissas of the quadrature rule $\mathcal{R}^l_{n+1}$ and for the second one use Corollary 11 for the abscissas of $\mathcal{R}^r_{n+1}$. □

6. Comments

1. The inequalities of Propositions 12 and 13 were earlier proved by Bessenyei and Páles [1–3].

2. In [2] these inequalities were proved for the weight function $w\equiv 1$ by the method of smoothing of convex functions of higher order. Namely, it is shown in [2, Theorem 5] that for an n-convex function $f:I\to\mathbb{R}$ and for any compact subinterval $J\subset\operatorname{Int}I$ there exists a sequence of n-convex functions of the $\mathcal{C}^\infty$ class convergent uniformly to f on J.

3. In more recent paper [3] Hadamard-type inequalities for convex functions with respect to Chebyshev systems are given. The results are proved for any weight function. The method of the proof was based on integration of the determinant defining the convexity of this kind. The paper [1] contains the same results. However, some assumption present in [3] was removed. Both quoted papers do not contain any results of support-type.

4. For some cases it is possible to give the inequalities of Hadamard-type which are better in some sense from the inequalities of Propositions 12 and 13. Some of them are presented in the next section.

7. Other Hadamard-type inequalities

In this section we consider real functions defined on $[-1,1]$ and the weight function $w\equiv 1$. In this setting

$$\mathcal{G}_2(f)=f\left(-\frac{\sqrt{3}}{3}\right)+f\left(\frac{\sqrt{3}}{3}\right),$$

$$\mathcal{L}_4(f)=\frac{1}{6}\big(f(-1)+f(1)\big)+\frac{5}{6}\left(f\left(-\frac{\sqrt{3}}{3}\right)+f\left(\frac{\sqrt{3}}{3}\right)\right).$$

The abscissas of these quadrature rules are the zeros of suitable orthogonal polynomials.

Remarks on even functions

1. If f is an even function then $\int_{-1}^1 f(x)\,dx=2\int_0^1 f(x)\,dx$.

2. If f is an n-convex function and n is an odd number then the function $f(-x)$ is also n-convex (cf. [8]). Then an even part of f, i.e. the function $f_e(x)=\frac{f(x)+f(-x)}{2}$, is n-convex as well.

3. Let f be the integrable function. Then $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} f_e(x)\,dx$. Indeed, since f_e is an even function we have

$$\int_{-1}^{1} f_e(x)\,dx = 2\int_{0}^{1} f_e(x)\,dx = \int_{0}^{1} \big(f(x)+f(-x)\big)\,dx = \int_{0}^{1} f(x)\,dx + \int_{0}^{1} f(-x)\,dx$$
$$= \int_{0}^{1} f(x)\,dx + \int_{-1}^{0} f(t)\,dt = \int_{-1}^{1} f(x)\,dx.$$

4. Fix $x_1,\dots,x_n \in (0,1]$ and for any function f define

$$\mathcal{T}(f) := \alpha_0 f(0) + \sum_{i=1}^{n} \alpha_i \big(f(x_i)+f(-x_i)\big).$$

Then $\mathcal{T}(f) = \mathcal{T}(f_e)$. Namely,

$$\mathcal{T}(f) = \alpha_0 f(0) + \sum_{i=1}^{n} \alpha_i \cdot 2 f_e(x_i) = \alpha_0 f_e(0) + \sum_{i=1}^{n} \alpha_i \big(f_e(x_i)+f_e(-x_i)\big) = \mathcal{T}(f_e).$$

5. Let n be an odd positive integer. Because of the above remarks the inequalities of the form $\mathcal{T}_1(f) \leqslant \int_{-1}^{1} f(x)\,dx \leqslant \mathcal{T}_2(f)$ hold for any n-convex function f if and only if they hold for any n-convex and even function f.

Inequalities for Chebyshev quadrature

Recall that the operator $\mathcal{C}(f) := \frac{2}{3}\big(f(-\frac{\sqrt{2}}{2}) + f(0) + f(\frac{\sqrt{2}}{2})\big)$ is connected with the 3-point Chebyshev quadrature rule (cf. [9,15]).

Proposition 14. *If $f:[-1,1]\to\mathbb{R}$ is 3-convex then*

$$\mathcal{G}_2(f) \leqslant \mathcal{C}(f) \leqslant \int_{-1}^{1} f(x)\,dx.$$

Proof. It is enough to prove the proposition for even functions.

1. By 3-convexity and (5) $D(-v,-u,0,u,v;\,f) \geqslant 0$ for any $0<u<v\leqslant 1$. Expanding this determinant by the last row we simply compute $v^2 f(u) \leqslant u^2 f(v) + (v^2-u^2) f(0)$. For $u = \frac{\sqrt{3}}{3}$, $v = \frac{\sqrt{2}}{2}$ we obtain $\mathcal{G}_2(f) \leqslant \mathcal{C}(f)$.

2. By Theorem 2 (for $n=3$, $k=3$, $x_1 = -\frac{\sqrt{2}}{2}$, $x_2 = 0$, $x_3 = \frac{\sqrt{2}}{2}$, $l_1 = l_2 = 1$, $l_3 = 2$) there exists a polynomial $p \in \Pi_3$ such that $p(-\frac{\sqrt{2}}{2}) = f(-\frac{\sqrt{2}}{2})$, $p(0) = f(0)$, $p(\frac{\sqrt{2}}{2}) = f(\frac{\sqrt{2}}{2})$ and $p \leqslant f$ on $[0,1]$. By Newton's Interpolation Formula (4)

$$p(x) = f\left(\frac{\sqrt{2}}{2}\right) + \left[-\frac{\sqrt{2}}{2}, 0; f\right]\left(x+\frac{\sqrt{2}}{2}\right) + \left[-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}; f\right]\left(x+\frac{\sqrt{2}}{2}\right)x$$
$$+ A\left(x+\frac{\sqrt{2}}{2}\right)x\left(x-\frac{\sqrt{2}}{2}\right)$$

for some constant A. Computing these divided differences we can easily see that $2\int_0^1 p(x)\,dx = \mathcal{C}(f)$, whence $\mathcal{C}(f) \leqslant 2\int_0^1 f(x)\,dx = \int_{-1}^1 f(x)\,dx$. □

Inequalities for 5-convex functions

Recall that the operator $\mathcal{S}(f) := \frac{1}{3}(f(-1) + 4f(0) + f(1))$ is connected with Simpson's quadrature rule (cf. [9,20]).

Proposition 15. *If $f : [-1, 1] \to \mathbb{R}$ is 5-convex then*

$$\int_{-1}^{1} f(x)\,dx \leqslant \frac{2}{5}\mathcal{S}(f) + \frac{3}{5}\mathcal{G}_2(f) \leqslant \mathcal{L}_4(f).$$

Proof. It is enough to prove the proposition for even functions.

1. By Theorem 2 ($n = 5$, $k = 5$, $x_1 = -1$, $x_2 = -\frac{\sqrt{3}}{3}$, $x_3 = 0$, $x_4 = \frac{\sqrt{3}}{3}$, $x_5 = 1$, $l_1 = l_2 = l_3 = 1$, $l_4 = 2$, $l_5 = 1$) there exists a polynomial $p \in \Pi_5$ such that $p(x_i) = f(x_i)$, $i = 1, 2, 3, 4, 5$, and $p \geqslant f$ on $[0, 1]$. Similarly as in the proof of Proposition 14 we use Newton's Interpolation Formula (4) for the abscissas x_1, x_2, x_3, x_4, x_5 and we compute $2\int_0^1 p(x)\,dx = \frac{2}{5}\mathcal{S}(f) + \frac{3}{5}\mathcal{G}_2(f)$, from which the first inequality follows.

2. To obtain the second inequality we also proceed similarly to the proof of Proposition 14. By 5-convexity and (5)

$$D\left(-1, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{5}}{5}, 0, \frac{\sqrt{5}}{5}, \frac{\sqrt{3}}{3}, 1; f\right) \geqslant 0.$$

Expanding this determinant by the last row and performing some computations we get the desired inequality. □

Other inequalities between the quadrature operators can be found in [12].

8. Error bounds of quadrature rules

Hadamard-type inequalities can be applied to estimate the errors of quadrature rules. We illustrate this for the quadrature $\mathcal{T}(f) := \frac{2}{5}\mathcal{S}(f) + \frac{3}{5}\mathcal{G}_2(f)$. Denote $\mathcal{I}(f) := \int_{-1}^1 f(x)\,dx$.

Proposition 16. *If $f \in \mathcal{C}^6([-1, 1])$ and $M := \sup\{|f^{(6)}(x)|: x \in [-1, 1]\}$, then $|\mathcal{T}(f) - \mathcal{I}(f)| \leqslant \frac{M}{28\,350}$.*

Proof. Let $g(x) := \frac{Mx^6}{6!}$. Then $g^{(6)}(x) = M$ and $|f^{(6)}(x)| \leqslant g^{(6)}(x)$. Therefore $(g + f)^{(6)} \geqslant 0$ and $(g - f)^{(6)} \geqslant 0$. By Theorem D the functions $g + f$, $g - f$ are 5-convex. By Proposition 15 we get

$$\mathcal{I}(g + f) \leqslant \mathcal{T}(g + f), \qquad \mathcal{I}(g - f) \leqslant \mathcal{T}(g - f).$$

Since the operators $\mathcal{T}$ and $\mathcal{I}$ are linear then

$$\mathcal{I}(g) - \mathcal{T}(g) \leqslant \mathcal{T}(f) - \mathcal{I}(f), \qquad \mathcal{T}(f) - \mathcal{I}(f) \leqslant \mathcal{T}(g) - \mathcal{I}(g).$$

Hence $|\mathcal{T}(f) - \mathcal{I}(f)| \leqslant \mathcal{T}(g) - \mathcal{I}(g)$. We conclude the proof by computing $\mathcal{T}(g) - \mathcal{I}(g) = \frac{M}{28\,350}$. $\square$

The method presented above can be applied for other quadrature rules. However, using it for Chebyshev, Gauss–Legendre, Lobatto, Radau and Simpson's quadratures we obtain the error bounds known from numerical analysis (cf. [9,15,17–20]).

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