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Note

On total domination vertex critical graphs of high connectivity

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ABSTRACT

A graph is called γ-critical if the removal of any vertex from the graph decreases the domination number, while a graph with no isolated vertex is γ_t-critical if the removal of any vertex that is not adjacent to a vertex of degree 1 decreases the total domination number. A γ_t-critical graph that has total domination number k, is called k-γ_t-critical. In this paper, we introduce a class of k-γ_t-critical graphs of high connectivity for each integer $k \geq 3$. In particular, we provide a partial answer to the question "Which graphs are γ-critical and γ_t-critical or one but not the other?" posed in a recent work [W. Goddard, T.W. Haynes, M.A. Henning, L.C. van der Merwe, The diameter of total domination vertex critical graphs, Discrete Math. 286 (2004) 255–261].

1. Introduction

For many graph parameters, criticality is a fundamental question. Much has been written about those graphs where a parameter (such as connectedness or chromatic number) goes up or down whenever an edge or vertex is removed or added. In this paper, we continue the study of total domination vertex critical graphs introduced by Goddard, Haynes, Henning, and van der Merwe [3].

Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [2] and is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two excellent domination books by Haynes, Hedetniemi, and Slater [5,6].

For notation and graph theory terminology we in general follow [5]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E. We denote the degree of v in G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from the context. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. The (open) neighborhood of vertex $v \in V$ is denoted by $N(v) = \{u \in V \mid uv \in E\}$ while $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. The set S is a *dominating set* if $N[S] = V$, and a *total dominating set* if $N(S) = V$. For sets $A, B \subseteq V$, we say that A *dominates* B if $B \subseteq N[A]$ while A *totally dominates* B if $B \subseteq N(A)$. The minimum cardinality of a total dominating set is the *total domination number*, denoted as $\gamma_t(G)$. A total dominating set of cardinality $\gamma_t(G)$ we call a $\gamma_t(G)$-set.

Let $G = (V, E)$ be a graph and let $S \subseteq V$ and $v \in S$. We denote the subgraph of G induced by S by $G[S]$. The minimum degree of G is denoted by $\delta(G)$. A vertex $w \in V$ is an *S-private neighbor of* v if $N(w) \cap S = \{v\}$. Further, the *S-private neighborhood of* v, denoted as $\mathrm{pn}(v, S)$, is the set of all S-private neighbors of v.

We say that a vertex v in a graph G is *critical* if $\gamma_t(G - v) < \gamma_t(G)$. Since total domination is undefined for a graph with isolated vertices, we say that a graph G is *total domination vertex critical*, or just γ_t-critical, if every vertex of G that is not adjacent to a vertex of degree 1 is critical. In particular, if $\delta(G) \geq 2$, then G is γ_t-critical if every vertex of G is critical. If G is γ_t-critical, and $\gamma_t(G) = k$, then we say that G is *k-γ_t-critical*. For example, the 5-cycle is 3-γ_t-critical as is the complement

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of the Petersen graph, and the 6-cycle is 4-γ_t-critical. We also note that K_2 is trivially 2-γ_t-critical. We remark that our terminology used for total domination vertex critical graphs is similar to that used for domination vertex critical graphs. For example, a graph G is *k-γ-critical* if $\gamma(G) = k$ and $\gamma(G - v) < k$ for all vertices v in G. A graph is γ-critical if it is k-γ-critical for some k. The study of γ-critical graphs was begun by Brigham, Chinn, and Dutton [1].

In [3], the authors pose the following open question:

Question 1 ([3]). *Which graphs are γ-critical and γ_t-critical or one but not the other?*

Our aim in this paper is twofold. Our first aim is to introduce for each integer $k \geq 3$, a class of k-γ_t-critical graphs of high connectivity. Our second aim is to provide a partial answer to Question 1, by presenting an infinite family of graphs that are γ_t-critical but not γ-critical.

1.1. Harary graphs

For $2 \leq k < n$, the Harary graph $H_{k,n}$ on n vertices is defined by West [9] as follows. Place n vertices around a circle, equally spaced. If k is even, $H_{k,n}$ is formed by making each vertex adjacent to the nearest $k/2$ vertices in each direction around the circle. If k is odd and n is even, $H_{k,n}$ is formed by making each vertex adjacent to the nearest $(k-1)/2$ vertices in each direction around the circle and to the diametrically opposite vertex. In both cases, $H_{k,n}$ is k-regular. If both k and n are odd, $H_{k,n}$ is constructed as follows. It has vertices $0, 1, \ldots, n-1$ and is constructed from $H_{k-1,n}$ by adding edges joining vertex i to vertex $i + (n-1)/2$ for $0 \leq i \leq (n-1)/2$.

2. Known results

Harary [4] showed that the graph $H_{k,n}$ is k-connected.

Theorem 1 ([4]). *For $2 \leq k < n$, the Harary graph $H_{k,n}$ is k-connected.*

We shall need the following trivial lower bound on the domination number of a graph in terms of its order and maximum degree.

Observation 1. *If G is a graph of order n, then $\gamma(G) \geq \left\lceil \frac{n}{1+\Delta(G)} \right\rceil$.*

Dominating sets in Harary graphs are investigated by Khodkar, Mojdeh and Kazemi [7]. The authors obtain exact values of the domination number of Harary graphs. In particular, they establish the following result.

Theorem 2 ([7]).
(a) *For $n \geq 2k+1 \geq 3$, $\gamma(H_{2k,n}) = \lceil n/(2k+1) \rceil$.*
(b) *For $n - (k+1) = (2k+2)t + r$, where $0 \leq r \leq 2k+1$,*

$$\gamma(H_{2k+1,2n}) = \begin{cases} \lceil n/(k+1) \rceil + 1 & \text{if } 2 \leq r \leq k+1 \text{ and } t + r \geq k+1 \\ \lceil n/(k+1) \rceil & \text{otherwise.} \end{cases}$$

Khodkar, Mojdeh and Kazemi [7] also study total dominating sets in Harary graphs. They construct total dominating sets for several Harary graphs in Theorem 13, Theorem 14, and Theorem 15 of [7], thereby establishing important upper bounds on the total domination number of certain Harary graphs. However in the proofs of [7] some details are missing. In this note, we obtain the total domination numbers in Harary graphs, and give a proof with complete details.

The k-γ_t-vertex-critical graphs with end-vertices are characterized in [3]. The *corona* $\mathrm{cor}(H)$ of a graph H (denoted $H \circ K_1$ in [5]) is that graph obtained from H by adding a pendant edge to each vertex of H.

Theorem 3 ([3]). *Let G be a connected graph of order at least 3 with at least one end-vertex. Then, G is k-γ_t-vertex-critical if and only if $G = \mathrm{cor}(H)$ for some connected graph H of order k with $\delta(H) \geq 2$.*

3. Main results

In view of Theorem 3, we henceforth only consider connected graphs of order at least 3 with minimum degree at least two. We shall prove:

Theorem 4. *For all integers $\ell \geq 1$ and $k \geq 2$, the Harary graph $H_{2k+1,2\ell(2k+1)+2}$ is a $(2k+1)$-connected graph that is $(2\ell+2)$-γ_t-critical.*

A proof of Theorem 4 is presented in Section 3.1. We remark that as a special case of Theorem 4 when $\ell = 1$, the Harary graph $H_{2k+1,4k+4}$ is a $(2k+1)$-connected graph that is a 4-γ_t-critical graph of diameter 2 for every integer $k \geq 2$. A different family of 4-γ_t-critical graphs of diameter 2 is constructed by Loizeaux and van der Merwe [8].

Theorem 5. *For all integers $\ell \geq 1$ and $k \geq 2$, the Harary graph $H_{2k,\ell(3k+1)+1}$ is a $2k$-connected graph that is $(2\ell+1)$-γ_t-critical.*

Theorem 5 can be proved in an identical manner to in the proof of Theorem 4, and we therefore omit a proof of Theorem 5. As a consequence of Observation 1 and Theorem 2, we can readily establish the following result, a proof of which is presented in Section 3.2.

Corollary 1. (a) *For $n > k+1$ and $n-(k+1) \equiv r \pmod{(2k+2)}$ for some integer r where $r \in \{0, 1\}$ or $k+2 \leq r \leq 2k+1$, the Harary graph $H_{2k+1,2n}$ is not γ-critical.*
(b) *For $n \geq 2k+1 \geq 3$, where $n \not\equiv 1 \pmod{(2k+1)}$, the Harary graph $H_{2k,n}$ is not γ-critical.*

Note that if $n = \ell(2k+1)+1$, then $n-(k+1) = (2k+2)(\ell-1)+(k-\ell+2)$. Hence as an immediate consequence of Theorems 4, 5 and Corollary 1, we have the following results.

Corollary 2. *For integers $k \geq \ell - 2 \geq 0$ where $k - \ell + 2 = r$ and $r \in \{0, 1\}$ or $k+2 \leq r \leq 2k+1$, the Harary graph $H_{2k+1,2\ell(2k+1)+2}$ is γ_t-critical and is not γ-critical.*

Corollary 3. *For all integers $\ell \geq 1$ and $k \geq 2$ such that $(2k+1) \nmid \ell(3k+1)$, the Harary graph $H_{2k+1,2\ell(2k+1)+2}$ is γ_t-critical and is not γ-critical.*

3.1. Proof of Theorem 4

To prove Theorem 4, let $G = H_{2k+1,2\ell(2k+1)+2}$. Then, G is a $(2k+1)$-regular, vertex-transitive graph of order $n = 2\ell(2k+1)+2$. By Theorem 1, the graph G is $(2k+1)$-connected. Let $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$. Then, v_i and v_j are adjacent if $i - k/2 \leq j \leq i + k/2$ or if $j = i + n/2$ where addition is taken modulo n.

We proceed further by proving the following five facts.

Fact 1. *Let S be a set of vertices of G such that $|S| = 2\ell$ and $G[S]$ contains no isolated vertex. If S contains two adjacent vertices that are not diametrically opposite in G, then S totally dominates at most $n-k-2$ vertices of G.*

Proof. Let v_i and v_j be two adjacent vertices in S that are not diametrically opposite in G. For notational convenience, we may assume that $i = k$ and $j = k+t$ where $1 \leq t \leq k$. Then, $v_s \in N(v_i) \setminus N(v_j)$ if $0 \leq s \leq t-1$ or $s \in \{k+t, k+n/2\}$ and $v_s \in N(v_j) \setminus N(v_i)$ if $2k+1 \leq s \leq 2k+t$ or $s \in \{k, k+t+n/2\}$. Further, $v_s \in N(v_i) \cap N(v_j)$ for $s \in \{t, t+1, \ldots, 2k\} \setminus \{k, k+t\}$. Thus, $|N(v_i) \setminus N(v_j)| = |N(v_j) \setminus N(v_i)| = t+2$, while $|N(v_i) \cap N(v_j)| = 2k-t-1$. Hence, $\{v_i, v_j\}$ totally dominates $2k+t+3 \leq 3(k+1)$ vertices. Since G is a $(2k+1)$-regular, the set $S \setminus \{v_i, v_j\}$ totally dominates at most $(|S|-2)(2k+1) = 2(\ell-1)(2k+1) = n-4(k+1)$ vertices. Hence the set S totally dominates at most $n-k-1$ vertices of G.

Assume the set S totally dominates exactly $n-k-1$ vertices of G. Then, $t = k$ (and so, $i = k$ and $j = 2k$) and the $k-1$ vertices between v_i and v_j, namely $v_{k+1}, \ldots, v_{2k-1}$, are the only common neighbors of v_i and v_j. Further, every vertex in $N(v_i) \setminus N(v_j)$ is an S-private neighbor of v_i, while every vertex in $N(v_j) \setminus N(v_i)$ is an S-private neighbor of v_j; that is, $|\mathrm{pn}(v_i, S)| = |N(v_i) \setminus N(v_j)| = k+2$ and $|\mathrm{pn}(v_j, S)| = |N(v_j) \setminus N(v_i)| = k+2$. Thus, $v_r \notin S$ for $r \in (\{0, 1, \ldots, 4k\} \setminus \{k, 2k\}) \cup \{n/2, n/2+1, \ldots, n/2+3k\} \cup \{n-k, \ldots, n-1\}$. Moreover, every neighbor of a vertex $v \in S \setminus \{v_i, v_j\}$ is an S-private neighbor of v. This implies that the $2(\ell-1)$ vertices in $S \setminus \{v_i, v_j\}$ form $\ell-1$ pairs of diametrically opposite vertices in G. Thus if $v_s \in S$ for some $s \in \{n/2-k, \ldots, n/2-1\}$, then $v_{s+n/2} \in S$ contradicting our earlier observation that $v_r \notin S$ for $r \in \{n-k, \ldots, n-1\}$. Hence, $v_s \notin S$ for $s \in \{n/2-k, \ldots, n/2-1\}$. Further since $v_r \notin S$ for $r \in \{3k+1, \ldots, 4k\}$, we have that $v_s \notin S$ for $s \in \{n/2+3k+1, \ldots, n/2+4k\}$. But then the vertex v_z is not totally dominated by S for $z \in \{n/2, n/2+1, \ldots, n/2+3k\} \setminus \{n/2+k, n/2+2k\}$. Thus there are at least $3k-1$ vertices not totally dominated by S. By assumption, exactly $k+1$ vertices of G are not totally dominated by S. Hence, $k+1 \geq 3k-1$, implying that $k \leq 1$, a contradiction. We deduce, therefore, that the set S totally dominates at most $n-k-2$ vertices of G. □

Fact 2. *Let S be a set of vertices of G such that $|S| = 2\ell$ and $G[S]$ contains no isolated vertex. If every pair of adjacent vertices in $G[S]$ is a pair of diametrically opposite vertices in G, then $G[S] = \ell K_2$ and S totally dominates at most $n-2$ vertices of G.*

Proof. Since every pair of adjacent vertices in $G[S]$ is a pair of diametrically opposite vertices in G, $G[S] = \ell K_2$. Since G is $(2k+1)$-regular, the set S totally dominates at most $|S|(2k+1) = 2\ell(2k+1) = n-2$ vertices of G. □

Fact 3. *Let S be a set of vertices of G such that $G[S]$ contains no isolated vertex and let $v \in V(G) \setminus S$. If v is adjacent to a vertex of S that is not diametrically opposite to it in G, then v totally dominates at most $k+1$ vertices that are not totally dominated by S.*

Proof. Let u be a vertex of S that is adjacent to v but is not diametrically opposite to v in G. For notational convenience, we may assume that $u = v_i$ and $v = v_j$ where $i = k$ and $j = k+t$ where $1 \leq t \leq k$. Then, $v_s \in N(v) \setminus N(u)$ if $2k+1 \leq s \leq 2k+t$ or $s \in \{k, k+t+n/2\}$. Hence, v totally dominates $t+2$ vertices, including the vertex u, that are not totally dominated by u. However since $G[S]$ contains no isolated vertex, the vertex u is totally dominated by S. Thus, v totally dominates at most $t+1 \leq k+1$ vertices that are not totally dominated by S. □

Fact 4. $\gamma_t(G) = 2\ell + 2$.

Proof. By Facts 1 and 2, no set of 2ℓ vertices is a total dominating set of G, and so $\gamma_t(G) \geq 2\ell+1$. Assume that $\gamma_t(G) = 2\ell+1$. Let D be a $\gamma_t(G)$-set. Then, $|D| = 2\ell + 1$, $G[D]$ contains no isolated vertex, and $N(D) = V$. Since $|D|$ is odd, at least one component F of $G[D]$ has odd order. Thus at least one vertex of F is not the diametrically opposite vertex of any other vertex in D. Among all such vertices in F, let v be one of smallest degree in F.

If $F - v$ contains an isolated vertex w, then $d_F(w) = 1$ and w is adjacent only to v in F. But since F has order at least 3, v is adjacent to at least one vertex in F different from w, and so $d_F(v) \geq 2$. Thus, $d_F(v) > d_F(w)$. Since v is not the diametrically opposite vertex of any other vertex in D, neither is w. This contradicts our choice of the vertex v. Hence, $F - v$ has no isolated vertex. Let $S = D \setminus \{v\}$. Then, $|S| = 2\ell$ and $G[S]$ contains no isolated vertex.

If S contains two adjacent vertices that are not diametrically opposite in G, then, by Fact 1, S totally dominates at most $n - k - 2$ vertices of G. By Fact 2, v totally dominates at most $k + 1$ vertices that are not totally dominated by S. Hence, D totally dominates at most $n - 1$ vertices of G, contradicting the fact that D is a total dominating set of G. Thus every pair of adjacent vertices in $G[S]$ is a pair of diametrically opposite vertices in G. Hence, by Fact 2, $G[S] = \ell K_2$ and S totally dominates at most $n - 2$ vertices of G. Further if v_i is a vertex not totally dominated by S, then $v_{i+n/2}$ is also not totally dominated by S (where addition is taken modulo n). Hence, G contains a pair v_i and $v_{i+n/2}$ of diametrically opposite vertices that are not totally dominated by S. Since two diametrically opposite vertices have no common neighbor, the vertex v totally dominates at most one of v_i and $v_{i+n/2}$. This contradicts the fact that D is a total dominating set of G. We deduce, therefore, that $\gamma_t(G) \geq 2\ell + 2$. The set

$$\bigcup_{i=0}^{\ell}\{v_{i(2k+1)}, v_{i(2k+1)+n/2}\},$$

for example, is a total dominating set of G of cardinality $2\ell + 2$, and so $\gamma_t(G) \leq 2\ell + 2$. Consequently, $\gamma_t(G) = 2\ell + 2$. $\square$

Fact 5. $\gamma_t(G - v) = 2\ell + 1$ *for every vertex* $v \in V$.

Proof. Since G is vertex-transitive, we may assume that $v = v_0$. Let

$$S = \bigcup_{i=1}^{\ell}\{v_{i(2k+1)}, v_{i(2k+1)+n/2}\}.$$

Then, $|S| = 2\ell$ and the set S totally dominates every vertex of $G - v$ except for the vertex $v_{n/2}$ that is diametrically opposite $v = v_0$ in G. Hence the set $S \cup \{v_j\}$ is a total dominating set of G for any vertex v_j adjacent to $v_{n/2}$ that is different from v, i.e., for any value of j satisfying $n/2 - k \leq j \leq n/2 + k$ and $j \neq n/2$. Thus, $\gamma_t(G - v) \leq 2\ell + 1$. The removal of a vertex can decrease the total domination number by at most 1, and so, by Fact 4, $\gamma_t(G - v) \geq \gamma_t(G) - 1 = 2\ell + 1$. Consequently, $\gamma_t(G - v) = 2\ell + 1$ for every vertex $v \in V$; that is, every vertex of G is critical. $\square$

As an immediate consequence of Facts 4 and 5, we have that the graph G is $(2\ell + 2)$-γ_t-critical. This completes the proof of Theorem 4. $\square$

3.2. Proof of Corollary 1

Recall the statement of Corollary 1(a).

Corollary 1(a). *For* $n > k+1$ *and* $n - (k+1) \equiv r \pmod{2k+2}$ *for some integer* r *where* $r \in \{0, 1\}$ *or* $k+2 \leq r \leq 2k+1$, *the Harary graph* $H_{2k+1,2n}$ *is not* γ*-critical.*

Proof. Let $G = H_{2k+1,2n}$, and so G is a $(2k+1)$-regular graph of order $2n$. Let $v \in V(G)$ and let $G_v = G - v$. Then, G_v is a graph of order $2n - 1$. To show that G is not γ-critical, it suffices for us to show that $\gamma(G_v) \geq \gamma(G)$. Since $n - (k+1) = (2k+2)t + r$ where $r \in \{0, 1\}$ or $k + 2 \leq r \leq 2k + 1$, we have by Theorem 2 that $\gamma(G) = \lceil n/(k + 1)\rceil = 2t + 1 + \lceil r/(k + 1)\rceil$. Since $n > k + 1$, we also note that $\Delta(G_v) = \Delta(G) = 2k + 1$. By Observation 1, we have that $\gamma(G_v) \geq \lceil (2n - 1)/2(k + 1)\rceil = 2t + 1 + \lceil (2r - 1)/2(k + 1)\rceil$. If $r = 0$, then $\gamma(G_v) \geq 2t + 1 = \gamma(G)$. If $r = 1$, then $\gamma(G_v) \geq 2t + 2 = \gamma(G)$. If $k + 2 \leq r \leq 2k + 1$, then $\gamma(G_v) \geq 2t + 3 = \gamma(G)$. In all cases, $\gamma(G_v) \geq \gamma(G)$, as desired. $\square$

Recall the statement of Corollary 1(b).

Corollary 1(b). *For* $n \geq 2k + 1 \geq 3$, *where* $n \not\equiv 1 \pmod{2k + 1}$, *the Harary graph* $H_{2k,n}$ *is not* γ*-critical.*

Proof. Let $G = H_{2k,n}$, and so G is a $(2k)$-regular graph of order n. Let $v \in V(G)$ and let $G_v = G - v$. Then, G_v is a graph of order $n - 1$. To show that G is not γ-critical, it suffices for us to show that $\gamma(G_v) \geq \gamma(G)$. If $n = 2k + 1$, then $G = K_n$, which is not γ-critical. Hence we may assume that $n > 2k + 1$, for otherwise the desired result holds. With this assumption, we note that $\Delta(G_v) = \Delta(G) = 2k$. Let $n = t(2k + 1) + r$, where $0 \leq r \leq 2k$. Suppose first that $r = 0$. Then by Theorem 2(a), $\gamma(G) = t$. By Observation 1, we have that $\gamma(G_v) \geq \lceil (n - 1)/(2k + 1)\rceil = t = \gamma(G)$, as desired. Therefore we may assume that $r \geq 1$. Since $n \not\equiv 1 \pmod{2k + 1}$, we have that $r \geq 2$. By Theorem 2(a), $\gamma(G) = t + 1$. By Observation 1, we have that $\gamma(G_v) \geq t + 1 = \gamma(G)$, as desired. $\square$

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References

[1] R.C. Brigham, P.Z. Chinn, R.D. Dutton, Vertex domination-critical graphs, Networks 18 (1988) 173–179.
[2] E.J. Cockayne, R.M. Dawes, S.T. Hedetniemi, Total domination in graphs, Networks 10 (1980) 211–219.
[3] W. Goddard, T.W. Haynes, M.A. Henning, L.C. van der Merwe, The diameter of total domination vertex critical graphs, Discrete Math. 286 (2004) 255–261.
[4] F. Harary, The maximum connectivity of a graph, Proc. Natl. Acad. Sci. USA 48 (1962) 1142–1146.
[5] T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Fundamentals of Domination in Graphs, Marcel Dekker, Inc., New York, 1998.
[6] T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker, Inc., New York, 1998.
[7] A. Khodkar, D.A. Mojdeh, A.P. Kazemi, Domination in Harary graphs, Bull. ICA 49 (2007) 61–78.
[8] M. Loizeaux, L. van der Merwe, A total domination vertex-critical graph of diameter two, Bull. ICA 48 (2006) 63–65.
[9] D.B. West, Introduction to Graph Theory, second edition, Prentice-Hall, Upper Saddle River, NJ, 2001.