Matrix Extensions of Liouville-Dirichlet-Type Integrals

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To A. S. Householder on his seventy-fifth birthday.

Submitted by Emeric Deutsch

ABSTRACT

The Dirichlet integral provides a formula for the volume over the k-dimensional simplex $\omega=\{x_1,\dots,x_k\colon x_i\geqslant 0,\ i=1,\dots,k,\ s\leqslant\Sigma_1^k x_i\leqslant T\}$. This integral was extended by Liouville. The present paper provides a matrix analog where now the region becomes $\Omega=\{V_1,\dots,V_k\colon V_i>0,\ i=1,\dots,k,\ 0\leqslant\Sigma V_i\leqslant t\}$, where now each V_i is a $p\times p$ symmetric matrix and $A\geqslant B$ means that $A-B$ is positive semidefinite.

1. INTRODUCTION

The well-known Dirichlet integral provides a formula for the volume over the k-dimensional simplex

$$\omega=\{x_1,\dots,x_k: x_i\geqslant 0\ (i=1,\dots,k),\ s\leqslant\Sigma x_i\leqslant t\}.$$

This integral was extended by Liouville:

$$\int_\omega f\left(\sum_1^k x_i\right)\prod_1^k x_i^{a_i-1}\Pi\,dx_i=\frac{\prod_1^k\Gamma(a_i)}{\Gamma(\Sigma a_i)}\int_s^t f(z)z^{\Sigma a_i-1}\,dz, \tag{1.1}$$

where $f(x)$ is continuous and $a_j>0$, $j=1,\dots,k$. The case $f(x)\equiv 1$ is the Dirichlet integral. For a general discussion of such integrals see [1].

The integral (1.1) has been generalized in several directions. Sivazlian [4] provides the following extension: Let $f(x)$ be continuous, a_j

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$b_j > 0$ $(j=1,\ldots,l)$, and

$$\omega_1 = \{x_1,\ldots,x_k,y_1,\ldots,y_l : 0<x_i \ (i=1,\ldots,k),\ 0<y_j \ (j=1,\ldots,l),\ \Sigma x_i + \Sigma y_j \leqslant t\}.$$

Then

$$\int_{\omega_1} f\left(\sum_1^k x_i\right) \prod_1^k x_i^{a_i-1} \prod_1^l y_i^{b_i-1} \prod_1^k dx_i \prod_1^l dy_i$$

$$= \frac{\prod_1^k \Gamma(a_i) \prod_1^l \Gamma(b_i)}{\Gamma\left(\sum_1^k a_i\right)\Gamma\left(\sum_1^l b_i + 1\right)} \int_0^t f(z) z^{\Sigma a_i - 1}(t-z)^{\Sigma b_i} f(z)\, dz. \quad (1.2)$$

Klamkin [2] and Sivazlian [5] offer several extensions of (1.2). The following is the simplest of these: if $f(x)$ and $g(x)$ are continuous, $a_i > 0$ $(i=1,\ldots,k)$, $b_j > 0$ $(j=1,\ldots,l)$, then

$$\int_{\omega_1} f\left(\sum_1^k x\right) g\left(\sum_1^l y\right) \prod_1^k x_i^{a_i} \prod_1^l y_i^{b_i-1} \prod_1^k dx_i \prod_1^l dy_i$$

$$= \frac{\prod_1^k \Gamma(a_i)}{\Gamma\left(\sum_1^k a_i\right)\Gamma\left(\sum_1^l b_i\right)} \iint_{0<v<w<t} f(v) g(w) v^{\Sigma a_i - 1}(t-w)^{\Sigma b_i - 1}\, dv\, dw. \quad (1.3)$$

A different type of generalization of (1.1) was obtained by Olkin [3], where now the integrand is extended to scalar functions of matrices. Such integrals arise quite naturally in statistical multivariate analysis.

In the matrix version the matrices are symmetric of dimension p. We write $A \geqslant B$ and $A > B$ to mean that $A - B$ is positive semidefinite and positive definite, respectively, and denote the determinant of A by $|A|$.

If $f(X)$ is a continuous scalar function of the $p \times p$ symmetric matrix X, $a_i > (p-1)/2$ $(i=1,\ldots,k)$, and

$$\Omega = \{V_1,\ldots,V_k : V_i > 0 \ (i=1,\ldots,k),\ A \leqslant \Sigma V_i \leqslant B\},$$

then

$$\int_{\Omega} f\left(\sum_1^k V_i\right)\prod_1^k |V_i|^{a_i-(p+1)/2}\prod_1^k dV_i$$

$$= \prod_{j=2}^{k} B_p(a_1+\cdots+a_{j-1},a_j)\int_{A<Z<B} f(Z)|Z|^{\Sigma a_i-(p+1)/2}dZ, \quad (1.4)$$

where

$$B_p(m,n)=\frac{\Gamma_p(m)\Gamma_p(b)}{\Gamma_p(m+n)}, \qquad \Gamma_p(m)=\pi^{p(p-1)/4}\prod_1^p \Gamma\left(m-\frac{i-1}{2}\right).$$

The result (1.4) is proved in [3]. In the present note we obtain matrix analogs, in the spirit of (1.4), of the integrals (1.2) and (1.3).

2. MATRIX ANALOGS OF LIOUVILLE-DIRICHLET TYPE INTEGRALS

The extension of (1.2) is given in the following

THEOREM 1. *If* $V_1,\dots,V_k$ *are* $p\times p$ *symmetric matrices,* $f(V)$ *is a continuous scalar function of the symmetric matrix* V, $a_i>(p-1)/2$ $(i=1,\dots,n)$, $b_j>(p-1)/2$ $(j=1,\dots,l)$, *and*

$$\Omega_1=\left\{V_1,\dots,V_k,W_1,\dots,W_l : V_i>0\ (i=1,\dots,k),\ W_j>0\ (j=1,\dots,l),\right.$$

$$\left. 0\leqslant \sum_1^k V_i+\sum_1^l W_j\leqslant T\right\},$$

then

$$\int_{\Omega_1} f\left(\sum_1^k V_i\right)\prod_1^k |V_i|^{a_i-(p+1)/2}\prod_1^l |W_i|^{b_i-(p+1)/2}\prod_1^k dV_i\prod_1^l dW_j$$

$$=\prod_{j=2}^{k} B_p(a_1+\cdots+a_{j-1},a_j)\prod_{j=2}^{l} B_p(b_1+\cdots+b_{j-1},b_j)$$

$$B_p\left(\sum_1^l b_1,\frac{p+1}{2}\right)\int_{0<Z<T} f(Z)|Z|^{\Sigma a_i-(p+1)/2}|T-Z|^{\Sigma b_i}dZ. \quad (2.1)$$

Proof. The proof is based on using (1.4) twice. First integrating $V_1,\dots,V_k$ over the region $V_i>0$ $(i=1,\dots,k)$, $0\leqslant\Sigma_1^k V_i\leqslant T-\Sigma_1^l W_i$, we obtain

$$\int f\left(\sum_1^k V_i\right)\prod_1^k |V_i|^{a_i-(p+1)/2}\prod_1^k dV_i = c_1\int_{0<Z<T-\Sigma_1^l W_i} f(Z)|Z|^{\Sigma_1^k a_i-(p+1)/2}dZ, \tag{2.2}$$

where $c_1=\Pi_{j=2}^k B_p(a_1+\cdots+a_{j-1},a_j)$. Thus, the left-hand side of (2.1) becomes

$$c_1\int_{0<Z+\Sigma W_i<T} f(Z)|Z|^{\Sigma_1^k a_i-(p+1)/2}\prod_1^l |W_i|^{b_i-(p+1)/2}\prod_1^l dW_i\,dZ. \tag{2.3}$$

Again invoking (1.4), (2.3) becomes

$$c_1c_2\int_{0<Z+S<T} f(Z)|Z|^{\Sigma_1^k a_i-(p+1)/2}|S|^{\Sigma_1^l b_i-(p+1)/2}dS\,dZ, \tag{2.4}$$

where $c_2=\Pi_{j=2}^l B_p(b_1+\cdots+b_{j-1},b_j)$.

We now make a sequence of transformations in order to evaluate the integral

$$\int_{0<Z+S<T}|S|^{\Sigma b_i(p+1)/2}dS. \tag{2.5}$$

First, let $Z+S=H$ be a transformation from S to H, yielding

$$\int_{Z<H<T}|H-Z|^{\Sigma b_i-(p+1)/2}dH$$

$$=\int_{Z<H<T}|Z|^{\Sigma b_i-(p+1)/2}|Z^{-1/2}HZ^{-1/2}-I|^{\Sigma b_i-(p+1)/2}dH. \tag{2.6}$$

Now let $Z^{-1/2}HZ^{-1/2}=G$. Then $dH=|Z|^{(p+1)/2}dG$, and we obtain

$$\int_{I<G<Z^{-1/2}TZ^{-1/2}}|Z|^{\Sigma b_i}|G-I|^{\Sigma b_i-(p+1)/2}dG. \tag{2.7}$$

Let $U=G-I$, and write $B=Z^{-1/2}TZ^{-1/2}-I$. Then (2.7) becomes

$$|Z|^{\Sigma b_i}\int_{0<U<B}|U|^{\Sigma b_i-(p+1)/2}dU. \tag{2.8}$$

Finally, let $B^{-1/2}UB^{-1/2}=Y$, so that $dU=|B|^{(p+1)/2}dY$, and (2.8) becomes

$$|Z|^{\Sigma b_i}|B|^{\Sigma b_i}\int_{0<Y<I}|Y|^{\Sigma b_i-(p+1)/2}dY. \tag{2.9}$$

The integral in (2.9) is a matrix analog of the Beta function and was evaluated by Olkin [3]:

$$\int_{0<Y<I}|Y|^{c-(p+1)/2}|I-Y|^{d-(p+1)/2}dY=B_p(c,d), \tag{2.10}$$

so that (2.9) is equal to $|Z|^{\Sigma b_i}|B|^{\Sigma b_i}B_p(\Sigma b_i,(p+1)/2)$. Combining (2.4) with (2.9) then yields the result. ■

A matrix extension of (1.3) can be obtained in a similar manner, namely, by a repeated application of (2.1). We state this result without proof.

THEOREM 2. *If f and g are continuous scalar functions of a symmetric matrix, then*

$$\int_{\substack{0<V_i\\0<W_i\\0<\Sigma V_i+W_i<T}} f\left(\sum_1^k V_i\right)g\left(\sum_1^l W_i\right)\prod_1^k|V_i|^{a_i-(p+1)/2}\prod_1^l|W_i|^{b_i-(p+1)/2}\prod_1^k dV_i\prod_1^l dW$$

$$=\prod_1^k B_p(a_1+\cdots+a_{j-1},a_j)$$

$$\times\int_{\substack{0<W_i\\0<X+\Sigma W_i<T}} f(X)g\left(\sum_1^l W_i\right)|X|^{\Sigma a_i-(p+1)/2}\prod|W_i|^{b_i-(p+1)/2}dX\prod_1^l dW_i$$

$$=\prod_1^k B_p(a_1+\cdots+a_{j-1},a_j)\prod_1^l B_p(b_1+\cdots+b_{j-1},b_j)$$

$$\times\int_{0<X+Y<T} f(X)g(Y)|X|^{\Sigma a_i-(p+1)/2}|Y|^{\Sigma b_i-(p+1)/2}dX\,dY. \tag{2.11}$$

Note that (2.11) is equivalent to (1.3) when $p=1$.

Matrix extensions of other integrals in [2] can be obtained in a similar manner.

REFERENCES

1 J. Edwards, *A Treatise on the Integral Calculus*, Vol. 2, Macmillan, New York, 1922.
2 M. S. Klamkin, Extensions of Dirichlet's multiple integral, *SIAM J. Math. Anal.* 2:467–469 (1971).
3 I. Olkin, A class of integral identities with matrix argument, *Duke Math. J.* 26:207–214 (1959).
4 B. D. Sivazlian, The generalized Dirichlet's multiple integral, *SIAM Rev.* 11:285–288 (1969).
5 B. D. Sivazlian, A class of multiple integrals, *SIAM J. Math. Anal.* 2:72–75 (1971).

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