Matrix Extensions of Liouville-Dirichlet-Type Integrals

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To A. S. Householder on his seventy-fifth birthday.

Submitted by Emeric Deutsch

ABSTRACT

The Dirichlet integral provides a formula for the volume over the k-dimensional simplex \( \omega = \{ x_1, \ldots, x_k : x_i \geq 0, \ i = 1, \ldots, k, \ s \leq \sum x_i \leq t \} \). This integral was extended by Liouville. The present paper provides a matrix analog where now the region becomes \( \Omega = \{ V_1, \ldots, V_k : V_i > 0, \ i = 1, \ldots, k, \ 0 \leq \sum V_i \leq t \} \), where now each \( V_i \) is a \( p \times p \) symmetric matrix and \( A > B \) means that \( A - B \) is positive semidefinite.

1. INTRODUCTION

The well-known Dirichlet integral provides a formula for the volume over the k-dimensional simplex

\[ \omega = \{ x_1, \ldots, x_k : x_i > 0 \ (i = 1, \ldots, k), \ s \leq \sum x_i \leq t \}. \]

This integral was extended by Liouville:

\[
\int_\omega f \left( \sum_{1}^{k} x_i \right) \prod_{1}^{k} x_i^{a_i - 1} \prod dx_i = \frac{\prod_{1}^{k} \Gamma(a_i)}{\Gamma(\sum a_i)} \int_{s}^{t} f(z) z^{\sum a_i - 1} dz, \quad (1.1)
\]

where \( f(x) \) is continuous and \( a_i > 0, \ i = 1, \ldots, k \). The case \( f(x) = 1 \) is the Dirichlet integral. For a general discussion of such integrals see [1].

The integral (1.1) has been generalized in several directions. Sivazlian [4] provides the following extension: Let \( f(x) \) be continuous, \( a_i \)
\[ b_j > 0 \ (j = 1, \ldots, l), \] and
\[ \omega_1 = \{ x_1, \ldots, x_k, y_1, \ldots, y_l : 0 < x_i \ (i = 1, \ldots, k), 0 < y_j \ (j = 1, \ldots, l), \Sigma x_i + \Sigma y_j < t \}. \]

Then
\[
\int_{\omega_1} f \left( \sum_{1}^{k} x_i \right) \prod_{1}^{k} x_i^{a-1} \prod_{1}^{l} y_i^{b-1} \prod_{1}^{l} dx_i \prod_{1}^{l} dy_i
\]
\[
= \frac{\prod_{1}^{k} \Gamma(a_i) \prod_{1}^{l} \Gamma(b_i)}{\Gamma \left( \sum_{1}^{k} a_i \right) \Gamma \left( \sum_{1}^{l} b_i + 1 \right)} \int_{0}^{t} f(z) z^{\sum_{1}^{k} a_i - 1} (t - z)^{\sum_{1}^{l} b_i} f(z) dz. \quad (1.2)
\]

Klamkin [2] and Sivazlian [5] offer several extensions of (1.2). The following is the simplest of these: if \( f(x) \) and \( g(x) \) are continuous, \( a_i > 0 \ (i = 1, \ldots, k) \), \( b_j > 0 \ (j = 1, \ldots, l) \), then
\[
\int_{\omega_1} f \left( \sum_{1}^{k} x \right) g \left( \sum_{1}^{l} y \right) \prod_{1}^{k} x_i^{a} \prod_{1}^{l} y_i^{b-1} \prod_{1}^{l} dx_i \prod_{1}^{l} dy_i
\]
\[
= \frac{\prod_{1}^{k} \Gamma(a_i)}{\Gamma \left( \sum_{1}^{k} a_i \right) \Gamma \left( \sum_{1}^{l} b_i \right)} \int \int_{0 < v < w < t} f(v) g(w) v^{\sum_{1}^{k} a_i - 1} (t - w)^{\sum_{1}^{l} b_i - 1} dv dw. \quad (1.3)
\]

A different type of generalization of (1.1) was obtained by Olkin [3], where now the integrand is extended to scalar functions of matrices. Such integrals arise quite naturally in statistical multivariate analysis.

In the matrix version the matrices are symmetric of dimension \( p \). We write \( A > B \) and \( A \geq B \) to mean that \( A - B \) is positive semidefinite and positive definite, respectively, and denote the determinant of \( A \) by \( |A| \).

If \( f(X) \) is a continuous scalar function of the \( p \times p \) symmetric matrix \( X \), \( a_i > (p - 1)/2 \ (i = 1, \ldots, k) \), and
\[ \Omega = \{ V_1, \ldots, V_k : V_i > 0 \ (i = 1, \ldots, k), A < \sum V_i < B \}, \]
where

\[ B_p(m, n) = \frac{\Gamma_p(m) \Gamma_p(b)}{\Gamma_p(m + n)}, \quad \Gamma_p(m) = \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma \left( m - \frac{i-1}{2} \right). \]

The result (1.4) is proved in [3]. In the present note we obtain matrix analogs, in the spirit of (1.4), of the integrals (1.2) and (1.3).

2. MATRIX ANALOGS OF LIOUVILLE-DIRICHLET TYPE INTEGRALS

The extension of (1.2) is given in the following

**Theorem 1.** If \( V_1, \ldots, V_k \) are \( p \times p \) symmetric matrices, \( f(V) \) is a continuous scalar function of the symmetric matrix \( V, a_i > (p-1)/2 \) (\( i = 1, \ldots, n \)), \( b_j > (p-1)/2 \) (\( j = 1, \ldots, l \)), and

\[
\Omega_1 = \left\{ V_1, \ldots, V_k, W_1, \ldots, W_l : V_i > 0 \ (i = 1, \ldots, k), \ W_j > 0 \ (j = 1, \ldots, l), \ 0 < \sum_{i=1}^{k} V_i + \sum_{j=1}^{l} W_j < T \right\},
\]

then

\[
\int_{\Omega_1} f \left( \sum_{i=1}^{k} V_i \right) \prod_{i=1}^{k} |V_i|^{-(p+1)/2} \prod_{j=1}^{l} |W_i|^{-(p+1)/2} dV_i \prod_{j=1}^{l} dW_i
\]

\[
= \prod_{i=2}^{k} B_p(a_1 + \cdots + a_{i-1}, a_i) \prod_{j=2}^{l} B_p(b_1 + \cdots + b_{j-1}, b_j)
\]

\[
B_p \left( \sum_{i=1}^{l} b_i, \frac{p+1}{2} \right) \int_{0 < Z < T} f(Z) |Z|^{-(p+1)/2} |T - Z|^{\Sigma b_i} dZ.
\]
Proof. The proof is based on using (1.4) twice. First integrating $V_1, \ldots, V_k$ over the region $V_i > 0$ ($i = 1, \ldots, k$), $0 < \Sigma_i^k V_i < T - \Sigma_i^k W_i$, we obtain

$$
\int f \left( \sum_{i=1}^k V_i \right) \prod_{i=1}^k V_i^{\alpha_i - (p+1)/2} \prod_{i=1}^k dV_i = c_1 \int_{0 < Z < T - \Sigma_i^k W_i} f(Z) |Z|^{\Sigma_i^k \alpha_i - (p+1)/2} dZ,
$$

(2.2)

where $c_1 = \prod_{i=2}^k B_p (a_1 + \cdots + a_i - 1, a_i)$. Thus, the left-hand side of (2.1) becomes

$$
c_1 \int_{0 < Z + \Sigma_i^k W_i < T} f(Z) |Z| \Sigma_i^k \alpha_i - (p+1)/2 \prod_{i=1}^k W_i^{\alpha_i - (p+1)/2} \prod_{i=1}^k dW_i dZ. \quad (2.3)
$$

Again invoking (1.4), (2.3) becomes

$$
c_1 c_2 \int_{0 < Z + s < T} f(Z) |Z| \Sigma_i^k \alpha_i - (p+1)/2 |S|^{\Sigma_i^k \alpha_i - (p+1)/2} dS dZ,
$$

(2.4)

where $c_2 = \prod_{i=2}^k B_p (b_1 + \cdots + b_i - 1, b_i)$.

We now make a sequence of transformations in order to evaluate the integral

$$
\int_{0 < Z + S < T} |S|^{\Sigma_i^k (p+1)/2} dS. \quad (2.5)
$$

First, let $Z + S = H$ be a transformation from $S$ to $H$, yielding

$$
\int_{Z < H < T} |H - Z|^{\Sigma_i^k (p+1)/2} dH
\quad = \int_{Z < H < T} |Z|^{\Sigma_i^k (p+1)/2} |Z - 1/2 HZ - 1/2 - I|^{\Sigma_i^k (p+1)/2} dH. \quad (2.6)
$$

Now let $Z^{-1/2} HZ^{-1/2} = G$. Then $dH = |Z|^{(p+1)/2} dG$, and we obtain

$$
\int_{I < G < Z^{-1/2} T Z^{-1/2}} |Z|^{\Sigma_i^k} |G - I|^{\Sigma_i^k (p+1)/2} dG. \quad (2.7)
$$
Let \( U = G - I \), and write \( B = Z^{-1/2}T Z^{-1/2} - I \). Then (2.7) becomes

\[
|Z|^{\Sigma h} \int_{0 < U < B} |U|^\Sigma h - (p+1)/2 \, dU. \tag{2.8}
\]

Finally, let \( B^{-1/2}UB^{-1/2} = Y \), so that \( dU = |B|^{(p+1)/2} \, dY \), and (2.8) becomes

\[
|Z|^{\Sigma h} |B|^{\Sigma h} \int_{0 < Y < l} |Y|^\Sigma h - (p+1)/2 \, dY. \tag{2.9}
\]

The integral in (2.9) is a matrix analog of the Beta function and was evaluated by Olkin [3]:

\[
\int_{0 < Y < l} |Y|^{c-(p+1)/2} |I - Y|^{d-(p+1)/2} \, dY = B_p(c,d), \tag{2.10}
\]

so that (2.9) is equal to \( |Z|^{\Sigma h} |B|^{\Sigma h} R_p(\Sigma b, (p + 1)/2) \). Combining (2.4) with (2.9) then yields the result.

\[\square\]

A matrix extension of (1.3) can be obtained in a similar manner, namely, by a repeated application of (2.1). We state this result without proof.

**Theorem 2.** If \( f \) and \( g \) are continuous scalar functions of a symmetric matrix, then

\[
\int_{0 < V_i} f \left( \sum_{i=1}^{k} V_i \right) g \left( \sum_{i=1}^{l} W_i \right) \prod_{i=1}^{k} |V_i|^{a - (p+1)/2} \prod_{i=1}^{l} |W_i|^{b - (p+1)/2} \prod_{i=1}^{k} \int \prod_{i=1}^{l} dV_i \prod_{i=1}^{l} dW_i \]  

\[= \prod_{i=1}^{k} B_p(a_1 + \cdots + a_i, a_i) \times \int_{0 < W_i} f(X) g \left( \sum_{i=1}^{l} W_i \right) |X|^{\Sigma a - (p+1)/2} \prod_{i=1}^{l} |W_i|^{b - (p+1)/2} \prod_{i=1}^{l} dX \prod_{i=1}^{l} dW_i \]  

\[= \prod_{i=1}^{k} B_p(a_1 + \cdots + a_i, a_i) \prod_{i=1}^{l} B_p(b_1 + \cdots + h_i, h_i) \times \int_{0 < X + Y < T} f(X) g(Y) |X|^{\Sigma a - (p+1)/2} |Y|^{\Sigma h - (p+1)/2} \prod_{i=1}^{l} dX \prod_{i=1}^{l} dY. \tag{2.11}
\]
Note that (2.11) is equivalent to (1.3) when $p = 1$.

Matrix extensions of other integrals in [2] can be obtained in a similar manner.

REFERENCES


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