Global existence and asymptotic convergence of weak solutions for the one-dimensional Navier–Stokes equations with capillarity and nonmonotonic pressure

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Abstract
We construct global weak solution of the Navier–Stokes equations with capillarity and nonmonotonic pressure. The volume variable \( v_0 \) is initially assumed to be in \( H^1 \) and the velocity variable \( u_0 \) to be in \( L^2 \) on a finite interval \([0, 1]\). We show that both variables become smooth in positive time and that asymptotically in time \( u \to 0 \) strongly in \( L^2([0, 1]) \) and \( v \) approaches the set of stationary solutions in \( H^1([0, 1]) \).

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1. Introduction

We study global weak solutions to the Navier–Stokes equations with capillarity

\[
\begin{align*}
  v_t &- u_x = 0, \\
  u_t + p(v)_x &= \left( \frac{u_x}{v} \right)_x - v_{xxx},
\end{align*}
\]

\( (1.1) \quad (1.2) \)

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with Cauchy data

\[ v(\cdot, t) = v_0 \in H^1([0, 1]), \quad u(\cdot, t) = u_0 \in L^2([0, 1]) \]  

(1.3)

and the boundary conditions

\[ v_x(0, t) = v_x(1, t) = 0, \quad u(0, t) = u(1, t) = 0. \]  

(1.4)

Here \( v \) is the specific volume, \( u \) is the velocity. Time \( t \) is assumed to be nonnegative and \( x \in [0, 1] \). The pressure \( p(v) \) is to satisfy the following assumptions:

\[ \lim_{v \to +0} p(v) = +\infty, \quad \lim_{v \to +\infty} p(v) \leq 0. \]  

(1.5)

The pressure \( p \) is closely related to a stored-energy function \( W(v) \) by means of the equation

\[ W'(v) = -p(v) + a, \]  

(1.6)

with a positive constant \( a \). We assume that \( W \geq 0 \) for all positive values of \( v \) and

\[ \lim_{v \to +0} W(v) = +\infty, \]  

(1.7)

as well as

\[ \lim_{v \to +\infty} W(v) = +\infty. \]  

(1.8)

We say that \( (v, u) \) is a weak solution of (1.1), (1.2) provided that \( \inf_{x \in [0, 1]} v(\cdot, t) > 0 \) and

\[ v(\cdot, t) \in C([0, \infty); H^1([0, 1])) \cap C((0, \infty); H^2([0, 1])), \]  

(1.9)

\[ u(\cdot, t) \in C([0, \infty); L^2([0, 1])) \cap C((0, \infty); H^1([0, 1])), \]  

(1.10)

\[ v(\cdot, 0) = v_0, \quad u(\cdot, 0) = u_0, \]  

(1.11)

\[ v_x(0, t) = v_x(1, t) = 0, \quad u(0, t) = u(1, t) = 0, \]  

(1.12)

and for all test functions \( \varphi \in C^1((-\infty, \infty) \times [0, 1]), \psi \in C^3_0((-\infty, \infty) \times [0, 1]) \), and all times \( t_2 > t_1 \geq 0 \),

\[ \int_0^{t_2} \int_0^1 v \varphi_t dx dt - \int_0^{t_1} \int_0^1 [v \varphi_t - u \varphi_x] dx dt = 0, \]  

(1.13)

\[ \int_0^{t_2} \int_0^1 u \psi_t dx dt - \int_0^{t_1} \int_0^1 [u \psi_t + p(v) \psi_x] dx dt = -\int_0^{t_2} \int_0^1 \left[ \frac{u_x}{v} \psi_x - v \psi_{xxx} \right] dx dt. \]  

(1.14)

As in the case of a \( p \)-system with capillarity, we can rightfully expect that a solution to (1.1), (1.2) becomes \( C^\infty \) in positive time. However, only minimal regularity of \( v \) and \( u \) is necessary to investigate their asymptotic behavior.
Stationary solutions of (1.1), (1.2), (1.4) are functions \((\tilde{v}(x), 0)\) solving
\[
0 < \tilde{v}(x), \quad p(\tilde{v})_x + \tilde{v}_{xxx} = 0, \quad \tilde{v}_x(0) = \tilde{v}_x(1) = 0.
\] (1.15)

Let \(S\) be the set of solutions of (1.15). Then \(S\) contains any positive constant. On the other hand, in the case of nonmonotonic pressure, nonconstant stationary solutions do occur and we will give an example of such a case.

We are concerned with subsets \(S_M \subset S\) of stationary solutions with a given total volume \(M\),
\[
S_M = \left\{ \tilde{v} \in S : \int_0^1 \tilde{v}(x) \, dx = M \right\},
\]
due to the conservation of total volume by the solutions of (1.1)–(1.4). It is quite natural to study \(S_M\) instead of \(S\). Firstly, \(S_M\) contains only one constant element. Secondly, unlike \(S\), \(S_M\) can be finite and this gives the prospect of studying the asymptotic convergence of \(v\) to a particular element of \(S_M\), for example a specific nonconstant one.

Our results are motivated by the work of Andrews and Ball [1], who proved global existence and asymptotic convergence of solutions of nonlinear viscoelasticity with capillarity. It was shown that the solution approaches the set of stationary solutions in an appropriate space as \(t\) goes to infinity. The analysis of (1.1), (1.2) is complicated, however, by the presence of additional nonlinearities in the Stokes viscosity, which seems to preclude direct application of the ideas from [1]. Moreover, there are significant difficulties in obtaining high-order time-independent estimates to show that \(\bigcup_{t \geq 0} (v(\cdot, t), u(\cdot, t))\) belongs to a compact subset of \(H^1 \times L^2\). This appears to prevent any straightforward application of the LaSalle invariance principle. Instead, we pursue the change of the dependent variables approach, which for the Navier–Stokes equations is given in Tsyganov [2] and was for nonlinear viscoelasticity in Pego [3]. The new dependent variables are defined as follows:

\[
r(x, t) = \int_0^x u(y, t) \, dy - \int_0^1 \int_0^y u(z, t) \, dz \, dy,
\] (1.16)

\[
q(x, t) = \ln(v(x, t)) - \int_0^1 \ln(v(y, t)) \, dy - r(x, t).
\] (1.17)

Definitions (1.16) and (1.17) provide a convenient equation for \(q_t\) by means of
\[
q_t(x, t) = p(v(x, t)) - \int_0^1 p(v(y, t)) \, dy + v_{xx}(x, t).
\] (1.18)

Careful considerations show that
\[
\int_0^\infty \int_0^1 \left( p(v(x, t)) - \int_0^1 p(v(y, t)) \, dy + v_{xx}(x, t) \right)^2 \, dx \, dt \leq \infty,
\] (1.19)
for a given solution \((v, u)\) of (1.1)–(1.4) and
\[
p(\tilde{v}(x)) - \int_{0}^{1} p(\tilde{v}(y)) \, dy + \tilde{v}_{xx}(x) = 0, \tag{1.20}
\]
for stationary solutions \((\tilde{v}, 0)\). Both (1.19) and (1.20) indicate possible directions of the asymptotic analysis, and, as will be seen later, (1.16) and (1.17) can be exploited in the proof of asymptotic convergence.

Our existence result is formulated in the following theorem:

**Theorem 1.1.** Assume that the initial data \((v_0, u_0)\) satisfy
\[
0 < v_0(x) \leq \bar{v}, \tag{1.21}
\]
\[
\int_{0}^{1} \left[ \frac{1}{2} u_0^2(x) + W(v_0(x)) + \frac{1}{2} v_0^2(x) \right] \, dx = C_0, \tag{1.22}
\]
for some positive constants \(v, \bar{v}, C_0\). Then there is a weak solution \((v, u)\) of (1.1)–(1.4) and positive constants \(C, C(T)\) depending on \(v, \bar{v}, C_0\) and \(v, \bar{v}, C_0, T\), respectively, such that
\[
C^{-1} \leq v(x,t) \leq C, \tag{1.23}
\]
\[
\int_{0}^{1} \left[ \frac{1}{2} u(x,t)^2(x) + W(v(x,t)) + \frac{1}{2} v^2(x,t) \right] \, dx + \int_{0}^{T} \int_{0}^{1} \frac{u^2(x,t)}{v} \, dx \, dt = C_0, \tag{1.24}
\]
\[
\sup_{0 \leq t \leq T} \left( t^2 \int_{0}^{1} \dot{v}_x(x,t)^2 \, dx + t^4 \int_{0}^{1} \dot{v}_{xx}(x,t)^2 \, dx \right) \leq C(T), \tag{1.25}
\]
\[
\sup_{0 \leq t \leq T} \left( \int_{0}^{1} u(x,t)^2 \, dx + t \int_{0}^{1} u_x(x,t)^2 \, dx \right) \leq C(T), \tag{1.26}
\]
\[
\sup_{0 \leq t \leq T} \left( t^2 \int_{0}^{1} \dot{u}(x,t)^2 \, dx + t^4 \int_{0}^{1} \dot{u}_x(x,t)^2 \, dx \right) \leq C(T), \tag{1.27}
\]
where \(\dot{v}\) and \(\dot{u}\) stand for \(v_t\) and \(u_t\). In addition, Eq. (1.18) and inequality (1.19) hold for any \((x,t), t \geq 0\).

As a consequence of (1.24), the functional
\[
I(t) \equiv \int_{0}^{1} \left[ \frac{1}{2} u(x,t)^2(x) + W(v(x,t)) + \frac{1}{2} v^2(x,t) \right] \, dx \tag{1.28}
\]
is nonincreasing in \(t\). The following theorem gives our main asymptotic convergence result.
Theorem 1.2. Let \((v, u)\) be as in the conclusion of Theorem 1.1 with total volume \(\int_0^1 v(x, t) \, dx = \int_0^1 v_0(x) = M\). Then

1. the distance between \((v(\cdot, t), u(\cdot, t))\) and \((S_M, 0)\) in \(H^1([0, 1]) \times L^2([0, 1])\) goes to zero as \(t \to \infty\);
2. there is a limit of \(\int_0^1 p(v(x, t)) \, dx\) as \(t \to \infty\);
3. if for every positive value of \(c\) the equation

\[
0 < v(x), \int_0^1 v(x) \, dx = M, \quad v_{xx} + p(v) - c = 0, \quad v_x(0) = v_x(1) = 0 \quad (1.29)
\]

has at most countably many solutions, then there is an element \(\tilde{v} \in S_M\) such that

\[
\text{dist}( (v(\cdot, t), u(\cdot, t)), (\tilde{v}, 0) ) \to 0 \text{ in } H^1([0, 1]) \times L^2([0, 1]).
\]

The assumption in (2) can be easily verified by ODE techniques (cf. for example Schaaf [4], Laugesen and Pugh [5]), but we will not present details of such verification in this paper.

The following theorems demonstrate that the time-asymptotics is not always given by constant-state solutions.

Theorem 1.3. There is a pressure \(p\) satisfying (1.5) and a sequence of initial data \(\{(v_0^n, u_{0n})\}\) with total volume \(M\) for which \(\{v_n\} \to v_0 \equiv M\) in \(H^1([0, 1])\) and the corresponding weak solutions satisfy

\[
\text{dist}( (v_n(\cdot, t), u_n(\cdot, t)), (M, 0) ) = C_n > 0 \text{ in } H^1([0, 1]) \times L^2([0, 1]) \text{ for } t \geq 0.
\]

Theorem 1.4. The two-phase global minimizers of \(\int_0^1 [W(v) + \frac{1}{2}v_x^2] \, dx\) identified by Carr, Gurtin, and Slemrod in [7] for a very wide class of van-der-Waals-like pressure functions \(p\) are dynamically stable in the following sense: for each such minimizer \(\tilde{v}\) and each \(\epsilon > 0\) there exists \(\delta > 0\) such that if \((v_0, u_0)\) satisfies

\[
\|v_0 - \tilde{v}\|_{H^1} + \|u_0\|_{L^2} < \delta,
\]

then the solution \((u, v)\) of (1.1)–(1.4) with initial data \((v_0, u_0)\) satisfies

\[
\text{dist}_{H^1 \times L^2}( (v(\cdot, t), u(\cdot, t)), (\tilde{v}, 0) ) < \epsilon.
\]

The plan of the paper is as follows. In Section 2 we develop approximate solutions of (1.1)–(1.4) by the method of semidiscrete difference schemes. This approach enables us to derive a number of energy estimates without having to deal with the issue of a priori regularity. At the same time we derive a discretized versions of (1.16)–(1.19). In Section 3 we find the limit as the discretization parameter tends to zero and prove that the limit is indeed a weak solution of (1.1)–(1.4). In Section 4 exploiting the inequality (1.19) and monotonicity of \(I\), we give a proof of Theorem 1.2. Even though Theorem 1.3 is an immediate consequence of Theorem 1.4, a very short proof of Theorem 1.3 which does not refer to the Carr–Gurtin–Slemrod minimizers opens the final Section 5; this is to elucidate how easily convergence to constant solutions does not occur. The proof of Theorem 1.4 concludes our analysis.
2. Difference approximations

In this section we construct approximate solutions to the system (1.1)–(1.4) and show that they satisfy various regularity estimates. We employ the method of semidiscrete difference schemes which can be considered standard for Navier–Stokes equations in Lagrangean coordinates. In fact, our analysis shares many ideas with the one in Hoff [6].

Let \( N \) be a positive integer number and \( h \) be an increment in \( x \) given by \( h = \frac{1}{N} \). We set \( x_n = nh \) for \( n = 0, 1, 2, \ldots, N \), and \( x_j = jh, \ j = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots, N + \frac{1}{2} \). Define the difference operator \( \delta \) by

\[
\delta w_l = \frac{w_{l+1/2} - w_{l-1/2}}{h}.
\]

We then use the following semidiscrete scheme to generate approximations \( v_j(t) \) and \( u_k(t) \):

\[
\dot{v}_j - \delta u_j = 0, \quad j = \frac{1}{2}, \frac{3}{2}, \ldots, N - \frac{1}{2}, \quad (2.1)
\]

\[
\dot{u}_n + \delta p_n = \delta \left( \frac{\delta u}{v} \right)_n - \delta (\delta v)_n, \quad n = 1, 2, \ldots, N - 1, \quad (2.2)
\]

\[
u_0(t) = u_N(t) = 0, \quad v_{-\frac{1}{2}}(t) = v_{\frac{1}{2}}(t), \quad v_{N-\frac{1}{2}}(t) = v_{N+\frac{1}{2}}(t). \quad (2.3)
\]

It is implied that \( v_j \) and \( u_n \) depend also on \( h \), but we will drop the superscript \( h \) throughout the section.

**Lemma 2.1.** Assume that for some positive constant \( C \) the initial data \((v_0, u_0)\) satisfies

\[
\frac{1}{2} \sum_n u_0(x_n)^2h + \sum_j W(v_0(x_j))h + \frac{1}{2} \sum_n \delta v_0^2nh \leq CC_0. \quad (2.4)
\]

Then there is a global solution of (2.1)–(2.3) satisfying

\[
\frac{1}{2} \sum_n u_n(t)^2h + \sum_j W(v_j(t))h + \frac{1}{2} \sum_n \delta v_n^2nh + \int_0^t \sum_j \frac{\delta u_j^2(s)}{v_j(s)}hds \leq CC_0. \quad (2.5)
\]

**Proof.** The system of ODEs (2.1)–(2.3) has a local solution defined up to some positive time. Define

\[
I(t) = \frac{1}{2} \sum_{n=1}^{N-1} u_n(t)^2h + \sum_{1/2}^{N-1/2} W(v_j(t))h + \frac{1}{2} \sum_{n=1}^{N-1} \delta v_n^2h + \int_0^t \sum_j \frac{\delta u_j^2(s)}{v_j(s)}hds.
\]

We then find that \( \dot{I} = 0 \), so that \( I(t) = I(0) \) as long as \( v_j(t) > 0 \) for all \( j \). But this together with (1.7) and (1.8) implies that there is a positive constant \( C = C(C_0, h) \) such that

\[
0 < C^{-1} \leq |v_j(t)| \leq C \quad (2.6)
\]
as long as \((v_j, u_n)\) are defined. The truncated solution therefore exists for all time and (2.5) holds for any \(t \geq 0\).  

Below we list a number of higher order estimates for \((v_j, u_n)\).

**Lemma 2.2.** There is a positive constant \(C(T)\) depending on \(C_0\) and \(T\) such that

\[
\int_0^T \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \delta(\delta v)_j(t)^2 h \, dt \leq C(T), \tag{2.7}
\]

\[
\sup_{0 \leq t \leq T} \left( t \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \delta u_j(t)^2 h + t \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} \delta(\delta v)_j(t)^2 h \right) + \int_0^T \sum_{n=1}^{N-1} \delta(\delta u)_n(t)^2 h \, dt \leq C(T), \tag{2.8}
\]

\[
\sup_{0 \leq t \leq T} \left( t \sum_{n=1}^{N-1} \delta v_n(t)^2 h + t \sum_{n=1}^{N-1} \delta(\delta v)_n(t)^2 h \right) + \int_0^T \sum_{n=1}^{N-1} \delta(\delta u)_n(t)^2 h \, dt \leq C(T), \tag{2.9}
\]

\[
\sup_{0 \leq t \leq T} \left( t^2 \sum_{n=1}^{N-1} \delta(\delta v)_n(t)^2 h \right) \leq C(T), \tag{2.10}
\]

\[
\sup_{0 \leq t \leq T} \left( t^2 \sum_{n=1}^{N-1} \delta(\delta u)_n(t)^2 h \right) \leq C(T). \tag{2.11}
\]

**Proof.** Multiplying (2.2) by \(\delta v_n h\), summing over \(n\), integrating with respect to \(t\), and rearranging, we arrive at

\[
\int_0^t \sum_j \delta(\delta v)_j(s)^2 h \, ds = - \sum_n u_n(t) \delta v_n(t) h + \sum_n u_n(0) \delta v_n(0) h
\]

\[
- \int_0^t \sum_j \delta u_j(s) \dot{v}_j(s) h \, ds - \int_0^t \sum_n \delta p_j(s) \delta v_j(s) h \, ds
\]

\[
+ \int_0^t \sum_j \frac{\delta u_j(s)}{v_j(s)} \delta(\delta v)_j(s) h \, ds.
\]

We use the estimate (2.5) to show that each term on the right side can be bounded by \(C(t)\) plus a term that can be absorbed into the left side. This will prove (2.7).

In order to prove (2.8), we multiply both sides of (2.2) by \(t \delta(\delta u)_n h\), sum over \(n\), and integrate with respect to \(t\). Summing by parts where necessary and rearranging terms, we obtain that
\[
\frac{1}{2} \sum_j \delta u_j(t)^2 h + \frac{1}{2} t \sum_j \delta (\delta v)_j(t)^2 h + \frac{1}{2} \int_0^t s \sum_n \frac{\delta (\delta u)_n(s)^2}{v_{n+\frac{1}{2}}(s)} h \, ds
\]

\[
= \frac{1}{2} \int_0^t \sum_j \delta u_j(s)^2 h \, ds + \int_0^t t \sum_n \delta p_n(s) \delta (\delta u)_n(s) h \, ds
\]

\[
+ \frac{1}{2} \int_0^t \sum_j \delta (\delta v)_j(s)^2 h \, ds + \int_0^t s \sum_n \frac{\delta v_n(s) \delta (\delta u)_n(s)}{v_{n+\frac{1}{2}}(s)v_{n-\frac{1}{2}}(s)} h \, ds,
\]

which leads to the following estimate:

\[
\sup_{0 \leq t \leq T} \left( t \sum_j \delta u_j(t)^2 h + \sum_j \delta (\delta v)_j(t)^2 h \right) + \int_0^T t \sum_n \delta (\delta u)_n(t)^2 h \, dt
\]

\[
\leq C(T) + C(T) \int_0^T t \sum_n \delta u_{n-\frac{1}{2}}(t)^2 \delta v_n(t)^2 h \, dt.
\]

The integral on the right is dominated by \( C(T) \sup_{0 \leq t \leq T} (t \max_n \delta v_n(t)^2) \), which in turn is bounded by

\[
C(T) \sup_{0 \leq t \leq T} \left( \left[ \sum_n \delta v_n(t)^2 h \right] \left[ t \sum_j \delta (\delta v)_j(t)^2 h \right] \right)^{1/2}.
\]

Therefore the second term on the right can be absorbed into the left side of (2.12) thus proving (2.8).

We now multiply both sides of (2.2) by \( t \dot{u}_n h \), sum by parts, and integrate with respect to \( t \) to arrive at

\[
\int_0^T t \sum_n \dot{u}_n(t)^2 h \, dt \leq C(T) + C(T) \int_0^T t \sum_j |\delta u_j(t)|^3 h \, dt
\]

\[
\leq C(T) + C(T) \sup_{0 \leq t \leq T} \left( t \max_j |\delta u_j(t)| \right).
\]

Next, by differentiating (2.2) with respect to \( t \), multiplying by \( t^2 \dot{u}_n h \), summing over \( n \), and integrating, we show that

\[
\sup_{0 \leq t \leq T} \left( t^2 \sum_n \dot{u}_n(t)^2 h + t^2 \sum_n \delta \dot{v}_n(t)^2 h \right) + \int_0^T t^2 \sum_n \delta \dot{v}_n(t)^2 h \, dt
\]

\[
\leq C(T) + C(T) \int_0^T t \sum_n \dot{u}_n(t)^2 h \, dt + C(T) \int_0^T t^2 \sum_n \delta u_n(t)^4 h \, dt.
\]
The sum of the two equalities gives us

\[
\sup_{0 \leq t \leq T} \left( t^2 \sum_n \dot{u}_n(t)^2 h + t^2 \sum_n \delta \dot{v}_n(t)^2 h \right) + \int_0^T t^2 \sum_n \delta \dot{v}_n(t)^2 h \, dt \\
\leq C(T) + C(T) \sup_{0 \leq t \leq T} \left( t^2 \max_j |\delta u_j(t)|^2 \right). \tag{2.13}
\]

We estimate \( \max_j |\delta u_j|^2 \) as follows:

\[
\max_j |\delta u_j|^2 \leq \left[ \sum_j \delta u_j^2 h \right] + C \left[ \sum_j \delta u_j^2 h \right]^{1/2} \left[ \sum_n \delta (\delta u)^2 n h \right]^{1/2}.
\]

This inequality allows us to derive (2.9) from (2.13).

We observe that (2.10) follows immediately from Eq. (2.2) by means of inequality (2.9). In order to prove (2.11), we first differentiate (2.2) with respect to \( t \), multiply by \( t^3 \delta \dot{v}_n \), sum by parts, rearrange, and apply inequalities (2.7)–(2.9) to arrive at

\[
\int_0^T t^3 \sum_j \delta (\delta \dot{v})_j (t)^2 h \, dt \leq C(T) + C \left( \sum_j \delta \dot{u}_j (T)^2 h \right).
\]

Differentiating (2.2) with respect to \( t \), multiplying by \( t^4 \delta (\delta \dot{u})_n \), integrating, and summing by parts, we can also obtain that

\[
\sup_{0 \leq t \leq T} \left( t^4 \sum_j \delta \dot{u}_j (t)^2 h + t^4 \sum_j \delta (\delta \dot{v})_j (t)^2 h \right) \leq C(T) + C \int_0^T t^3 \sum_j \delta (\delta \dot{v})_j (t)^2 h \, dt.
\]

We now add the last two inequalities and this will complete the proof of (2.11).

Next we introduce new dependent variables \( r \) and \( q \) which will play a crucial role in the large-time behavior analysis. The values of these variables on mesh points are given by

\[
r_n = \sum_{k=1}^n u_k h - \sum_{k=1}^N \left( \sum_{m=1}^k u_m h \right) h, \quad n = 1, \ldots, N, \tag{2.14}
\]

\[
q_j = \ln(v_j) - \sum_{i=1 \frac{1}{2}}^{N - \frac{1}{2}} \ln(v_i) h - r_{j - \frac{1}{2}}, \quad j = \frac{1}{2}, \ldots, N + \frac{1}{2}. \tag{2.15}
\]

The first important expression to derive is the one for \( \dot{q}_j \):

\[
\dot{q}_j = p_j - \sum_{i=1 \frac{1}{2}}^{N - \frac{1}{2}} p_i h + \delta (\delta v)_j. \tag{2.16}
\]
Next we introduce a functional $E = \sum_{n=0}^{N-1} r_n \dot{q}_{n+\frac{1}{2}} h$. First we observe that

$$
E = \sum_{n=0}^{N-1} \left( \sum_{k=0}^{n} u_k h - \sum_{k=0}^{N} \left( \sum_{m=0}^{k} u_m h \right) h \right) \left( p_{n+\frac{1}{2}} - \sum_{i=1}^{N-\frac{1}{2}} \frac{N-\frac{1}{2}}{2} \right) \left( \delta(\dot{v})_{n+\frac{1}{2}} \right) h
$$

$$
= \sum_{n=0}^{N-1} \left( \sum_{k=0}^{n} u_k h - \sum_{k=0}^{N} \left( \sum_{m=0}^{k} u_m h \right) h \right) \left( p_{n+\frac{1}{2}} - \sum_{i=1}^{N-\frac{1}{2}} \frac{N-\frac{1}{2}}{2} \right) h - \sum_{n=0}^{N} \delta v_n u_n h,
$$

which together with (2.5) proves $|E(t)| \leq C$. Another important property of $E$ comes from the computation of $\dot{E}$:

$$
\dot{E} = \sum_{n=0}^{N-1} \left( \sum_{k=0}^{n} u_k h - \sum_{k=0}^{N} \left( \sum_{m=0}^{k} u_m h \right) h \right) \left( \dot{p}_{n+\frac{1}{2}} - \sum_{i=1}^{N-\frac{1}{2}} \frac{N-\frac{1}{2}}{2} \right) h
$$

Then we rearrange terms and sum by parts to obtain

$$
\dot{E} + \sum_{n=0}^{N-1} \dot{q}_{n+\frac{1}{2}}^2 h = \sum_{n=0}^{N-1} \left( \frac{\dot{v}_{n+\frac{1}{2}}}{v_{n+\frac{1}{2}}} - \sum_{i=1}^{\frac{N-\frac{1}{2}}{2}} \frac{\dot{v}_i}{v_i} \right) \dot{q}_{n+\frac{1}{2}} h + \sum_{i=1}^{\frac{N-\frac{1}{2}}{2}} \dot{v}_i \delta u_i h
$$

$$
\leq C^{-1} \sum_{n=0}^{N-1} \dot{q}_{n+\frac{1}{2}}^2 h + C \sum_{i=\frac{1}{2}}^{\frac{N-\frac{1}{2}}{2}} \delta u_i^2 h.
$$

Therefore

$$
\int_{0}^{T} \sum_{n=0}^{N-1} \dot{q}_{n+\frac{1}{2}}^2 h dt \leq C(E(0) - E(T)) + C \int_{0}^{T} \sum_{i=\frac{1}{2}}^{\frac{N-\frac{1}{2}}{2}} \delta u_i^2 h dt \leq C
$$

and

$$
\int_{0}^{\infty} \sum_{n=0}^{N-1} \dot{q}_{n+\frac{1}{2}}^2 h dt \leq C. \quad (2.17)
$$
3. Weak solution

We construct approximate solutions $v^h, u^h$ as follows: $v^h(x_j, t) = v_j(t)$ for $j = -\frac{1}{2}, \frac{1}{2}, \ldots, N + \frac{1}{2}$, $u^h(x_n, t) = u_n(t)$ for $n = 0, \ldots, N$, and take $v^h(\cdot, t)$ and $u^h(\cdot, t)$ to be linear on each interval $[x_j, x_{j+1})$ and $[x_n, x_{n+1})$, respectively. Below we state a number of regularity properties of $v^h$ and $u^h$.

**Lemma 3.1.** There are positive constants $C, C(T)$ depending on $C_0$ and $C_0, T$, respectively, and independent of $h$ such that

\[ C^{-1} \leq v^h(x, t) \leq C, \quad (3.1) \]

\[ \sup_{0 \leq t \leq T} \left( \int_0^1 v^h_x(x, t)^2 \, dx + t \int_0^1 v^h_{xx}(x, t)^2 \, dx + t^2 \int_0^1 v^h_{x}(x, t)^2 \, dx \right) \leq C(T), \quad (3.2) \]

\[ \sup_{0 \leq t \leq T} \left( \int_0^1 u^h(x, t)^2 \, dx + t \int_0^1 u^h_x(x, t)^2 \, dx \right) \leq C(T), \quad (3.3) \]

\[ \sup_{0 \leq t \leq T} \left( t^2 \int_0^1 u^h(x, t)^2 \, dx + t^4 \int_0^1 u^h_{x}(x, t)^2 \, dx \right) \leq C(T), \quad (3.4) \]

\[ \| v^h(\cdot, t_2) - v^h(\cdot, t_1) \|_{L^2} \leq C |t_1 - t_2|^{1/2}. \quad (3.5) \]

There is also a positive constant $C(T, \tau)$ depending on $C_0, T$, and $\tau$, such that for any $0 < \tau \leq t_1 < t_2 \leq T$

\[ |v^h(x_1, t_1) - v^h(x_2, t_2)| \leq C(T, \tau)(|x_1 - x_2|^{1/2} + |t_1 - t_2|), \quad (3.6) \]

\[ |u^h(x_1, t_1) - u^h(x_2, t_2)| \leq C(T, \tau)(|x_1 - x_2|^{1/2} + |t_1 - t_2|), \quad (3.7) \]

\[ |u^h_{x}(x_1, t_1) - u^h_{x}(x_2, t_2)| \leq C(T, \tau)(|x_1 - x_2|^{1/2} + |t_1 - t_2|^{1/4} + h^{1/2}). \quad (3.8) \]

**Proof.** Inequalities (3.1)–(3.4) follow immediately from (2.5), (2.6), (2.8), and (2.11). To prove (3.5) we refer to the inequality

\[ \int_0^1 (v^h(x, t_2) - v^h(x, t_1))^2 \, dx \leq \int_0^1 \left| \int_{t_1}^{t_2} v^h(x, \tau) \, d\tau \right|^2 \, dx \]

\[ \leq |t_2 - t_1| \int_{t_1}^{t_2} \int_0^1 u^h_{x}(x, \tau)^2 \, dx \, d\tau \leq C |t_2 - t_1|. \]

Next, $v^h$ and $u^h$ are Hölder continuous with respect to $x$ due to (3.2) and (3.3). Also,
\[ |u^h(x, t_1) - u^h(x, t_2)| \leq \int_{t_1}^{t_2} \|\dot{u}^h(\cdot, t)\|_{L^\infty} \, dt \]

\[ \leq \int_{t_1}^{t_2} \|\dot{u}^h(\cdot, t)\|_{L^2}^{1/2} \|\ddot{u}^h_x(\cdot, t)\|_{L^2}^{1/2} \, dt \leq C(T, \tau)|t_1 - t_2| \]

by (3.4). This completes the proof of (3.6). The proof of (3.7) is similar.

The derivation of (3.8) is the most interesting. First, the estimate

\[ |u^h_x(x_j, t) - u^h_x(x_{j'}, t)| \leq \left| \sum_{j < n < j'} \delta(\delta u)_n \right| \leq C(T, \tau)|x_j - x_{j'}|^{1/2} \]  

(3.9)

follows directly from (2.9). Next we derive time-regularity bound. Let \( J \) be a large positive integer. Then by (3.9)

\[ |u^h_x(x_j, t_1) - u^h_x(x_j, t_2)| \leq \frac{1}{Jh} \sum_{|i-j| < J} |u^h_x(x_i, t_1) - u^h_x(x_i, t_2)| + C(T, \tau)(Jh)^{1/2} \]

\[ \leq \sqrt{\frac{t_2 - t_1}{Jh}} \left( \int_{t_1}^{t_2} \sum_{|i-j| < J} \dot{u}^h_x(x_j, t)^2 \, dt \right)^{1/2} + C(T, \tau)(Jh)^{1/2} \]

\[ \leq C(T, \tau) \left( \frac{\sqrt{t_2 - t_1}}{Jh} + (Jh)^{1/2} \right). \]

We choose \( J \) so that \( Jh - h < \sqrt{t_2 - t_1} \leq Jh \). So (3.8) holds for \( x_1 = x_j, x_2 = x_{j'} \), and arbitrary \( t_1, t_2 > 0 \). To complete the proof, it remains to show that

\[ |u^h_x(x, t) - u^h_x(x_j, t)|, \quad |u^h_x(x, t) - u^h_x(x_{j+1}, t)| \leq C(T, \tau)h^{1/2} \]

when \( x \in (x_j, x_{j+1}) \). The above inequalities are indeed true because \( \|u^h_x(\cdot, t)\|_{L^\infty} = \max_j (|\delta u_j(t)|) \) and

\[ \max_j (|\delta u_j|) \leq C \left( \sum_j \delta u_j h \right)^{1/2} + C \left( \sum_j \delta u_j^2 h \right)^{1/4} \left( \sum_n \delta(\delta u)_n^2 h \right)^{1/4}. \]

Now we define a new variable \( w^h \) by \( w^h(x, t) = \delta v_j \) and taking \( w^h \) to be linear on \([x_j, x_{j+1}]\). We also introduce \( w^h \):

\[ w^h(x, t) = v^h(0, t) + \int_0^x w^h_n(y, t) \, dy. \]  

(3.10)

Both variables are necessary to show that the limit of \( v^h \) as \( h \to 0 \) satisfies high order regularity estimates. Below we list several properties of \( w^h \).
Proposition 3.1. There is a positive constant $C(T, \tau)$ depending on $C_0$, $T$, and $\tau$, such that for any $0 < \tau \leq t_1 < t_2 \leq T$

$$\sup_{0 \leq t \leq T} \left( t \int_0^1 w_{x}^h(x,t)^2 \, dx + t^4 \int_0^1 \dot{w}_{x}^h(x,t)^2 \, dx \right) \leq C(T), \quad (3.11)$$

$$\left| w_{x}^h(x_1, t_1) - w_{x}^h(x_2, t_2) \right| \leq C(T, \tau) (|x_1 - x_2|^{1/2} + |t_1 - t_2|), \quad (3.12)$$

$$\left| w_{xx}^h(x_1, t_1) - w_{xx}^h(x_2, t_2) \right| \leq C(T, \tau) (|x_1 - x_2|^{1/2} + |t_1 - t_2|^{1/4} + h^{1/2}). \quad (3.13)$$

Proof. Inequality (3.11) follows from (2.8) and (2.11) and the proof of (3.12) and (3.13) is similar to those of (3.7) and (3.8). \(\square\)

Now we are in a position to extract convergent subsequences as $h \to 0$.

Lemma 3.2. There is a sequence $h = \{h_l\} \to 0$ and functions $u, v, w_\ast \in H^1([0, 1])$, such that $u^h - u, v^h - v, w_\ast^h - w_\ast \to 0$ uniformly on compact sets in $\mathbb{R} \times (t > 0)$, $\quad (3.14)$

$v_x^h - v_x, u_x^h - u_x \to 0$ uniformly on compact sets in $\mathbb{R} \times (t > 0)$, $\quad (3.15)$

$w_{xx}^h \to w_{xx}$ uniformly on compact sets in $\mathbb{R} \times (t > 0)$. $\quad (3.16)$

There is also a function $w \in H^2([0, 1])$ such that

$w^h - w, w_x^h - w_x, w_{xx}^h - w_{xx} \to 0$ uniformly on compact sets in $\mathbb{R} \times (t > 0)$. $\quad (3.17)$

Proof. Hölder estimates (3.7), (3.8), (3.12), and (3.13) together with (3.1), (3.3), and (3.11) prove uniform convergence (3.14)–(3.16). (3.17) now follows from (3.10) with $w(x,t) = v(0,t) + \int_0^x w_\ast(y,t) \, dy$. \(\square\)

The following proposition gives us the main property of $w$:

Proposition 3.2. $v \equiv w$.

Proof. We observe that $v^h(x_n, t) = w^h(x_n, t)$ for any $n$. We now recall the space regularity of $v^h(\cdot, t)$ to complete the proof. \(\square\)

Now we have all the necessary elements to prove Theorem 1.1.

Proof of Theorem 1.1. First we show that the limit $(v, u)$ satisfies the pointwise and the energy inequalities. The bounds in (1.23) follow immediately from (3.1) and (3.14). The proofs of (1.25)–(1.27) are similar, so we show how to derive one of them. From (3.17) and Proposition 3.2, $w^h \to v$ in $D'(\mathbb{R} \times (0, \infty))$. So that $\dot{w}_x^h \to \dot{v}_x$ and $\dot{w}_{xx}^h \to \dot{v}_{xx}$ in $D'$. According to (3.2) and (3.11), there is a subsequence $h = h_{lm}$ for which $\dot{w}_x^h$ and $\dot{w}_{xx}^h$ converge weakly in $L^2([0, 1] \times (\tau, \infty))$ for any positive $\tau$. Therefore $\dot{w}_x^h \to \dot{w}_x^\ast$ and $\dot{w}_{xx}^h \to \dot{w}_{xx}^\ast$ weakly in
Since weak convergence is norm-decreasing, we also obtain from (3.2) and (3.11) that
\[
\int_{t_1}^{t_2} \int_{[0]}^{1} t^2 v_x^2 \, dx \, dt + \int_{t_1}^{t_2} \int_{[0]}^{1} t^4 v_{xx}^2 \, dx \, dt \leq C(T)(t_2 - t_1).
\]
Thus for a simple function \( \chi(t) \),
\[
\left| \int_{0}^{T} \chi(t) \left( \int_{0}^{1} t^2 v_x^2 \, dx + t^4 v_{xx}^2 \, dx \right) \, dt \right| \leq C(T) \| \chi \|_{L^1}.
\] (3.18)

The inequality (3.18) can be extended to all \( \chi \in L^1([0, T]) \), thus proving (1.25). We will now show that \((v, u)\) is a weak solution of (1.1)–(1.4). Let \( \psi \) be a smooth test function and set \( \varphi_j(t) = \varphi(x_j, t) \). We multiply (2.1) by \( \varphi_j h \), sum, and integrate to obtain
\[
\sum_j v_j \varphi_j h \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_j v_j \varphi_j h \, dt + \int_{t_1}^{t_2} \sum_n u_n \delta \varphi_n h \, dt = 0.
\] (3.19)

We need to show that each term in (3.19) converges to the corresponding term in (1.13). Indeed, the equalities
\[
\int_{t_1}^{t_2} \int_{x_j}^{x_{j+1}} u^h \varphi_x \, dx \, dt = \int_{t_1}^{t_2} \sum_j \int_{x_j}^{x_{j+1}} u^h \varphi_x \, dx \, dt
\]
\[
= \int_{t_1}^{t_2} \left( \sum_n u_n \varphi_x(x_n, t) \right) h + O(h) \sum_j \int_{x_j}^{x_{j+1}} \left( u^h \varphi_x \right)_x \, dx \, dt
\]
\[
= \int_{t_1}^{t_2} \sum_n u_n \varphi_x(x_n, t) h \, dt + O(h)
\]
prove that for the last term in (3.19). The other terms in (3.19) are treated in the same way. We multiply now (2.2) by \( \psi_n h \), sum, and integrate to obtain
\[
\sum_n u_n \psi_n h \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_n u_n \dot{\psi}_n h \, dt - \int_{t_1}^{t_2} \sum_j p_j \delta \psi_j h \, dt
\]
\[
= -\int_{t_1}^{t_2} \sum_j \frac{\delta u_j}{v_j} \delta \psi_j h \, dt + \int_{t_1}^{t_2} \sum_j v_j \delta (\delta \psi)_j \, h \, dt
\] (3.20)
and give the argument only for the most difficult term, which is the first term on the right.

\[ \int_{t_1}^{t_2} \int_0^1 \frac{u^h}{v^h} \psi_x dx dt = \int_{t_1}^{t_2} \sum_n \int_0^1 \frac{u^h}{v^h} \psi_x dx dt \]

\[ = \int_{t_1}^{t_2} \sum_j \delta u_j \left( \frac{\psi_x(x_j, t)}{v_j} h + O(h) \int_{x_{j-1/2}}^{x_{j+1/2}} \left| \frac{\psi_x}{v^h} \right| dx \right) dt \]

\[ = \int_{t_1}^{t_2} \sum_j \delta u_j \frac{\psi_x(x_j, t)}{v_j} h dt + O(h) \]

by (3.1) and (3.2), thus showing that the first term on the right in (3.20) approaches the corresponding term in (1.14) as \( h \to 0 \).

An argument similar to the one above proves (1.24) from (2.5).

Next we prove the regularity conditions (1.9) and (1.10). First, \( v \in C((0, \infty); H^2([0, 1])) \) and \( u \in C((0, \infty); H^1([0, 1])) \) by (3.6)–(3.8) and (3.12)–(3.13). Then, (3.5) shows that \( v_x(\cdot, t) \to v_{0x} \) in \( D'([0, 1]) \). The argument based on the uniform estimate (1.25) shows that

\[ v_x(\cdot, t) \to v_{0x} \quad \text{weakly in } L^2 \text{ as } t \to 0. \]

(3.21)

To prove strong convergence in (3.21), we fix \( t_2 \) and let \( \varphi \in C_0^2([0, 1]) \). Then

\[ \left. \int_0^1 v_x \varphi \right|_{t_1}^{t_2} - \int_0^1 [v_{x\varphi_t} + u_x \varphi_x] dx dt = 0. \]

We set \( t_1 = 0 \) and let \( \varphi \) approach \( v_x(\cdot, t_1) \) to obtain

\[ \int_0^1 v_x(x, t_1)^2 dx = \int_0^1 v_x(x, t_1) v_{0x}(x) dx + \int_0^1 u_x(x, t) v_{xx}(x, t_1) dx dt. \]

The first term on the right approaches \( \int_0^1 v_{0x}(x)^2 dx \) by (3.21) and the second term approaches zero as \( t_1 \to 0 \) due to (3.2). Therefore, \( v^h_x(\cdot, t) \to v_{0x} \) in \( L^2([0, 1]) \) and \( v \in C([0, \infty); H^1([0, 1])) \). A similar argument yields \( u(\cdot, t) \in C([0, \infty); L^2([0, 1])) \).

It is remains to see that (1.18) and (1.19) follow from (2.15)–(2.17) and (3.14)–(3.16). □

4. Large-time behavior

In this section we will prove Theorem 1.2.

First we show that \( \text{dist}((v(\cdot, t), u(\cdot, t)), (S_M, 0)) \to 0 \) in \( H^1([0, 1]) \times L^2([0, 1]) \). It follows from (1.19) and (1.24) that there is a sequence \( t = \{t_k\} \to \infty \) such that
$$p(\cdot, t_k) - \int_0^1 p(x, t_k) \, dx - v_{xx}(\cdot, t_k), u_x(\cdot, t_k) \to 0$$

strongly in $L^2([0, 1])$ as $k \to \infty$. Thus $v_{xx}(\cdot, t_k)$ is bounded in $L^2([0, 1])$ and this, together with (1.24), gives a further subsequence $\{t_{kn}\}$ for which $v(\cdot, t_{kn})$ and $u(\cdot, t_{kn})$ converge uniformly on $[0, 1]$, say to $\bar{v}_x$ and $\bar{v}$, respectively. Then

$$\lim_{n \to \infty} \left( \int_0^x \left[ p(v(y, t_{kn})) - \int_0^1 p(v(z, t_{kn})) \, dz - v_{xx}(y, t_{kn}) \right] \, dy \right)$$

$$= \lim_{n \to \infty} \left( \int_0^x \left[ (v(y, t_{kn})) - \int_0^1 p(v(z, t_{kn})) \, dz \right] \, dy - v_x(x, t_{kn}) \right)$$

$$= \int_0^x \left[ p(\bar{v}(y)) - \int_0^1 p(\bar{v}(z)) \, dz \right] \, dy - \bar{v}_x(x) = 0,$$

which immediately shows that $\bar{v} \in S_M$. In addition, $u(\cdot, t_{kn}) \to 0$ uniformly on $[0, 1]$ as $n \to \infty$. Consequently, $(v(\cdot, t_{kn}), u(\cdot, t_{kn}))$ converges to $(\bar{v}, 0)$ in $H^1([0, 1]) \times C([0, 1])$ and from the definition (1.28) of $I$,

$$I(v(\cdot, t_{kn}), u(\cdot, t_{kn})) \searrow I(\bar{v}(\cdot), 0), \quad n \to \infty.$$

The limit of $I$ over a subsequence can be improved to the limit as $t \to \infty$ due to the monotonicity of $I$. Thus

$$I(v(\cdot, t), u(\cdot, t)) \searrow I(\bar{v}(\cdot), 0), \quad t \to \infty. \quad (4.1)$$

Suppose that $\text{dist}(v(\cdot, t), S_M)$ does not go to zero in $C([0, 1])$ as $t \to \infty$. Then there exist a positive constant $c$ and a sequence $\{t_k\}$ such that $0 < c \leq \text{dist}(v(\cdot, t_k), S_M) \leq 2c$. Now there is a further subsequence $\{t'_{kn}\}$ and a function $v^* \in H^1([0, 1]), v^* \notin S_M$ for which $v(\cdot, t'_{kn}) \to v^*$ in $C([0, 1])$. Next we show that $\frac{d}{dt}(\|v(\cdot, t) - v^*\|^2_{L^2})$ is bounded. This will follow from the estimate

$$\left| \frac{d}{dt} \int_0^1 (v(x, t) - v^*(x))^2 \, dx \right| = \left| 2 \int_0^1 (v(x, t) - v^*(x))v_t(x, t) \, dx \right|$$

$$= \left| -2 \int_0^1 v_x(x, t)u(x, t) \, dx + 2 \int_0^1 v_x^*(x)u(x, t) \, dx \right|$$

$$\leq C.$$

This makes it possible for a fixed positive $\varepsilon$ to find $\tau > 0$ such that $\|v(\cdot, t) - v^*\|_{L^2} \leq \varepsilon$ when $t \in [t'_{kn}, t'_{kn} + \tau]$ and $n$ is large enough. Now we can construct yet another subsequence $\{t''_{kn}\}$
with the following properties: \( t_{k_n} \in [t_{k_n}, t_{k_n} + \tau] \) and \( \{(p - \int_0^1 p + v_{xx})(\cdot, t_{k_n})\} \) converges to zero in \( L^2([0, 1]) \). By taking a further subsequence \( \{t_{k_{n_m}}\} \), we can find a function \( \tilde{v} \in S_M \) such that \( \|v(\cdot, t_{k_{n_m}}) - \tilde{v}\|_{H^1} \to 0 \) as \( m \to \infty \) and \( I(\tilde{v}) = I(\tilde{v}) \). However, \( \|\tilde{v} - v^*\|_{L^2} \leq \varepsilon \) and, therefore, \( \tilde{v} \notin S_M \) if \( \varepsilon \) is small enough. This contradiction proves that \( \text{dist}_C(v(\cdot, t), S_M) \to 0 \) as \( t \to \infty \).

Define a subset \( T_v \subset S_M \) by

\[
T_v = \{ \tilde{v} \colon \tilde{v} \in S_M, \text{ and there is a sequence } \{t_k\} \to \infty \text{ such that } \|v(\cdot, t_k) - \tilde{v}\|_C \to 0 \}. \tag{4.2}
\]

First we observe that if \( \{t_k\} \) and \( \tilde{v} \in T_v \) are such that \( \|v(\cdot, t_k) - \tilde{v}\|_C \to 0 \) as \( t_k \to \infty \), then \( v_x(\cdot, t_k) \to \tilde{v}_x \) weakly in \( L^2([0, 1]) \) and thus

\[
I(\tilde{v}) = \int_0^1 W(\tilde{v}(x)) \, dx + \frac{1}{2} \int_0^1 \tilde{v}_x^2 \, dx \leq \lim_{t \to \infty} I(v(\cdot, t), u(\cdot, t)),
\]

which shows that \( T_v \) is bounded in \( C([0, 1]) \). Furthermore, for any \( \tilde{v}_n \in T_v \) we can apply the argument presented in this section to find a sequence \( \{\tilde{v}_n\} \subset T_v \) such that \( I(\tilde{v}_n) = \lim_{t \to \infty} I(v(\cdot, t), u(\cdot, t)) \) and \( \tilde{v}_n \to \tilde{v}_n \) strongly in \( L^2([0, 1]) \). Then we take a further subsequence \( \{n_k\} \) for which \( \tilde{v}_{nk} \) converges in \( C([0, 1]) \) and the limit is \( \tilde{v}_n \). We now resort to (1.20) to obtain

\[
\frac{1}{2} \tilde{v}_x^2 = W(\tilde{v}(x)) - W(\tilde{v}(0)), \tag{4.3}
\]

where \( W \) is given by (1.6) with \( a = \int_0^1 p(\tilde{v}(x)) \, dx \) and \( \tilde{v} \) is a stationary solution. Then

\[
I(\tilde{v}) = \int_0^1 W(\tilde{v}) \, dx + \int_0^1 \frac{1}{2} \tilde{v}_x^2 \, dx = 2 \int_0^1 W(\tilde{v}) \, dx - W(\tilde{v}(0)), \tag{4.4}
\]

and, therefore, \( I(\tilde{v}_{nk}) \to I(\tilde{v}_n) \) as \( k \to \infty \). This proves that all the elements of \( T_V \) have the same value of \( I \).

Suppose that \( \text{dist}_{H^1}(v(\cdot, t), S_M) \to 0 \) as \( t \to \infty \). Then we can find a sequence \( \{t_k\} \to \infty \) and a function \( \tilde{v} \in T_V \) such that \( \|v(\cdot, t_k) - \tilde{v}\|_C \to 0 \) and \( \|v(\cdot, t_k) - \tilde{v}\|_{H^1} \to 0 \) as \( t_k \to \infty \). Then \( v_x(\cdot, t_k) \to \tilde{v}_x \) weakly in \( L^2([0, 1]) \) and

\[
\int_0^1 \tilde{v}_x^2(x) \, dx \leq \lim_{n \to \infty} \left( \int_0^1 v_x^2(x, t_k) \, dx \right).
\]

We refer to the sequence of inequalities

\[
\int_0^1 \left( W(\tilde{v}) + \frac{1}{2} \tilde{v}_x \right) \, dx = \lim_{k \to \infty} \left( \int_0^1 \left( \frac{1}{2} u^2(\cdot, t_k) + W(v(\cdot, t_k)) + \frac{1}{2} v_x^2(\cdot, t_k) \right) \, dx \right)
\]
\[
\begin{align*}
\geq & \lim_{k \to \infty} \left( \int_0^1 \frac{1}{2} u^2(x, t_k) \, dx \right) + \int_0^1 W(\tilde{v}(x)) \, dx \\
& + \lim_{k \to \infty} \left( \int_0^1 \frac{1}{2} v^2(x, t_k) \, dx \right) \geq \int_0^1 \left( W(\tilde{v}) + \frac{1}{2} \tilde{v}_x \right) \, dx
\end{align*}
\]

to conclude that
\[
\lim_{k \to \infty} \int_0^1 v(x, t_k) \, dx = \int_0^1 \tilde{v}(x) \, dx, \quad \lim_{k \to \infty} \int_0^1 u(x, t_k) \, dx = 0,
\]
which prove that \( v(\cdot, t_k) \to \tilde{v} \) in \( H^1([0, 1]) \) and thus \( \text{dist}_{H^1} (v(\cdot, t), S_M) \to 0 \) as \( t \to \infty \). An argument similar to the above gives \( \lim_{t \to \infty} \int_0^1 u^2(x, t) \, dx = 0 \) and this will conclude the proof of part (1) of Theorem 1.2.

We now let \( \tilde{\tilde{v}} \in TV \) and \( W_0 = W(\tilde{\tilde{v}}(0)) \). Then by (4.3) there exist two consecutive roots \( v_1 = \alpha \) and \( v_2 = \beta \) of \( W(v) - W_0 \) such that
\[
W(v) - W_0 > 0 \quad \text{when } v \in [\alpha, \beta], \quad W(\alpha) - W_0 = W(\beta) - W_0 = 0,
\]
and
\[
x(\tilde{v}) = \frac{1}{\sqrt{2}} \int_\alpha^\beta \frac{dv}{\sqrt{W(v) - W_0}}, \quad \tilde{v} \in [\alpha, \beta]. \quad (4.5)
\]
In order for the integral in (4.5) to be integrable, both roots \( \alpha \) and \( \beta \) must be simple. This implies that there is a positive integer \( k \) such that \( x(\beta) = \frac{1}{k} \). Then the total volume \( M \) is given by
\[
M = k \sqrt{2} \int_\alpha^\beta \frac{v \, dv}{\sqrt{W_0 - W(v)}}. \quad (4.6)
\]
First we observe that the set \( TV \) is connected in \( C([0, 1]) \) and \( I \) is continuous. If \( c = \int_0^1 p(\tilde{v}) \) is nonconstant on \( TV \), then there is a sequence \( \{\tilde{v}_n\} \subset TV \) such that \( \tilde{v}_n \to \tilde{v} \) in \( C([0, 1]) \) and \( c_n = \int_0^1 p(\tilde{v}_n) \, dx \neq c = \int_0^1 p(\tilde{v}) \, dx \) for any \( n \). We now compute the limit \( \frac{dI}{dc_n} = \lim_{n \to \infty} \frac{I(\tilde{v}) - I(\tilde{v}_n)}{c - c_n} \):
\[
\frac{d}{dc_n} I(\tilde{v}) = \frac{d}{dc_n} \left( 2 \int_0^1 W(\tilde{v}) \, dx - W_0 \right) = \frac{d}{dc_n} \left( \sqrt{2k} \int_\alpha^\beta \frac{v \, dv}{\sqrt{W(v) - W_0}} + W_0 \right)
\]
\[
= k \int_\alpha^\beta \frac{v \, dv}{\sqrt{W(v) - W_0}} + \sqrt{W(\beta) - W_0} \frac{dB}{dc_n} - \sqrt{W(\alpha) - W_0} \frac{dA}{dc_n}
\]
\[
+ \frac{dW_0}{dc_n} = M \frac{dA}{dc_n} = M
\]
by (4.4), (4.6), the definition of $\alpha$, $\beta$, and the fact that $x(\beta) = \frac{1}{k}$. However, this is a contradiction to the fact that $I$ is constant on $T_V$. Thus we have shown that $a$ is constant on $T_V$ which together with the conclusion of part (1) proves (2).

In part (3) of Theorem 1.2 we can assume that $T_V$ contains more than one element, otherwise the proof is immediate. Then it follows from the proofs of parts (1) and (2) that $T_V$ contains a non-trivial continuum and that $\int_0^1 p(\tilde{v})$ is constant on $T_V$. This together with (1.20) contradicts the assumptions in (3).

5. Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3. For a functional $I[\tilde{v}] = \int_0^1 W(\tilde{v}) \, dx + \frac{1}{2} \int_0^1 \tilde{v}_x^2 \, dx$ we compute its second variation at $\tilde{v} \equiv c$, where $c$ is a constant:

$$\delta^2 I[\lambda] = \frac{d^2}{dt^2} I[c + t\lambda]_{t=0} = - \int_0^1 p'(c) \lambda^2 \, dx + \frac{1}{2} \int_0^1 \lambda_x^2 \, dx.$$  

We choose $\lambda(x) = \cos \pi x$, so that the total volume of $c + t\lambda$ is preserved. Then

$$\delta^2 I[\cos \pi x] = - \frac{1}{2} p'(c) + \frac{\pi^2}{2},$$

which can be negative if the value of $p'$ at $c$ is positive and large. In this case there is a sequence $\{t_n\} \to 0$ such that $I[c + t_n \cos(\pi x)] < I[c]$ and $c + t_n \cos(\pi x) > 0$. We now set $v_{0n} = c + t_n \cos(\pi x), u_{0n} = 0$. Then the solutions $(v_{n}, u_{n})$ of (1.1)--(1.4) with the initial data $(v_{0n}, u_{0n})$ approach $T_{V, n}$ with $M_n = c$ as $t \to \infty$. It is clear that $T_{V, n}$ are nonempty and do not contain $\tilde{v} \equiv c$. □

Proof of Theorem 1.4. The proof is based upon the fact that there are exactly two global minimizers of $I$ for pressure functions $p$ identified in [7].

If the conclusion of the theorem does not hold, then there are a positive $\varepsilon$, a sequence of initial data $\{(v_{0n}, u_{0n})\}$, a global minimizer $\tilde{v}$ of $I[v]$, and a sequence of positive values $\{t_n\}$ such that

$$(v_{0n}, v_{0n}) \to (\tilde{v}, 0) \quad \text{in} \quad H^1([0, 1]) \times L^2([0, 1])$$

and weak solutions $(v_n, u_n)$ of (1.1)--(1.4) with initial data $(v_{0n}, v_{0n})$ satisfy

$$\varepsilon \leq \|v_n(\cdot, t_n) - \tilde{v}\|_{H^1} + \|u(\cdot, t_n)\|_{L^2} \leq 2\varepsilon.$$  

Without loss of generality we can assume that $\varepsilon$ is small. Then we can find a subsequence $\{(v_{nk}, u_{nk})\}$ and a function $v^* \in H^1([0, 1])$ for which $v_{nk}(\cdot, t_{nk}) \to v^*$ in $L^2([0, 1])$ and $v^*$ is different from the other global minimizer of $I[v]$. We can now conclude that

$$\lim_{k \to \infty} I(v_{nk}, u_{nk}) \geq I(v^*, 0)$$

and $I[\tilde{v}] \geq I[v^*]$ due to the monotonicity property of $I$. Therefore, $v^*$ is a global minimizer different from the other two. □
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References