Growth theorems and coefficient bounds for univalent holomorphic mappings which have parametric representation

Hidetaka Hamada a,*,1, Tatsuhiro Honda b,2,3, Gabriela Kohr c

a Faculty of Engineering, Kyushu Sangyo University, 3-1 Matsukadai 2-Chome, Higashi-ku, Fukuoka 813-8503, Japan
b Department of Mathematics, Ariake National College of Technology, 150 Higashihagio-machi, Onuta-shi, Fukuoka 836-8585, Japan
c Faculty of Mathematics and Computer Science, Babeş-Bolyai University, I M. Kogălniceanu Str., 400084 Cluj-Napoca, Romania

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Abstract

Let $B$ be the unit ball in $\mathbb{C}^n$ with respect to an arbitrary norm and let $f(z, t)$ be a $g$-Loewner chain such that $e^{-t} f(z, t) - z$ has a zero of order $k + 1$ at $z = 0$. In this paper, we obtain growth and covering theorems for $f(\cdot, 0)$. Moreover, we consider coefficient bounds and examples of mappings in $S^{0}_{g,k+1}(B)$.© 2005 Elsevier Inc. All rights reserved.

* Corresponding author.
E-mail addresses: h.hamada@ip.kyusan-u.ac.jp (H. Hamada), thonda@cc.it-hiroshima.ac.jp (T. Honda), gkohr@math.ubbcluj.ro (G. Kohr).
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2 Partially supported by Grant-in-Aid for Scientific Research (C) No. 15540193 from Japan Society for the Promotion of Science, 2005.
3 Current address: Department of Civil and Architectural Engineering, Hiroshima Institute of Technology, 2-1-1 Miyake, Saeki-ku, Hiroshima 731-5193, Japan.

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1. Introduction and preliminaries

Let $\mathbb{C}^n$ denote the space of $n$ complex variables $z = (z_1, \ldots, z_n)$ with respect to an arbitrary norm $\| \cdot \|$. Let $B = \{ z \in \mathbb{C}^n : \| z \| < 1 \}$. Let $U$ be the unit disc in $\mathbb{C}$. Let $H(G)$ denote the set of holomorphic mappings from an open set $G \subset \mathbb{C}^n$ into $\mathbb{C}^n$. Further, let $L(\mathbb{C}^n, \mathbb{C}^m)$ be the space of all continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm. Let $I$ be the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$. A mapping $f \in H(B)$ is called normalized if $f(0) = 0$ and $Df(0) = I$.

Let $S(B)$ be the set of normalized univalent holomorphic mappings in $H(B)$. Also let $K(B)$, respectively $S^*(B)$, be the sets of normalized convex, respectively starlike, mappings on $B$. When $n = 1$, the sets $S(U)$, $S^*(U)$ and $K(U)$ are denoted by $S$, $S^*$ and $K$, respectively.

For each $z \in \mathbb{C}^n \setminus \{0\}$, we set $T(z) = \{ l_z \in L(\mathbb{C}^n, \mathbb{C}) : l_z(z) = \| z \|, \| l_z \| = 1 \}$. Then this set is nonempty by the Hahn–Banach theorem.

If $f, g \in H(B)$, we say that $f$ is subordinate to $g$, and write $f \prec g$, if there exists a Schwarz mapping $v$ (i.e., $v \in H(B)$, $v(0) = 0$, and $\| v(z) \| < 1$, $z \in B$) such that $f = g \circ v$ on $B$. If $g$ is univalent on $B$, this condition is equivalent to $f(0) = g(0)$ and $f(B) \subset g(B)$.

We recall that a mapping $f : B \times [0, \infty) \to \mathbb{C}^n$ is called a Loewner chain if $f(\cdot, t)$ is univalent holomorphic on $B$, $f(0, t) = 0$, $Df(0, t) = e^t I$ for $t \geq 0$, and

$$f(z, s) \prec f(z, t), \quad z \in B, \quad 0 \leq s \leq t < \infty.$$  

The above condition is equivalent to the fact that there exists a unique univalent Schwarz mapping $v = v(z, s, t)$, called the transition mapping of $f(z, t)$, such that $f(z, s) = f(v(z, s, t), t)$, $z \in B$, $t \geq s \geq 0$. The normalization of $f(z, t)$ implies the normalization $Dv(0, s, t) = e^{s-t} I$ for $t \geq s \geq 0$.

A fundamental role in the study of Loewner chains and the Loewner differential equation in several complex variables is played by the following set:

$$\mathcal{M} = \left\{ p \in H(B) : p(0) = 0, \quad Dp(0) = I, \quad \text{Re} \|l_z(p(z))\| > 0, \quad z \in B \setminus \{0\}, \quad l_z \in T(z) \right\},$$

which is the generalization of the Carathéodory set in one complex variable.

In [8] (see also [7]; cf. [5]), it is proved the following result:

Lemma 1. Let $f(z, t)$ be a Loewner chain and $v = v(z, s, t)$ be the transition mapping of $f(z, t)$. Then $f(z, \cdot)$ is locally Lipschitz continuous on $[0, \infty)$ locally uniformly with respect to $z \in B$, and there exists a mapping $h = h(z, t)$ such that $h(\cdot, t) \in \mathcal{M}$, $t \geq 0$, $h(z, \cdot)$ is measurable on $[0, \infty)$, and

$$\frac{\partial f}{\partial t}(z, t) = Df(z, t)h(z, t), \quad a.e. \ t \geq 0, \ \forall z \in B. \quad (1)$$

Also $v(z, s, t)$ satisfies the initial value problem

$$\frac{\partial v}{\partial t} = -h(v, t), \quad a.e. \ t \geq s, \quad v(z, s, s) = z. \quad (2)$$
for all \( z \in B \) and \( s \geq 0 \). Moreover, if \( \{ e^{-t} f(z, t) \}_{t \geq 0} \) is a normal family on \( B \), then for every \( s \geq 0 \),
\[
\lim_{t \to \infty} e^{t} v(z, s, t) = f(z, s)
\]
and the above limit holds locally uniformly on \( B \).

**Definition 2.** Let \( f : B \to \mathbb{C}^n \) be a normalized holomorphic mapping. We say that \( f \) has parametric representation if there exists a mapping \( h = h(z, t) \) which satisfies the conditions in Lemma 1 such that \( f(z) = \lim_{t \to \infty} e^{t} v(z, t) \) locally uniformly on \( B \), where \( v = v(z, t) \) is the unique solution of the initial value problem
\[
\frac{\partial v}{\partial t} = -h(v, t), \quad \text{a.e. } t \geq 0, \quad v(z, 0) = z,
\]
for all \( z \in B \) (see [5]; cf. [19,27]).

Let \( S^0(B) \) be the set of all mappings which have parametric representation on \( B \). Then \( S^0(B) \subset S(B) \) (see [5,27]). It is known that in the case of one complex variable, \( S^0(U) = S \) (see [26, Theorems 6.1 and 6.3]). However, in higher dimensions, \( S(B) \) is a larger set than \( S^0(B) \) (see [5]).

**Remark 3.** Let \( f(z, t) \) be a Loewner chain such that \( \{ e^{-t} f(z, t) \}_{t \geq 0} \) is a normal family on \( B \). According to Lemma 1, we deduce that \( f = f(\cdot, 0) \in S^0(B) \). Graham–Hamada–Kohr [5] (cf. [28]) proved that the converse result is also true. That is, if \( f \in S^0(B) \), then there exists a Loewner chain \( f(z, t) \) such that \( \{ e^{-t} f(z, t) \}_{t \geq 0} \) is a normal family on \( B \) and \( f = f(\cdot, 0) \).

**Assumption 4.** Let \( g : U \to \mathbb{C} \) be a univalent holomorphic function such that \( g(0) = 1 \), \( g(\zeta) = g(\zeta) \) for \( \zeta \in U \) (so, \( g \) has real coefficients in its power series expansion), \( \Re g(\zeta) > 0 \) on \( U \). We assume that \( g \) satisfies the conditions
\[
\begin{align*}
\min_{|\zeta|=r} \Re g(\zeta) &= \min\{g(r), g(-r)\}, \\
\max_{|\zeta|=r} \Re g(\zeta) &= \max\{g(r), g(-r)\},
\end{align*}
\]
for \( r \in (0, 1) \).

We mention that there are many functions which satisfy the above assumption (see [5]). As in [5,19], we shall introduce various subsets of \( \mathcal{M} \). Let
\[
\mathcal{M}_g = \left\{ p \in H(B) : p(0) = 0, \quad Dp(0) = I, \quad \frac{1}{\|z\|} l_z(p(z)) \in g(U), \quad z \in B \setminus \{0\}, \quad l_z \in T(z) \right\}.
\]
If \( g(\zeta) = (1 + \zeta)/(1 - \zeta) \), then \( \mathcal{M}_g = \mathcal{M} \). However, there are other choices of \( g \) which provide interesting properties of the set \( \mathcal{M}_g \).

The basic existence theorem for the Loewner differential equation on \( B \), originally due to Pfaltzgraff (see [23, Theorem 2.1]), can be improved by omitting the boundedness assumption on \( h(z, t) \). The following proposition is due to [5, Theorem 1.4] (cf. [27]).
Proposition 5. Let $g : U \to \mathbb{C}$ satisfy the conditions of Assumption 4 and let $h = h(z,t) : \mathcal{B} \times [0, \infty) \to \mathbb{C}^n$ satisfy the following conditions:

(i) for each $t \geq 0$, $h(\cdot, t) \in \mathcal{M}_g$;
(ii) for each $z \in \mathcal{B}$, $h(z, t)$ is a measurable function of $t \in [0, \infty)$.

Then the limit
$$\lim_{t \to \infty} e^t v(z, s, t) = f(z, s)$$
exists locally uniformly on $\mathcal{B}$ for each $s \geq 0$, where $v = v(z, s, t)$ is the unique solution of the initial value problem (2). The mapping $v(z, s, t) = e^{s-t} z + \cdots$ is a univalent Schwarz mapping on $\mathcal{B}$ and is a locally Lipschitz function of $t \geq s$ locally uniformly with respect to $z \in \mathcal{B}$. Moreover, $f(z, t)$ is a Loewner chain and $v(z, s, t)$ is the transition mapping associated to $f(z, t)$. Further, $f$ and $h$ satisfy the differential equation (1).

In view of the above result, Graham–Hamada–Kohr [5] (cf. [19,27]) have recently introduced the following definition.

Definition 6. Let $g$ satisfy the conditions of Assumption 4 and let $f \in \mathcal{H}(\mathcal{B})$. We say that $f$ has $g$-parametric representation on $\mathcal{B}$ if there exists a mapping $h(z, t) : \mathcal{B} \times [0, \infty) \to \mathbb{C}^n$, which satisfies the assumptions of Proposition 5 such that
$$\lim_{t \to \infty} e^t v(z, s, t) = f(z, s)$$
locally uniformly on $\mathcal{B}$, where $v = v(z, t)$ is the unique solution of the initial value problem (3).

Let $S^0_\mathcal{B}(\mathcal{B})$ be the set of all mappings which have $g$-parametric representation on $\mathcal{B}$ [5] (cf. [19]). If $g(\zeta) = (1 + \zeta)/(1 - \zeta)$, then $S^0_\mathcal{B}(\mathcal{B})$ reduces to the set $S^0(\mathcal{B})$ of mappings which have parametric representation on $\mathcal{B}$. Clearly, $S^0_\mathcal{B}(\mathcal{B}) \subset S^0(\mathcal{B}) \subset S(\mathcal{B})$. On the other hand, we remark that, in several complex variables, there exist mappings which can be imbedded in Loewner chains without having parametric representation [5, Example 2.12].

Remark 7. According to Definition 6, Proposition 5 and Lemma 1, a mapping $f$ belongs to $S^0_\mathcal{B}(\mathcal{B})$ if and only if there exist a Loewner chain $f(z, t)$ and a mapping $h(z, t)$ such that $f = f(\cdot, 0)$, $\{e^{-t} f(z, t)\}_{t \geq 0}$ is a normal family on $\mathcal{B}$, $h(\cdot, t) \in \mathcal{M}_g$, $t \geq 0$, $h(z, \cdot)$ is measurable on $[0, \infty)$ for $z \in \mathcal{B}$, and $f(z, t)$ satisfies the Loewner differential equation (1). Such a Loewner chain is also called a $g$-Loewner chain (cf. [5,19]). This equivalence provides many examples of mappings which have $g$-parametric representation on $\mathcal{B}$.

It is known that certain subsets of $S(\mathcal{B})$ can be characterized in terms of Loewner chains. For example, a mapping $f$ belongs to $S^*(\mathcal{B})$ if and only if $f(z, t) = e^t f(z)$ is a Loewner chain. On the other hand, according to [24], we say that a normalized locally biholomorphic mapping $f \in \mathcal{H}(\mathcal{B})$ is said to be close-to-starlike if there exists a mapping $g \in S^*(\mathcal{B})$ such that
$$\Re_l(z) \left( [Df(z)]^{-1} g(z) \right) > 0$$
for all \( z \in B \setminus \{0\} \) and \( l_z \in T(z) \). Let \( C(B) \) denote the set of all close-to-starlike mappings on \( B \). It is known that every mapping \( f \in C(B) \) is univalent on \( B \). Moreover, close-to-starlikeness can also be characterized in terms of Loewner chains, in the sense that \( f \in C(B) \) if and only if there exists a mapping \( g \in S^*(B) \) such that \( f(z,t) = f(z) + \left( e^t - 1 \right) g(z) \) is a Loewner chain.

Let \( 0 < p < 1 \). A normalized locally biholomorphic mapping \( f \in H(B) \) is said to be starlike of order \( p \) if
\[
\frac{1}{\|z\|} l_z \left( \left[ Df(z) \right]^{-1} f(z) \right) - \frac{1}{2p} < \frac{1}{2p}
\]
for all \( z \in B \setminus \{0\} \) and \( l_z \in T(z) \). We denote by \( S_p^*(B) \) the set of all starlike mappings of order \( p \) on \( B \). It is clear that \( S_p^*(B) \subset S^*(B) \) for \( p \in (0,1) \).

Another set of special interest in our discussion is that consisting of quasi-convex mappings. This subset of \( S(B) \) was introduced by Roper–Suffridge [29] as a natural generalization to higher dimensions of convexity. Let \( u \in \mathbb{C}^n \) with \( \|u\| = 1 \) and \( l_u \in T(u) \).

Let \( G \) denote the set of all normalized locally biholomorphic mappings \( f \) on \( B \) that satisfy the condition \( \Re G_f(\alpha,\beta) > 0 \), for all \( \alpha, \beta \in U, u \in \mathbb{C}^n \) with \( \|u\| = 1 \) and \( l_u \in T(u) \). Roper and Suffridge [29] proved the inclusion relation
\[
K(B) \subset G \subset S_{1/2}^*(B)
\]
and obtained several properties of the mappings in \( G \). In particular, they obtained the sharp growth result for mappings in \( G \) on the Euclidean unit ball. In [5], it is shown that this result is also valid in the case of an arbitrary norm. We shall refer to the set \( G \) as the set of quasi-convex mappings.

Let \( \alpha \in (-\pi/2, \pi/2) \) and \( f \in H(B) \) be a normalized locally biholomorphic mapping. According to [11], we say that \( f \) is spirallike of type \( \alpha \) if
\[
\Re l_z \left( e^{-i\alpha} \left[ Df(z) \right]^{-1} f(z) \right) > 0, \quad z \in B \setminus \{0\}, \ l_z \in T(z).
\]

Let \( \tilde{S}_\alpha(B) \) be the set of spirallike mappings of type \( \alpha \). In [11], it is proved that every \( f \in \tilde{S}_\alpha(B) \) is univalent on \( B \) and also the following alternative characterization of spirallikeness of type \( \alpha \) is proved: \( f \) is spirallike of type \( \alpha \) if and only if \( f(z,t) = e^{(1-i\alpha)t} f(e^{i\alpha}z) \) is a Loewner chain, where \( \alpha = \tan \alpha \).

We have the following inclusion relations:
\[
S^*(B) \subset C(B) \subset S^0(B) \subset S(B)
\]
and
\[
\tilde{S}_\alpha(B) \subset S^0(B), \quad |\alpha| < \frac{\pi}{2}.
\]

Let \( f \in H(B) \) and let \( k \) be a positive integer. Then \( f \) is said to be \( k \)-fold symmetric if the image of \( f \) is unchanged when it multiplied by the scalar complex number \( \exp(2\pi i/k) \). We say that \( z = 0 \) is a zero of order \( k \) of \( f(z) \) if \( f(0) = 0, \ldots, D^{k-1} f(0) = 0 \) and \( D^k f(0) \neq 0 \).
and } f(0) \text{ is the } j\text{th Fréchet derivative of } f \text{ at } z = 0. \text{ We note that } z = 0 \text{ is a zero of order } m \text{ of } f(z) - z \text{ for some } m \text{ with } m \geq k + 1 \text{ if } f \text{ is } k\text{-fold symmetric and } f(z) \neq z.

We denote by } S^0_{k+1}(B) \text{ (respectively } S^0_{g,k+1}(B)) \text{ the subset of } S^0(B) \text{ (respectively } S^g(B)) \text{ consisting of mappings } f \text{ for which there exists a Loewner chain (respectively a } g\text{-Loewner chain) } f(z,t) \text{ such that } \{e^{-t} f(z,t)\}_{t \geq 0} \text{ is a normal family on } B, f = f(\xi,0) \text{ and } z = 0 \text{ is a zero of order } k + 1 \text{ of } e^{-t} f(z,t) - z \text{ for each } t \geq 0. \text{ Also, we denote by } S_{k+1}(B) \text{ (respectively } S_{g,k+1}(B), K_{k+1}(B), G_{k+1}(B), S_p^{g,k+1}(B), \hat{S}_{\alpha,k+1}(B)) \text{ the subset of } S(B) \text{ (respectively } S^{g}(B), K(B), G(B), S_p^{g}(B), \hat{S}_{\alpha}(B)) \text{ of mappings } f \text{ such that } z = 0 \text{ is a zero of order } k + 1 \text{ of } f(z) - z. \text{ Moreover, we denote by } C_{k+1}(B) \text{ the subset of } C(B) \text{ of mappings } f \text{ such that } z = 0 \text{ is a zero of order } k + 1 \text{ of } e^{-t} f(z) + e^{-t}(e^t - 1)g(z) - z \text{ for each } t \geq 0.

In the rest of this paper, we shall obtain growth and covering theorems, as well as coefficient bounds for mappings in } S^0_{g,k+1}(B). \text{ These results were obtained by Graham, Hamada and Kohr [5] in the case } k = 1 \text{ (cf. [19,27]). Some of the growth and covering theorems in this paper are generalizations of the results in Liu–Liu [20].}

2. A growth theorem for mappings in } S^0_{g,k+1}(B)

In this section, we will give a growth theorem for mappings in } S^0_{g,k+1}(B). \text{ To this end, we need to use the following lemma.}

**Lemma 8.** Let } g \text{ satisfy the conditions of Assumption 4, } h \text{ satisfy the assumptions of Proposition 5 and } f(z,t) \text{ be a } g\text{-Loewner chain satisfying the differential equation (1) such that } z = 0 \text{ is a zero of order } k + 1 \text{ of } e^{-t} f(z,t) - z. \text{ Then}

\[
\|z\| \min\left\{g(\|z\|^k), g(-\|z\|^k)\right\} \leq \Re l_z(h(z,t)) \leq \|z\| \max\left\{g(\|z\|^k), g(-\|z\|^k)\right\}
\]

for } z \in B \setminus \{0\}, l_z \in T(z) \text{ and a.e. } t \geq 0.

**Proof.** Fix } t \geq 0 \text{ such that the differential equation (1) holds. We take a point } z_0 \text{ with } \|z_0\| = 1. \text{ Let } p(\xi, t) : U \to \mathbb{C} \text{ be given by}

\[
p(\xi, t) = \begin{cases} \xi l_{z_0}(h(\xi z_0, t)), & \xi \neq 0, \\ 1, & \xi = 0. \end{cases}
\]

Then } p(\xi, t) \in H(U), p(0, t) = g(0) = 1. \text{ Since } z = 0 \text{ is a zero of order } k + 1 \text{ of } e^{-t} f(z,t) - z \text{ for each } t \geq 0, \text{ there exists a holomorphic mapping } F(z,t) \text{ on a neighbourhood of } 0 \text{ such that } f(\xi z_0, t) = e^t \xi z_0 = \xi^{k+1} F(\xi z_0, t). \text{ Then}

\[
\frac{\partial f}{\partial t}(\xi z_0, t) = e^t \xi z_0 = \xi^{k+1} \frac{\partial F}{\partial t}(\xi z_0, t).
\]

Therefore, we obtain that

\[
p(\xi, t) = l_{z_0} \left( \left[Df(\xi z_0, t)^{-1} e^t \right]^{-1} - l_{z_0} \left( \left[Df(\xi z_0, t)^{-1} e^t \right]^{-1} \frac{\partial F}{\partial t}(\xi z_0, t) \right) \xi^k.\right.
\]
Thus, there exists a holomorphic function \( \tilde{p}(\xi, t) \) on a neighbourhood of 0 such that
\[
p(\xi, t) = 1 + \xi^k \tilde{p}(\xi, t).
\]
Since \( h(z, t) \in M_g \), we deduce that \( p(\xi, t) \in g(U) \) for \( \xi \in U \). Therefore, \( g^{-1} \circ p(\cdot, t) : U \to U \) and \( g^{-1} \circ p(0, t) = 0 \). Since \( g^{-1}(1) = 0 \), there exists a holomorphic function \( G(w) \) on a neighbourhood of 1 such that
\[
g^{-1}(w) = (w - 1)G(w).
\]
Therefore, we obtain that
\[
g^{-1} \circ p(\xi, t) = \xi^k \tilde{p}(\xi, t)G(p(\xi, t))
\]
on a neighbourhood of 0.

Then, by the Schwarz lemma, we obtain that
\[
|g^{-1} \circ p(\xi, t)| \leq |\xi|^k \text{ for } \xi \in U.
\]
Thus, there exists a holomorphic function \( H(\xi, t) \) on \( U \) such that
\[
g^{-1} \circ p(\xi, t) = \xi^k H(\xi, t)
\]
on \( U \) and
\[
|H(\xi, t)| \leq 1 \text{ on } U.
\]
Then, \( p(\xi, t) = g(\xi^k H(\xi, t)) \). Next, in view of the maximum and minimum principle of harmonic functions, we conclude that
\[
\min\{g(\|\xi\|^k), g(-\|\xi\|^k)\} \leq \Re p(\xi, t) \leq \max\{g(\|\xi\|^k), g(-\|\xi\|^k)\}, \ \xi \in U.
\]
(5)

Putting \( \xi = \|z\| \) in (5), we obtain (4). This completes the proof.  

The following lemma generalizes [5, Lemma 2.1] (cf. [19]).

**Lemma 9.** Let \( f, g \) and \( h \) be as in Lemma 8. Also let \( v = v(z, s, t) \) be the solution of the initial value problem (2). Then
\[
e^t\|z\| \exp \int_{\|v(z, s, t)\|} \left[ \frac{1}{\max\{g(x^k), g(-x^k)\}} - 1 \right] \frac{dx}{x}
\]
\[
\leq e^t\|v(z, s, t)\| \leq e^t\|z\| \exp \int_{\|v(z, s, t)\|} \left[ \frac{1}{\min\{g(x^k), g(-x^k)\}} - 1 \right] \frac{dx}{x}
\]
(6)
for \( z \in B \) and \( t \geq s \geq 0 \).

**Proof.** We will prove the upper bound. The proof of the lower bound is similar. Fix \( s \geq 0 \) and \( z \in B \setminus \{0\} \) and let \( v(t) = v(z, s, t) \). Since \( v(t) \) is locally Lipschitz continuous on \([s, \infty)\), it follows that \( \|v(t)\| \) is also locally Lipschitz continuous for \( t \in [s, \infty) \). Thus \( \|v(t)\| \) is differentiable a.e. on \([s, \infty)\). Moreover,
\[
\frac{d\|v\|}{dt} = \Re\left[l_v\left(\frac{dv}{dt}\right)\right]
\]
for \( l_v \in T(v(t)) \) a.e. on \([s, \infty)\) by [16, Lemma 1.3]. Equivalently,
\[
\frac{d\|v\|}{dt} = -\Re[l_v(h(v, t))], \quad \text{a.e. on } [s, \infty).
\]

By Lemma 8, we obtain that
\[
1 \leq -\frac{1}{\|v(t)\| \min\{g(\|v(t)\|^k), g(-\|v(t)\|^k)\}} \cdot \frac{d\|v(t)\|}{dt}, \quad \text{a.e. } t \geq s.
\]
Since \( \|v(t)\| \) is locally absolutely continuous, we may integrate both sides of the above inequalities and make a change of variable, to obtain that
Finally, straightforward computations in the above relations yield (6), as desired. This completes the proof. □

We are now able to obtain the following growth result for the set $S_{g,k+1}^0(B)$. This result generalizes [5, Theorem 2.2] and [19, Theorem 2.3].

**Theorem 10.** Let $g: U \rightarrow \mathbb{C}$ satisfy the conditions of Assumption 4 and $f \in S_{g,k+1}^0(B)$. Then

$$
\|z\| \exp \int_0^\|z\| \left[ \frac{1}{\max \{g(x^k), g(-x^k)\}} - 1 \right] \frac{dx}{x} 
\leq \|f(z)\| \leq \|z\| \exp \int_0^\|z\| \left[ \frac{1}{\min \{g(x^k), g(-x^k)\}} - 1 \right] \frac{dx}{x}, \quad z \in B. 
$$

(7)

**Proof.** First, we mention that the above integrals exist and are finite since $g(0) = 1$ and $\text{Re} \ g(\zeta) > 0$ for $|\zeta| < 1$. Also since $f \in S_{g}^0(B)$, we have

$$
f(z) = \lim_{t \rightarrow \infty} e^t v(z, t) 
$$

(8)

locally uniformly on $B$, where $v = v(z, t)$ is the solution of the initial value problem (3). Taking into account the relations (6), one deduces that

$$
\|z\| \exp \int_{\|v(z,t)\|}^\|z\| \left[ \frac{1}{\max \{g(x^k), g(-x^k)\}} - 1 \right] \frac{dx}{x} 
\leq e^t \|v(z, t)\| \leq \|z\| \exp \int_{\|v(z,t)\|}^\|z\| \left[ \frac{1}{\min \{g(x^k), g(-x^k)\}} - 1 \right] \frac{dx}{x}, 
$$

(9)

for all $z \in B$ and $t \geq 0$. Since $\lim_{t \rightarrow \infty} e^t \|v(z, t)\| = \|f(z)\| < \infty$, we must have $\lim_{t \rightarrow \infty} \|v(z, t)\| = \lim_{t \rightarrow \infty} e^{-t} e^t \|v(z, t)\| = 0$. Letting $t \rightarrow \infty$ in (9) and using (8), we obtain the estimate (7), as desired. This completes the proof. □
We remark that if \( f(z, t) \) is a \( g \)-Loewner chain such that \( z = 0 \) is a zero of order \( k + 1 \) of \( e^{-t} f(z, t) - z \) for each \( t \geq 0 \), then using a similar reasoning as above, we obtain the following growth result (cf. [5, Corollary 2.3]).

**Corollary 11.** Let \( g: U \to \mathbb{C} \) satisfy the conditions of Assumption 4 and \( f(z, t) \) be a \( g \)-Loewner chain such that \( z = 0 \) is a zero of order \( k + 1 \) of \( e^{-t} f(z, t) - z \) for each \( t \geq 0 \). Then

\[
\|z\| \exp \left( \int_0^\|z\| \left[ \frac{1}{\max\{g(x^k), g(-x^k)\}} - 1 \right] \frac{dx}{x} \right) \leq \|f(z, t)\| \leq \|z\| \exp \left( \int_0^\|z\| \left[ \frac{1}{\min\{g(x^k), g(-x^k)\}} - 1 \right] \frac{dx}{x} \right), \quad z \in B, \ t \geq 0.
\]

3. **Examples of mappings in** \( S^0_{g,k+1}(B) \)

First, we remark that the following inclusion relations hold:

\[
K_{k+1}(B) \subset G_{k+1}(B) \subset S_{1/2,k+1}^+(B) \subset S_{k+1}^+(B) \subset C_{k+1}(B) \subset S^0_{k+1}(B)
\]

and \( \hat{S}_{\alpha,k+1}(B) \subset S^0_{k+1}(B) \) for \( |\alpha| < \pi/2 \).

A particular interest in Theorem 10 consists in the case \( g(\zeta) = (1 + \zeta)/(1 - \zeta) \). We have the following growth result for the set \( S^0_{k+1}(B) \) (cf. [5,19,27]).

**Theorem 12.** If \( f \in S^0_{k+1}(B) \), then

\[
\frac{\|z\|}{(1 + \|z\|^k)^{2/k}} \leq \|f(z)\| \leq \frac{\|z\|}{(1 - \|z\|^k)^{2/k}}, \quad z \in B.
\]

Consequently, \( f(B) \supset B_{2^{-2/k}} \).

As corollaries to the above theorem, we have the following growth results for the sets \( \hat{S}_{\alpha,k+1}(B) \) (cf. [11]) and \( C_{k+1}(B) \).

**Corollary 13.** If \( f \in \hat{S}_{\alpha,k+1}(B) \) or \( f \in C_{k+1}(B) \), then

\[
\frac{\|z\|}{(1 + \|z\|^k)^{2/k}} \leq \|f(z)\| \leq \frac{\|z\|}{(1 - \|z\|^k)^{2/k}}, \quad z \in B.
\]

Consequently, \( f(B) \supset B_{2^{-2/k}} \).

In particular, we have the following growth result for the set \( S_{k+1}^+(B) \) due to Liu–Liu [20, Theorem 1] (cf. [1,3,10]).
Corollary 14. Let $f \in S^*_k(B)$. Then
\[
\frac{\|z\|}{(1 + \|z\|^k)^{2/k}} \leq \|f(z)\| \leq \frac{\|z\|}{(1 - \|z\|^k)^{2/k}}, \quad z \in B.
\]
Consequently, $f(B) \supset B_{2^{-2/k}}$.

We have the following growth result for the set $S^*_p,k+1(B)$ due to Liu–Liu [20, Theorem 2] (cf. [13,17]).

Theorem 15. If $f \in S^*_p,k+1(B)$, then
\[
\frac{\|z\|}{(1 + \|z\|^k)^{2(1-p)/k}} \leq \|f(z)\| \leq \frac{\|z\|}{(1 - \|z\|^k)^{2(1-p)/k}}, \quad z \in B.
\]
Consequently, $f(B) \supset B_{2^{-2(1-p)/k}}$.

Proof. Let $f \in S^*_p,k+1(B)$. Then $f \in S^0_{g,k+1}(B)$, where $g(\zeta) = (1 + \zeta)/(1 + (2p - 1)\zeta)$. Therefore, we obtain the claimed result from Theorem 10. This completes the proof. \qed

In particular, we obtain the following corollary (cf. [5,29]. See also [4,12,15,17], [20, Corollary 1] and [31]). We mention that if $k = 1$, then the growth result contained in Corollary 16 is sharp (see [29]).

Corollary 16. If $f \in G_{k+1}(B)$, then
\[
\frac{\|z\|}{(1 + \|z\|^k)^{1/k}} \leq \|f(z)\| \leq \frac{\|z\|}{(1 - \|z\|^k)^{1/k}}, \quad z \in B.
\]
Consequently, $f(B) \supset B_{2^{-1/k}}$.

Remark 17. Liu–Liu [20] showed that the results of Corollary 14 and Theorem 15 are sharp. The sharpness of these results yield the sharpness of Theorem 12.

Example 18. (i) Let $B^2(p)$ be the unit ball in $\mathbb{C}^2$ with respect to a $p$-norm, where $2 \leq p < \infty$. Muir–Suffridge [22] proved that if $a \in \mathbb{C}$, $k \in \mathbb{N}^*$ and $f: B^2(p) \to \mathbb{C}^2$ is given by $f(z) = (z_1 + az_2^{k+1}, z_2)$, $z = (z_1, z_2) \in B^2(p)$, then $f$ is convex if and only if
\[
|a| \leq \begin{cases} 
0 & \text{if } k < p - 1, \\
\frac{1}{k(k+1)} & \text{if } k = p - 1 \in \mathbb{N}, \\
\frac{1}{k(k+1)} \left( \frac{k^k}{(p-1)^{p-1}(k+1-p)^{k+1-p}} \right)^{1/p} & \text{if } k > p - 1.
\end{cases}
\]
Under these conditions, $f$ is $k$-fold symmetric and $f \in K_{k+1}(B^2(p))$. Hence $f \in G_{k+1}(B^2(p))$ too.

(ii) Let $\mathbb{B}^n$ denote the Euclidean unit ball in $\mathbb{C}^n$. As in [29, Theorem 3.4], it is possible to prove that if $f_j(z_j)$ is a $k$-fold symmetric normalized convex function on $U$ for $j = 1, \ldots, n$ such that $f_j(z_j) \neq z_j$ for some $j$, then $f: \mathbb{B}^n \to \mathbb{C}^n$ given by $f(z) = (f_1(z_1), \ldots, f_n(z_n))$, $z = (z_1, \ldots, z_n) \in \mathbb{B}^n$, belongs $G_m(\mathbb{B}^n)$ for some $m$ with $m \geq k + 1$. 

For the Euclidean unit ball $B^n$ in $\mathbb{C}^n$, we have the following theorems and examples. The next theorem generalizes [23, Theorem 2.4] (cf. [2]).

**Theorem 19.** Let $f \in H(B^n)$ be a normalized locally biholomorphic mapping which satisfies

$$1 - \|z\|^2 \|\left[Df(z)\right]^{-1} D^2 f(z)(z, \cdot)\| \leq c \quad \text{for all } z \in B^n. \quad (10)$$

If $c \leq 1$ and $z = 0$ is a zero of order $k + 1$ of $f(z) - z$, then $f \in S_{k+1}^0(B^n)$ and

$$\frac{\|z\|}{(1 + c\|z\|^k)^{2/k}} \leq \|f(z)\| \leq \frac{\|z\|}{(1 - c\|z\|^k)^{2/k}}, \quad z \in B^n. \quad (11)$$

Consequently, $f(B^n) \supset B^n_{(1+c)^{-2/k}}$.

**Proof.** Pfaltzgraff [23, Theorem 2.4] proved that

$$f(z, t) = f(ze^{-t}) + (e^t - e^{-t})Df(ze^{-t})(z), \quad t \geq 0,$$

is a Loewner chain. Moreover, since $\lim_{t \to \infty} e^{-t} f(z, t) = z$ locally uniformly on $B^n$, we deduce that $\{e^{-t} f(z, t)\}_{t \geq 0}$ is a normal family, and thus $f \in S^0(B^n)$. Further, since $z = 0$ is a zero of order $k + 1$ of $e^{-t} f(z, t) - z$ for each $t \geq 0$, it follows that $f \in S^0_{k+1}(B^n)$. We will prove the bound (11). Let

$$E(z, t) = -(1 - e^{-2t})[Df(ze^{-t})]^{-1} D^2 f(ze^{-t})(ze^{-t}, \cdot)$$

and $h(z, t) = (I - E(z, t))^{-1}(I + E(z, t))(z)$. Then $f$ and $h$ satisfy the differential equation (1). Since $z = 0$ is a zero of order $k$ of $E(z, t)$ and $\|E(z, t)\| \leq c$ from (10), we obtain that $\|E(z, t)\| \leq c\|z\|^k$ by the Schwarz lemma. Therefore,

$$\|h(z, t) - z\| = \|E(z, t)(h(z, t) + z)\| \leq c\|z\|^k \|h(z, t) + z\|. \quad (12)$$

This implies that

$$\Re[h(z, t), z] \leq \|h(z, t)\| \cdot \|z\| \leq \|z\| \frac{1 + c\|z\|^k}{1 - c\|z\|^k}. \quad (13)$$

Also, from (12), we obtain that

$$\|z\| \frac{1 - c\|z\|^k}{1 + c\|z\|^k} \leq \|h(z, t)\| \quad (14)$$

and

$$\|h(z, t) - z\|^2 \leq c^2 \|z\|^{2k} \|h(z, t) + z\|^2. \quad (15)$$

From (14) and (15), we obtain that

$$(1 + c^2\|z\|^{2k})2\Re[h(z, t), z] \geq \frac{1 - c^2\|z\|^{2k}}{(1 + c\|z\|^k)^2} 2(1 + c^2\|z\|^{2k})\|z\|^2. \quad (16)$$

Thus, from (13) and (16), we obtain that

$$\|z\| \frac{1 - c\|z\|^k}{1 + c\|z\|^k} \leq \Re[h(z, t), z] \leq \|z\| \frac{1 + c\|z\|^k}{1 - c\|z\|^k}. \quad (17)$$
Let \( \alpha \in [0, 1] \) and \( \beta \in [0, 1/2] \) be such that \( \alpha + \beta \leq 1 \). Graham, Hamada, Kohr and Suffridge [6, Theorem 2.1] showed that if \( f \in S \), then \( \Psi_{n,\alpha,\beta}(f) \in S^0(B^n) \), where

\[
\Psi_{n,\alpha,\beta}(f)(z) = \left( f(z_1), z' \left( \frac{f(z_1)}{z_1} \right)^\alpha \left( f'(z_1) \right)^\beta \right)
\]  

for \( z = (z_1, z') \in B^n \). The branches of the power functions are chosen so that

\[
\left( \frac{f(z_1)}{z_1} \right)^\alpha \bigg|_{z_1=0} = 1 \quad \text{and} \quad \left( f'(z_1) \right)^\beta \bigg|_{z_1=0} = 1.
\]

We will generalize the above result to \( f \in S^0_{k+1}(U) \). This result gives many examples of mappings in \( S^0_{k+1}(B^n) \).

**Theorem 20.** Let \( \alpha \in [0, 1] \) and \( \beta \in [0, 1/2] \) be such that \( \alpha + \beta \leq 1 \). Let \( \Psi_{n,\alpha,\beta}(f) \) be as in (17). If \( f \in S^0_{k+1}(U) \), then \( \Psi_{n,\alpha,\beta}(f) \in S^0_{k+1}(B^n) \).

**Proof.** It suffices to give the proof when \( n = 2 \). Since \( f \in S^0_{k+1}(U) \), there exists a Loewner chain \( f(z_1, t) \) such that \( f(z_1) = f(z_1, 0) \) for all \( z_1 \in U \) and \( z_1 = 0 \) is a zero of order \( k + 1 \) of \( e^{-t}f(z_1, t) - z_1 \) for each \( t \geq 0 \). Let \( F_{\alpha,\beta}(z, t) \) be defined by

\[
F_{\alpha,\beta}(z, t) = \left( f(z_1, t), e^{(1-\alpha-\beta)t}z_2 \left( \frac{f(z_1, t)}{z_1} \right)^\alpha \left( f'(z_1, t) \right)^\beta \right),
\]

for \( z = (z_1, z_2) \in B^2 \) and \( t \geq 0 \). In [6, Theorem 2.1], it is shown that \( F_{\alpha,\beta}(z, t) \) is a Loewner chain such that \( \{e^{-t}F_{\alpha,\beta}(z, t)\}_{t \geq 0} \) is a normal family on \( B^2 \). Also, \( z = 0 \) is a zero of order \( k + 1 \) of \( e^{-t}F_{\alpha,\beta}(z, t) - z \) for each \( t \geq 0 \). This completes the proof. \( \qed \)

In the following result, we shall denote by \( \mathcal{L}S_n \) the set of normalized locally biholomorphic mappings in \( B^n \). Also for \( n \geq 1 \), let \( z' = (z_1, \ldots, z_n) \) so that \( z = (z', z_{n+1}) \in C^{n+1} \). Pfaltzgraff–Suffridge [25] defined the following extension operator \( \Phi_n : \mathcal{L}S_n \to \mathcal{L}S_{n+1} \) given by

\[
\Phi_n(f)(z) = F(z) = (f(z'), z_{n+1}[J_f(z')]^{1/(n+1)}), \quad z = (z', z_{n+1}) \in B^{n+1},
\]

where \( J_f(z') = \det Df(z') \) for \( z' \in B^n \). On the other hand, Graham–Kohr–Pfaltzgraff [9] have recently proved that the class \( S^0(B^n) \) is preserved by the above operator, i.e., if \( f \in S^0(B^n) \) then \( \Phi_n(f) \in S^0(B^{n+1}) \). In particular, they proved that if \( f \in S^\ast(B^n) \) then \( \Phi_n(f) \in S^\ast(B^{n+1}) \). Related to the class \( S^0_{k+1}(B^n) \), we have the following result, which also provides examples of mappings in \( S^0_{k+1}(B^{n+1}) \).

**Theorem 21.** If \( f \in S^0_{k+1}(B^n) \), then \( \Phi_n(f) \in S^0_{k+1}(B^{n+1}) \).
**Proof.** Since \( f \in S_{k+1}^0(\mathbb{B}^n) \), there exists a Loewner chain \( f(z^t, t) \) such that \( f(z^t) = f(z^t, 0) \), \( z^t \in \mathbb{B}^n \), and \( z^t = 0 \) is a zero of order \( k + 1 \) of \( e^{-t} f_t(z^t) - z^t \) for \( t \geq 0 \), where \( f_t(z^t) = f(z^t, t) \). Let \( F(z, t) \) be given by

\[
F(z, t) = \left( f(z^t, t), z_{n+1} e^{t/(n+1)} [J f_t(z^t)]^{1/(n+1)} \right)
\]

for \( z = (z^t, z_{n+1}) \in \mathbb{B}^{n+1} \) and \( t \geq 0 \). In [9, Theorem 2.1], it is shown that \( F(z, t) \) is a Loewner chain such that \( \{ e^{-t} F(z, t) \}_{t \geq 0} \) is a normal family. We next prove that \( z = 0 \) is a zero of order \( k + 1 \) of \( e^{-t} F(z, t) - z \) for \( t \geq 0 \). To this end, fix \( t \geq 0 \) and let \( G_t(z) = e^{-t} F(z, t) - z \) for \( z \in \mathbb{B}^{n+1} \). Then it is clear that \( G_t(0) = 0 \) and \( D G_t(0) = 0 \). A straightforward computation, based on the facts that \( \Phi(\zeta) \) is a normalized univalent holomorphic mapping of \( \mathbb{B} \) for each \( \zeta \), and thus \( F = F(\cdot, 0) \in S_{k+1}^0(\mathbb{B}^{n+1}) \), as claimed. This completes the proof. \( \square \)

We have seen that \( \tilde{S}_{a,k+1}(B) \subset S_{k+1}^0(B) \), \( |a| < \pi/2 \). However, in general, a spirallike mapping relative to a linear operator need not belong to \( S_{k+1}^0(B) \). In other words, there exist mappings in \( S_{k+1}(B) \setminus S_{k+1}^0(B) \). We have the following example on the Euclidean unit ball \( \mathbb{B}^2 \) of \( \mathbb{C}^2 \):

**Example 22.** Let \( n = 2 \) and \( f(z) = (z_1 + a z_2^{k+1}, z_2) \) for \( z = (z_1, z_2) \in \mathbb{B}^2 \). Let \( A(z) = ((k + 1) z_1, z_2) \) for \( z = (z_1, z_2) \in \mathbb{B}^2 \). Then \( m(A) > 0 \) and \( D f(z)^{-1} A f(z) = ((k + 1) z_1, z_2) \) for \( z = (z_1, z_2) \in \mathbb{B}^2 \). Hence \( f \) is a spirallike mapping relative to \( A \) for all \( a \in \mathbb{C} \) (see [30]). In particular, \( f \in S_{k+1}(\mathbb{B}^2) \). Let \( a \in \mathbb{R} \) with \( a > 2/16^{1/k} - 1 \). Let \( z_0 = (0, 1/2^{1/k}) \). Then \( \| f(z_0) \| > 2^{1/k} = \| z_0 \|/(1 - \| z_0 \|)2^{1/k} \). Taking into account Theorem 12, one deduces that \( f \not\in S_{k+1}^0(\mathbb{B}^2) \).

These observations suggest that one should consider another subset \( S_{k+1}^1(B) \) of \( S_{k+1}(B) \). That is, \( f \in S_{k+1}^1(B) \) if and only if there exists a Loewner chain \( f(z, t) \) such that \( z = 0 \) is a zero of order \( k + 1 \) of \( e^{-t} f(z, t) - z \) for each \( t \geq 0 \) and \( f(z, 0) = f(z) \) for \( z \in B \).

Combining Proposition 5 and Definition 6, we have the following inclusion relations:

\[
S_{k+1}^0(B) \subset S_{k+1}^1(B) \subset S_{k+1}(B).
\]

The next example shows that, in higher dimensions, \( S_{k+1}^1(\mathbb{B}^2) \) is a strictly larger set than \( S_{k+1}^0(\mathbb{B}^2) \). However, in the case of one complex variable, these sets are identical (cf. [26]).

**Example 23.** Let \( \Phi : \mathbb{C}^2 \to \mathbb{C}^2 \) be given by \( \Phi(z) = (z_1, z_2 + z_1^{k+1}) \), \( z = (z_1, z_2) \in \mathbb{C}^2 \). Then it is easy to see that \( \Phi \) is a normalized univalent holomorphic mapping of \( \mathbb{C}^2 \) onto \( \mathbb{C}^2 \). Further, we consider the Loewner chain

\[
f(z, t) = \left( \frac{e^t z_1}{(1 - z_1^k)^2/k}, \frac{e^t z_2}{(1 - z_2^{k+1})^{2/k}} \right), \quad z = (z_1, z_2) \in \mathbb{B}^2, \ t \geq 0,
\]
whose initial element \( f(z) = f(z, 0) \) satisfies \( \| f(r, 0) \| = r/(1 - r^k)^{2/k} \) for \( 0 \leq r < 1 \). Then it is not difficult to deduce that \( (\Phi \circ f)(z, t) \) is also a Loewner chain such that \( z = 0 \) is a zero of order \( k + 1 \) of \( e^{-t} \Phi(f(z, t)) - z, t \geq 0 \). Thus \( \Phi \circ f \in S_{k+1}^1(\mathbb{B}^2) \) and

\[
\| \Phi(f(r, 0)) \| = \sqrt{\frac{r^2}{(1 - r^k)^{4/k} + \frac{r^{2(k+1)}}{(1 - r^k)^{4(k+1)/k}}} \geq \frac{r}{(1 - r^k)^{2/k}}
\]

for \( r \in (0, 1) \). Therefore, from Theorem 12, we conclude that \( \Phi \circ f \notin S_{k+1}^0(\mathbb{B}^2) \).

4. Coefficient bounds for mappings in \( S_{g,k+1}^0(B) \)

We now prove an estimate for the \((k + 1)\)th order coefficients of mappings in the set \( S_{g,k+1}^0(B) \) (cf. [5, Theorem 2.14], [19, Theorem 2.4], [27, Theorem 3]).

**Theorem 24.** Let \( g \) satisfy the conditions of Assumption 4 and \( f \in S_{g,k+1}^0(B) \). Then

\[
\left| \frac{1}{(k + 1)!} I_w(D^{k+1}f(0)(w^{k+1})) \right| \leq \frac{1}{k} |g'(0)|, \quad \|w\| = 1, \quad I_w \in T(w).
\]

**Proof.** Since \( f \in S_{g,k+1}^0(B) \), there exist a mapping \( h_t(z) = h(z, t) \in M_g \) and a Loewner chain \( f(z, t) \) such that \( f(z, s) = \lim_{t \to \infty} e^{t}v(z, s, t) \) locally uniformly on \( B \), where \( v(t) = v(z, s, t) \) is the solution of the initial value problem (2), for each \( s \geq 0, z = 0 \) is a zero of order \( k + 1 \) of \( e^{-t} f(z, t) - z \) and \( f(z) = f(z, 0) \).

Fix \( z \in B \setminus \{0\} \), \( I_z \in T(z) \) and \( t_0 \geq 0 \). Let

\[
p_{t_0}(0) = \begin{cases} \frac{1}{\xi} I_z(h_{t_0}(\xi, \frac{z}{\|z\|})), & \xi \in U \setminus \{0\}, \\ 1, & \xi = 0. \end{cases}
\]

Then \( p_{t_0}(0) \) is a holomorphic function on \( U \). As in the proof of Lemma 8, we have \( p_{t_0}(\xi) = g(|\xi|^k H_{t_0}(\xi)) \) for \( \xi \in U \), where \( H_{t_0}(\xi) \) is a holomorphic function on \( U \) such that \( |H_{t_0}(\xi)| \leq 1 \) on \( U \). Hence we obtain that \( |p_{t_0}^{(k)}(0)| \leq k! |g'(0)| \).

Since

\[
\frac{1}{k!} p_{t_0}^{(k)}(0) = \frac{1}{(k + 1)!} I_z(D^{k+1}h_{t_0}(0)\left(\left(\frac{z}{\|z\|}\right)^{k+1}\right))
\]

by identifying the coefficients in the power series expansions, we deduce that

\[
\left| \frac{1}{(k + 1)!} I_z(D^{k+1}h_{t_0}(0)\left(\left(\frac{z}{\|z\|}\right)^{k+1}\right)) \right| \leq |g'(0)|.
\]

On the other hand, since \( f(z, t) \) is a Loewner chain, it follows from Lemma 1 that \( f(z, t) \) is differentiable for almost all \( t \in [0, \infty) \). Moreover, by differentiating the equality \( f(z, s) = f(v(z, s, t), t) \) with respect to \( t \) and using (2), we obtain that \( f \) and \( h \) satisfy the differential equation (1). Integrating both sides of the equality (1), we obtain that

\[
f(z, T) - f(z, 0) = \int_0^T Df(z, t)h(z, t) dt
\]
for $T > 0$. After simple computations, using the fact that $z = 0$ is a zero of order $k + 1$ of $e^{-t} f(z, t) - z$, we deduce that

$$D^{k+1} f(0, T)(z^{k+1}) - D^{k+1} f(0, 0)(z^{k+1})$$

$$= \int_0^T [(k + 1)D^{k+1} f(0, t)(z^{k+1}) + e^t D^{k+1} h(0, t)(z^{k+1})] dt. \tag{20}$$

Let

$$q(T) = e^{-(k+1)T} D^{k+1} f(0, T)(z^{k+1}) - D^{k+1} f(0, 0)(z^{k+1})$$

$$- \int_0^T e^{-kt} D^{k+1} h(0, t)(z^{k+1}) dt.$$ 

Since $q'(T) = 0$ for almost all $T > 0$ by (20), we have $q(T) = q(0) = 0$. This implies that

$$e^{-(k+1)T} l_z(D^{k+1} f(0, T)(z^{k+1})) - l_z(D^{k+1} f(0, 0)(z^{k+1}))$$

$$= \int_0^T l_z(e^{-kt} D^{k+1} h(0, t)(z^{k+1})) dt. \tag{21}$$

As in Corollary 11, we have the estimate

$$\|f(z, T)\| \leq e^T \|z\| \exp \int_0^{\|z\|} \left[ \frac{1}{\min\{g(x^k), g(-x^k)\}} - 1 \right] dx \tag{22}.$$

Next, using the Cauchy formula

$$\frac{1}{(k + 1)!} D^{k+1} f(0, T)(u^{k+1}) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi u, T)}{\xi^{k+2}} d\xi, \quad r < 1,$$

for $u \in \mathbb{C}^n$, $\|u\| = 1$, and taking into account (22), we easily obtain that

$$\lim_{T \to \infty} e^{-(k+1)T} D^{k+1} f(0, T)(z^{k+1}) = 0.$$

Letting $T \to \infty$ in (21) and using the above equality and (19), we deduce that

$$\left| \frac{1}{(k + 1)!} l_z(D^{k+1} f(0, 0) \left( \frac{z}{\|z\|} \right)^{k+1}) \right| \leq \frac{1}{k} |g'(0)|.$$

Since $l_z = l_z/\|z\|$ and $f(z, 0) = f(z)$ for $z \in B \setminus \{0\}$, the proof is complete. \qed

For the norm of the $(k + 1)$th order Fréchet derivative of a mapping in $S_{g, k+1}^0(B)$, we have the following estimate (cf. [5, Corollary 2.15]).
Corollary 25. Let \( g \) satisfy the conditions of Assumption 4 and \( f \in S^0_{g,k+1}(B) \). Then
\[
\left\| \frac{1}{(k+1)!} D^{k+1} f(0)(w^{k+1}) \right\| \leq b_k |g'(0)|, \quad \|w\| = 1,
\]
where \( b_k = (k + 1)^{(k+1)/k} / k \).

Proof. Let \( P_m = D^m f(0)/m! \). Then \( P_m \) is a homogeneous polynomial of degree \( m \). Let \( |V(P_m)| = \lim_{s \to 1-0} \sup \{|\lambda| : \lambda \in V(P_{m,s})\} \) be the numerical radius of \( P_m \), where \( P_{m,s}(z) = P_m(sz) \) and \( V(P_{m,s}) = \{l_z(P_{m,s}(z)) : l_z \in T(z), \|z\| = 1\} \) is the numerical range of \( P_{m,s} \). Then we obtain that \( \|P_{k+1}\| \leq (k + 1)^{(k+1)/k}|V(P_{k+1})| \) by [14, Theorem 1]. Taking into account (18) and the above relations, we easily deduce that \( |V(P_{k+1})| \leq \frac{1}{k} |g'(0)| \) and the result now follows. This completes the proof.

Corollary 26. If \( f \in S^0_{k+1}(B) \), then
\[
\left\| \frac{1}{(k+1)!} D^{k+1} f(0)(w^{k+1}) \right\| \leq 2 b_k, \quad \|w\| = 1, \quad l_w \in T(w).
\]
Moreover, for \( \|w\| = 1 \), we have
\[
\left\| \frac{1}{m!} D^m f(0)(w^m) \right\| < \left[ \frac{e(m+1)}{2} \right]^{2/k}, \quad m \in \mathbb{N}, \quad m \geq k + 2.
\] (23)

Proof. It suffices to prove the bounds (23). To this end, fix \( m \in \mathbb{N}, \quad m \geq k + 2 \), and \( w \in \mathbb{C}^n, \quad \|w\| = 1 \). Using the Cauchy formula
\[
\frac{1}{m!} D^m f(0)(w^m) = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(zw)}{z^m+1} \, dz, \quad 0 < r < 1,
\]
and taking into account Theorem 12, we easily obtain that
\[
\left\| \frac{1}{m!} D^m f(0)(w^m) \right\| \leq \frac{1}{2\pi r^{m}} \int_{0}^{2\pi} \| f(re^{i\theta}w) \| \, d\theta \leq \frac{1}{r^{m-1}(1-r^k)^{2/k}} \] (24)
for \( 0 < r < 1 \). If we set \( r^k = (m-1)/(m+1) \) in the inequality (24), then we obtain (23). This completes the proof.

For \( g(\zeta) = 1 + \zeta, \quad \zeta \in U \), we obtain the following bound for the \((k+1)\)th order coefficients of mappings in \( S^0_{g,k+1}(B) \). In particular, this result is satisfied by all mappings.
in $K_{k+1}(B)$ and $G_{k+1}(B)$ (cf. [5, Corollary 2.19], [18]). For the proof, it suffices to use arguments similar to those in the proof of Corollary 26.

**Corollary 27.** If $f \in S_{g,k+1}^0(B)$ with $g(\zeta) = 1 + \zeta$, $\zeta \in U$, then

$$\left| \frac{1}{(k+1)!} l_w \left(D^{k+1} f(0)(w^{k+1})\right) \right| \leq \frac{1}{k}, \quad \|w\| = 1, \ l_w \in T(w).$$

Moreover, for $\|w\| = 1$, we have

$$\left\| \frac{1}{(k+1)!} D^{k+1} f(0)(w^{k+1}) \right\| \leq b_k, \quad \|w\| = 1,$$

and

$$\left\| \frac{1}{m!} D^m f(0)(w^m) \right\| < (em)^{1/k}, \quad m \in \mathbb{N}, \ m \geq k + 2.$$

**References**


