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Resonances in Loewner equations

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Abstract

We prove that given a Herglotz vector field on the unit ball of $\mathbb{C}^n$ of the form $H(z,t) = (a_1 z_1, \ldots, a_n z_n) + O(|z|^2)$ with $\operatorname{Re} a_j < 0$ for all j, its evolution family admits an associated Loewner chain, which is normal if no real resonances occur. Hence the Loewner–Kufarev PDE admits a solution defined for all positive times.

Keywords: Loewner chains in several variables; Loewner equations; Evolution families; Resonances

Contents

1. Introduction 1414
2. Preliminaries 1417
3. Discrete evolution families and discrete Loewner chains 1419
4. Triangular discrete evolution families 1420
5. Existence of discrete dilation Loewner chains: nearly-triangular case 1422
6. Existence of discrete dilation Loewner chains: general case 1425
7. Essential uniqueness 1429
8. Continuous case 1430
9. Counterexamples 1433

Acknowledgments 1435
References 1435

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1. Introduction

Classical Loewner theory in the unit disc $\mathbb{D} \subset \mathbb{C}$ was introduced by C. Loewner in 1923 [14] and developed with contributions of P.P. Kufarev in 1943 [12] and C. Pommerenke in 1965 [16], and has been since then used to prove several deep results in geometric function theory [11]. Loewner theory is one of the main ingredients of the proof of the Bieberbach conjecture given by de Branges [6] (see also [8]) in 1985.

Among the extensions of classical Loewner theory we recall the chordal Loewner theory [13], the celebrated theory of Schramm–Loewner evolution [18] introduced in 1999 and the theory of Loewner chains in several complex variables [7,10,15].

In [3,4] it is proposed a generalization of both the radial and chordal theories. It is shown that on complete hyperbolic manifolds there is a one-to-one correspondence between certain semicomplete non-autonomous holomorphic vector fields (called *Herglotz vector fields* and denoted $H(z,t)$) and families $(\varphi_{s,t})_{0\leqslant s\leqslant t}$ of holomorphic self-maps called *evolution families*. Indeed, if $H(z,t)$ is a Herglotz vector field, then the family $(\varphi_{s,t})$ of evolution operators for the Loewner–Kufarev ODE

$$\dot{z}(t) = H(z,t), \quad t \geqslant 0,\ z \in \mathbb{B}, \tag{1.1}$$

is an evolution family. Conversely, any evolution family is the family of evolution operators for some Loewner–Kufarev ODE.

In [5] it is proved that in dimension one evolution families are (up to biholomorphism) in one-to-one correspondence with image-growing families $(f_s)_{s\geqslant 0}$ of univalent mappings $f_s : \mathbb{D} \to \mathbb{C}$ called *Loewner chains*. Namely given any Loewner chain (f_s) the family $(\varphi_{s,t})$ defined by

$$\varphi_{s,t} = f_t^{-1} \circ f_s$$

is an evolution family, which is said to be *associated* to (f_s). Conversely given any evolution family, there exists an associated Loewner chain. Composing the two correspondences above we obtain the correspondence between Loewner chains and Herglotz vector fields: $H(z,t)$ is associated to (f_s) if and only if the mapping $t \mapsto f_t$ is a global solution for the Loewner–Kufarev PDE

$$\frac{\partial f_t(z)}{\partial t} = -f_t'(z)H(z,t), \quad t \geqslant 0,\ z \in \mathbb{D}. \tag{1.2}$$

Let N be an integer greater or equal to 2, and let $\mathbb{B}$ be the unit ball of $\mathbb{C}^N$. A Loewner chain on $\mathbb{B}$ is an image-growing family $(f_s)_{s\geqslant 0}$ of univalent mappings $f_s : \mathbb{B} \to \mathbb{C}^N$. Every Loewner chain admits an associated evolution family, but it is not known whether the converse is true. In [1] it is proposed an abstract approach to the notion of Loewner chain. Let M be an N-dimensional complete hyperbolic complex manifold. An abstract Loewner chain is an image-growing family (f_s) of univalent mappings defined on M which are allowed to take values on an arbitrary N-dimensional complex manifold. In [1] it is shown that to any evolution family $(\varphi_{s,t})$ on M there corresponds a unique (up to biholomorphisms) abstract Loewner chain (f_s). In this way one can define the *Loewner range manifold* of $(\varphi_{s,t})$

$$\mathsf{Lr}(\varphi_{s,t}) = \bigcup_{s\geqslant 0} f_s(M)$$

which is well defined and unique up to biholomorphism. Hence the classical problem of finding a Loewner chain (with values in $\mathbb{C}^N$) associated to a given evolution family $(\varphi_{s,t})$ of the unit ball $\mathbb{B} \subset \mathbb{C}^N$ corresponds to investigating whether the Loewner range manifold $\mathsf{Lr}(\varphi_{s,t})$ embeds holomorphically in $\mathbb{C}^N$.

In this paper we investigate this problem for a special type of evolution families on $\mathbb{B}$. Let Λ be an $(N \times N)$-complex matrix

$$\Lambda = \mathrm{Diag}(\alpha_1, \ldots, \alpha_N), \quad \text{where } \mathsf{Re}\,\alpha_N \leqslant \cdots \leqslant \mathsf{Re}\,\alpha_1 < 0. \tag{1.3}$$

We define a *dilation evolution family* as an evolution family $(\varphi_{s,t})$ on the unit ball $\mathbb{B} \subset \mathbb{C}^N$ satisfying

$$\varphi_{s,t}(z) = e^{\Lambda(t-s)}z + O\big(|z|^2\big).$$

A *normal Loewner chain* is a Loewner chain (f_s) such that

(1) $f_s(z) = e^{-\Lambda s}z + O(|z|^2)$,
(2) (h_s) is a normal family.

Notice that each $h_s = e^{\Lambda s} f_s$ fixes the origin and is tangent to identity in the origin (we say $h_s \in \mathrm{Tang}_1(\mathbb{C}^N, 0)$).

Problem 1.1. Given a dilation evolution family, does there exist an associated Loewner chain (with values in $\mathbb{C}^N$)?

An affirmative answer to Problem 1.1 would yield as a consequence that any Loewner–Kufarev PDE

$$\frac{\partial f_t(z)}{\partial t} = -d_z f_t H(z,t), \quad t \geqslant 0,\ z \in \mathbb{B}, \tag{1.4}$$

where $H(z,t) = \Lambda z + O(|z|^2)$ (in this case the equation is known as the Loewner PDE), admits global solutions. A partial answer may be obtained by simply combining [7, Theorem 3.1] and [10, Theorems 2.3, 2.6]:

Theorem 1.2. *Let $(\varphi_{s,t})$ be a dilation evolution family such that the eigenvalues of Λ satisfy*

$$2\,\mathsf{Re}\,\alpha_1 < \mathsf{Re}\,\alpha_N. \tag{1.5}$$

Then there exists a normal Loewner chain (f_s) associated to $(\varphi_{s,t})$, such that $\bigcup_s f_s(\mathbb{B}) = \mathbb{C}^N$, hence $\mathsf{Lr}(\varphi_{s,t}) = \mathbb{C}^N$. This chain is given by

$$f_s = \lim_{t \to +\infty} e^{-\Lambda t}\varphi_{s,t}, \tag{1.6}$$

where the limit is taken in the topology of uniform convergence on compacta, and it is the unique normal Loewner chain associated to $(\varphi_{s,t})$. A family of univalent mappings (g_s) is a Loewner *chain associated to* $(\varphi_{s,t})$ *if and only if there exists an entire univalent mapping* Ψ *on* $\mathbb{C}^N$ *such that*

$$g_s = \Psi \circ f_s.$$

The main result of this paper gives an affirmative answer to Problem 1.1, without assuming condition (1.5).

Theorem 8.6. *Let* $(\varphi_{s,t})$ *be a dilation evolution family. Then there exists a Loewner chain* (f_s) *associated to* $(\varphi_{s,t})$, *such that* $\bigcup_s f_s(\mathbb{B}) = \mathbb{C}^N$, *hence* $\mathsf{Lr}(\varphi_{s,t}) = \mathbb{C}^N$. *If no real resonances occur among the eigenvalues of* Λ, *then* (f_s) *is a normal chain, not necessarily unique. A family of univalent mappings* (g_s) *is a Loewner chain associated to* $(\varphi_{s,t})$ *if and only if there exists an entire univalent mapping* Ψ *on* $\mathbb{C}^N$ *such that*

$$g_s = \Psi \circ f_s.$$

Notice that (1.5) is a classical condition which ensures the existence of a solution for the Schröder functional equation. In fact we will see that normal Loewner chains correspond to solutions of a parametric Schröder equation. Let us first recall some facts about linearization of germs.

Let $\varphi(z) = e^{\Lambda} z + O(|z|^2)$ be a holomorphic germ at the origin of $\mathbb{C}^N$, where Λ is a matrix satisfying (1.3). If $h \in \mathrm{Tang}_1(\mathbb{C}^N, 0)$ is a solution of the Schröder equation

$$h \circ \varphi = e^{\Lambda} h, \tag{1.7}$$

we say that h linearizes φ. It is not always possible to solve this equation, indeed there can occur complex resonances among the eigenvalues of Λ, that is algebraic identities

$$\sum_{j=1}^{N} k_j \alpha_j = \alpha_l,$$

where $k_j \geqslant 0$ and $\sum_j k_j \geqslant 2$, which are obstructions to linearization (the term "complex" is not standard and is here used to distinguish from real resonances, defined below). Indeed a celebrated theorem of Poincaré (see for example [17, pp. 80–86]) states that if no complex resonances occur, then there exists a solution h for (1.7). If moreover $2\,\mathsf{Re}\,\alpha_1 < \mathsf{Re}\,\alpha_N$ then h is given by $\lim_{n\to+\infty} e^{-\Lambda n} \varphi^{\circ n}$.

In our case we are interested in the following parametric Schröder equation

$$h_m \circ \varphi_{n,m} = e^{\Lambda(m-n)} h_n, \tag{1.8}$$

where $(\varphi_{n,m})$ is the discrete analogue of a dilation evolution family. We search for a solution (h_n) which is a normal family of univalent mappings in $\mathrm{Tang}_1(\mathbb{C}^N, 0)$. The parametric Schröder equation admits such a solution (h_n) if and only if $(\varphi_{n,m})$ admits a discrete normal Loewner chain (f_n), and

$$(f_n) = \big(e^{-\Lambda n} h_n\big).$$

There are surprising differences between the Schröder functional equation (1.7) and (1.8). Namely, while in the first complex resonances are obstructions to the existence of formal solutions, in the latter there always exists the holomorphic solution $h_n = e^{\Lambda n}\varphi_{0,n}^{-1}$, but the domain of definition of the mapping h_n shrinks as n grows. If, as we need, we look for solutions which are all defined in the unit ball $\mathbb{B}$, then we find as obstructions *real resonances* among the eigenvalues of Λ, that is algebraic identities

$$\mathsf{Re}\left(\sum_{j=1}^{N} k_j\alpha_j\right) = \mathsf{Re}\,\alpha_l,$$

where $k_j \geqslant 0$ and $\sum_j k_j \geqslant 2$. If real resonances occur we solve a slightly different equation:

$$h_m \circ \varphi_{n,m} = T_{n,m} \circ h_n,$$

where $(T_{n,m})$ is a suitable triangular evolution family, finding this way a non-necessarily normal discrete Loewner chain associated to $(\varphi_{n,m})$.

Once we solved the problem for discrete times, we solve the problem for continuous times: we discretize a given continuous dilation evolution family $(\varphi_{s,t})$ obtaining a discrete dilation evolution family $(\varphi_{n,m})$, and we take the associated discrete Loewner chain (f_n). Then we extend (f_n) to all real positive times obtaining this way a Loewner chain (f_s) and Theorem 8.6 above.

We give examples of

(1) a dilation evolution family with no real resonances and several associated normal Loewner chains,
(2) a semigroup-type dilation evolution family with complex resonances which does not admit any associated normal Loewner chain,
(3) a discrete dilation evolution family with pure real resonances (real non-complex resonances) which does not admit any discrete normal Loewner chain,
(4) a discrete evolution family not of dilation type which does not admit any associated discrete Loewner chain (with values in $\mathbb{C}^N$).

2. Preliminaries

The following is a several variables version of the Schwarz Lemma [11, Lemma 6.1.28].

Lemma 2.1. *Let $M > 0$ and $f : \mathbb{B} \to \mathbb{C}^N$ be a holomorphic mapping fixing the origin and bounded by M. Then for z in the ball, $|f(z)| \leqslant M|z|$. If there is a point $z_0 \in \mathbb{B} \setminus \{0\}$ such that $|f(z_0)| = M|z_0|$, then $|f(\zeta z_0)| = M|\zeta z_0|$ for all $|\zeta| < 1/|z_0|$. Moreover, if $f(z) = O(|z|^k)$, $k \geqslant 2$, then for z in the ball, $|f(z)| \leqslant M|z|^k$.*

Let $\mathcal{F}_{r,M,A}$ be the family of holomorphic mappings $f : r\mathbb{B} \to \mathbb{C}^N$, bounded by M, fixing the origin and with common differential Az at the origin satisfying $\|A\| < 1$.

Lemma 2.2. *For each $f \in \mathcal{F}_{r,M,A}$, we have $|f(z) - Az| \leqslant C|z|^2$, where $C = C(r, M, A)$. If moreover $f(z) - Az = O(|z|^k)$ for $k \geqslant 3$, then $|f(z) - Az| \leqslant C_k|z|^k$, where $C_k = C_k(r, M, A)$.*

Proof. Setting $C_k = M/r^k + \|A\|/r^{k-1}$, the result follows from the previous lemma. $\square$

As a consequence we get the following

Lemma 2.3. *For each $f \in \mathcal{F}_{r,M,A}$ we have the following estimate: to each $\|A\| < \alpha < 1$ there corresponds $s > 0$, $s = s(r, M, A)$ such that $|f(z)| \leqslant \alpha|z|$, if $|z| \leqslant s$.*

Proof. We proceed by contradiction: assume there exist a sequence $f_n \in \mathcal{F}_{r,M,A}$ and a sequence of points z_n converging to the origin verifying $|f_n(z_n)| > \alpha|z_n|$. We have

$$\left|f_n(z_n)\right| = \left|Ax_n + f_n(z_n) - Az_n\right| \leqslant |Az_n| + C|z_n|^2,$$

thus

$$\alpha < \frac{|f_n(z_n)|}{|z_n|} \leqslant \frac{|Az_n|}{|z_n|} + C|z_n|,$$

but the right-hand term has lim sup less or equal than $\|A\|$, which is the desired contradiction. $\square$

Lemma 2.4. *For each $f \in \mathcal{F}_{r,r,A}$ we have the following estimate: to each $s < r$ there corresponds $K < 1$, $K = K(r, A)$, such that $|f(z)| \leqslant K|z|$, if $|z| \leqslant s$.*

Proof. Assume the contrary: suppose there exist a sequence $f_n \in \mathcal{F}_{r,r,A}$ and a sequence of points z_n in $\overline{s\mathbb{B}}$ verifying $|f_n(z_n)| > (1 - 1/n)|z_n|$. Up to subsequences we have $z_n \to z'$ for some z' such that $|z'| \leqslant s$, and $f_n \to f$ uniformly on compacta since $\mathcal{F}_{r,r,A}$ is a normal family. If $z' \neq 0$ we have

$$1 - \frac{1}{n} < \frac{|f_n(z_n)|}{|z_n|} \to \frac{|f(z')|}{|z'|},$$

and $|f(z')|/|z'| < 1$ by Lemma 2.1, which is a contradiction. If $z' = 0$, using again Lemma 2.2 we get

$$1 - \frac{1}{n} < \frac{|f_n(z_n)|}{|z_n|} \leqslant \frac{|Az_n|}{|z_n|} + C|z_n|,$$

and the right-hand term has lim sup less than or equal to $\|A\|$, contradiction. $\square$

Lemma 2.5. *Suppose that D is an open set in $\mathbb{C}^N$ containing the origin. Suppose we have a uniformly bounded family $\mathcal{H}$ of holomorphic mappings $h : D \to \mathbb{C}^N$ in $\mathrm{Tang}_1(\mathbb{C}^N, 0)$. Then there exist a ball $r\mathbb{B} \subset D$ such that every $h \in \mathcal{H}$ is univalent on $r\mathbb{B}$, and a ball $s\mathbb{B}$ such that $s\mathbb{B} \subset h(r\mathbb{B})$ for all $h \in \mathcal{H}$.*

Proof. Suppose there does not exist a ball $r\mathbb{B} \subset D$ such that every $h \in \mathcal{H}$ is univalent on $r\mathbb{B}$. Since $\mathcal{H}$ is a normal family there exists a sequence $h_n \to f$ uniformly on compacta, and such that there does not exist a ball $r\mathbb{B} \subset D$ with the property that every h_n is univalent on $r\mathbb{B}$. Since $f \in \mathrm{Tang}_1(\mathbb{C}^N, 0)$ there exists a ball where f is univalent. We can now apply [11, Theorem 6.1.18], getting a contradiction. Assume now there does not exist a ball contained in each $h(r\mathbb{B})$. Again there is a sequence $h'_n \to f'$ uniformly on compacta, such that there does not exist a ball $s\mathbb{B} \subset \bigcap h'_n(r\mathbb{B})$. The contradiction is then given by [1, Proposition 3.1]. □

3. Discrete evolution families and discrete Loewner chains

Let A be a complex $(N \times N)$-matrix

$$A = \mathrm{Diag}(\lambda_1, \dots, \lambda_n), \quad 0 < |\lambda_N| \leqslant \cdots \leqslant |\lambda_1| < 1. \tag{3.1}$$

Definition 3.1. We define a *discrete evolution family* on a domain $D \subseteq \mathbb{C}^N$ as a family $(\varphi_{n,m})_{n \leqslant m \in \mathbb{N}}$ (sometimes denoted by $(\varphi_{n,m}; D)$) of univalent self-mappings of D, which satisfies the *semigroup conditions*:

$$\varphi_{n,n} = \mathsf{id}, \qquad \varphi_{l,m} \circ \varphi_{n,l} = \varphi_{n,m},$$

where $0 \leqslant n \leqslant l \leqslant m$. Such a family is clearly determined by the subfamily $(\varphi_{n,n+1})$. A *dilation discrete evolution family* is a discrete evolution family such that

$$0 \in D, \quad \varphi_{n,n+1}(0) = 0, \qquad \varphi_{n,n+1}(z) = Az + O\big(|z|^2\big) \quad \text{for all } n \geqslant 0. \tag{3.2}$$

Definition 3.2. A family $(f_n)_{n \in \mathbb{N}}$ of holomorphic mappings $f_n : D \to \mathbb{C}^N$ is a *discrete subordination chain* if for each $n < m$ the mapping f_n is *subordinate* to f_m, that is, there exists a holomorphic mapping (called *transition mapping*) $\varphi_{n,m} : D \to D$ such that

$$f_n = f_m \circ \varphi_{n,m}.$$

It is easy to see that the family of transition mappings of a subordination chain satisfies the semigroup property. If a subordination chain (f_n) admits transition mappings $\varphi_{n,m}$ which form a discrete evolution family (namely, $\varphi_{n,m}$ are univalent) we say that (f_n) is *associated* to $(\varphi_{n,m})$.

Definition 3.3. We define a *discrete Loewner chain* as a subordination chain (f_n) such that every f_n is univalent. In this case every *transition mapping* $\varphi_{n,m}$ is univalent and uniquely determined. Thus the transition mappings form a discrete evolution family. A Loewner chain (f_n) is *normalized* if $f_0(0) = 0$ and $d_0 f_0 = \mathrm{Id}$. A *dilation discrete Loewner chain* is a discrete Loewner chain such that

$$f_n(z) = A^{-n} z + O\big(|z|^2\big).$$

Following Pommerenke [16], we call a dilation Loewner chain *normal* if $(A^n f_n)$ is a normal family.

4. Triangular discrete evolution families

Recall that a *triangular automorphism* is a mapping $T : \mathbb{C}^N \to \mathbb{C}^N$ of the form

$$\begin{aligned}
T^{(1)}(z) &= \lambda_1 z_1, \\
T^{(2)}(z) &= \lambda_2 z_2 + t^{(2)}(z_1), \\
T^{(3)}(z) &= \lambda_3 z_3 + t^{(3)}(z_1, z_2), \\
&\vdots \\
T^{(N)}(z) &= \lambda_N z_N + t^{(N)}(z_1, z_2, \ldots, z_{N-1}),
\end{aligned}$$

where $0 < |\lambda_j| < 1$, and $t^{(i)}$ is a polynomial in $i-1$ variables, with all terms of degree greater or equal 2. This is indeed an automorphism, since we can iteratively write its inverse, which is still a triangular automorphism:

$$\begin{aligned}
z_1 &= \frac{w_1}{\lambda_1}, \\
z_2 &= \frac{w_2}{\lambda_2} - \frac{1}{\lambda_2} t^{(2)}(z_1), \\
&\vdots \\
z_N &= \frac{w_N}{\lambda_N} - \frac{1}{\lambda_N} t^{(N)}(z_1, z_2, \ldots, z_{N-1}).
\end{aligned} \tag{4.1}$$

Definition 4.1. The *degree* of T is $\max_i \deg T^{(i)}$. We define a *triangular evolution family* as a discrete dilation evolution family $(T_{n,m})$ of $\mathbb{C}^N$ such that each $T_{n,n+1}$, and hence every $T_{n,m}$, is a triangular automorphism. We denote $T^{(j)}_{n,n+1}(z) = \lambda_j z_j + t^{(j)}_{n,n+1}(z_1, z_2, \ldots, z_{j-1})$ for all $n \geqslant 0$ and all $1 \leqslant j \leqslant N$. We denote

$$T_{m,n} = T^{-1}_{n,m}, \quad 0 \leqslant n \leqslant m.$$

We say that a triangular evolution family $(T_{n,m})$ has *bounded coefficients* if the family $(T_{n,n+1})$ has uniformly bounded coefficients, and we say that it has *bounded degree* if $\sup_n \deg T_{n,n+1} < \infty$.

We can easily find a Loewner chain associated to a triangular evolution family:

$$(f_n) = (T_{n,0}) = \left(T^{-1}_{0,n}\right).$$

Indeed,

$$f_m \circ T_{n,m} = T_{m,0} \circ T_{n,m} = T_{n,0} = f_n.$$

The following lemmas are just adaptations of [17, Lemma 1, p. 80].

Lemma 4.2. *Assume that* $\sup_n \deg T_{n,n+1} < \infty$, *then* $\sup_n \deg T_{0,n} < \infty$.

Proof. Set for $j = 1, \dots, N$,

$$\mu^{(j)} = \max_n \deg T^{(j)}_{n,n+1}.$$

We denote by $S(m,k)$ the property

$$\deg T^{(j)}_{0,k} \leqslant \mu^{(1)} \cdots \mu^{(j)}, \quad 1 \leqslant j \leqslant m.$$

Since $T_{0,k+1} = T_{k,k+1} \circ T_{0,k}$, we have

$$T^{(j)}_{0,k+1} = \lambda_j T^{(j)}_{0,k} + t^{(j)}_{k,k+1}\big(T^{(1)}_{0,k}, \dots, T^{(j-1)}_{0,k}\big), \quad 2 \leqslant j \leqslant N,$$

thus $S(m,k+1)$ follows from $S(m,k)$ and $S(m-1,k)$. Since $S(1,k)$ and $S(m,1)$ are obviously true for all k and m (note that $\mu^{(1)} = 1$, and $\mu^{(j)} \geqslant 1$, for every j), $S(N,k)$ follows by induction. Hence

$$\deg T_{0,k} \leqslant \mu^{(1)} \cdots \mu^{(N)}. \qquad \square$$

Lemma 4.3. *Let* $(T_{n,m})$ *be a triangular evolution family of bounded degree and bounded coefficients. Let* Δ *be the unit polydisc. Then there exists a constant* $\gamma > 0$ *such that*

$$T_{k,0}(\Delta) \subset \gamma^k \Delta, \quad k \geqslant 1.$$

Proof. The family $(T_{n+1,n})$ of inverses of $(T_{n,n+1})$ has bounded coefficients. Indeed the family $(T_{n,n+1})$ has bounded coefficients, and the assertion follows by looking at (4.1). Likewise, $\sup_n \deg T_{n,0} < \infty$, since $\sup_n \deg T_{0,n} < \infty$. Hence there exists $C \geqslant 1$ such that $|T^{(j)}_{n+1,n}(z)| \leqslant C$ for $z \in \Delta$, $1 \leqslant j \leqslant N$, and there exists $d = \max_n \deg T_{n,0}$. Let M be the number of multi-indices $I = (i_1, \dots, i_N)$ with $|I| \leqslant d$, and set $\gamma = MC^d$, we claim that

$$\big|T^{(j)}_{k,0}(z)\big| \leqslant \gamma^k, \quad \text{for } z \in \Delta,\ j = 1, \dots, N. \tag{4.2}$$

We proceed by induction on k. Since $C \leqslant \gamma$, (4.2) holds for $k = 1$. Assume (4.2) holds for some $k \geqslant 1$. By Cauchy estimates the coefficients in $T^{(j)}_{k,0}(z) = \sum_{|I| \leqslant d} a_I z^I$ satisfy

$$|a_I| \leqslant \gamma^k.$$

Since $T_{k+1,0} = T_{k,0} \circ T_{k+1,k}$, we have

$$T^{(j)}_{k+1,0} = T^{(j)}_{k,0}\big(T^{(1)}_{k+1,k}, \dots, T^{(N)}_{k+1,k}\big) = \sum_{|I| \leqslant d} a_I \big(T^{(1)}_{k+1,k}\big)^{i_1} \cdots \big(T^{(N)}_{k+1,k}\big)^{i_N}.$$

Then

$$\big|T^{(j)}_{k+1,0}\big| \leqslant MC^d \gamma^k = \gamma^{k+1}. \qquad \square$$

Corollary 4.4. *Let $(T_{n,m})$ be a triangular evolution family of bounded degree and bounded coefficients. Let $\frac{1}{2}\Delta$ be the polydisc of radius $(1/2)$. Then there exists $\beta \geqslant 0$ such that for all $k \geqslant 1$ and all $z, z' \in \frac{1}{2}\Delta$,*

$$\left|T_{k,0}(z) - T_{k,0}\left(z'\right)\right| \leqslant \beta^k \left|z - z'\right|.$$

Proof. Recall that $\Delta \subset \sqrt{N}\mathbb{B}$ and that if $B = (b_{ij})$ is a complex $(N \times N)$-matrix, then

$$\|B\| \leqslant N \max_{i,j} |b_{ij}|.$$

If $z \in (1/2)\Delta$, then by Cauchy estimates and Lemma 4.3,

$$\|d_z T_{k,0}\| \leqslant 2N\sqrt{N}\gamma^k.$$

The result follows setting $\beta = 2N\sqrt{N}\gamma$. □

Lemma 4.5. *Let $(T_{n,m})$ be a triangular evolution family, with bounded degree and bounded coefficients. Then $T_{0,n}(z) \to 0$ uniformly on compacta. Hence for each neighborhood V of 0 we have*

$$\bigcup_{n=1}^{\infty} T_{n,0}(V) = \mathbb{C}^N.$$

Proof. Let K be a compact set in $\mathbb{C}^N$. We proceed by induction on i. Notice that $T^{(1)}_{0,k}(z) = \lambda_1^k z_1$, hence if $\|\cdot\|$ denotes the sup-norm on K, we have $\|T^{(1)}_{0,k}\| \to 0$. Let $1 < i \leqslant N$ and assume that $\lim_{k\to\infty} \|T^{(j)}_{0,k}\| = 0$, for $1 \leqslant j < i$. On K

$$\lim_{k\to\infty} \left\|t^{(i)}_{k,k+1}\left(T^{(1)}_{0,k}, \ldots, T^{(i-1)}_{0,k}\right)\right\| = 0,$$

since the family $(t^{(i)}_{k,k+1})$ has uniformly bounded coefficients and uniformly bounded degree. Notice that

$$T^{(i)}_{0,k+1} = \lambda_i T^{(i)}_{0,k} + t^{(i)}_{k,k+1}\left(T^{(1)}_{0,k}, \ldots, T^{(i-1)}_{0,k}\right), \quad 2 \leqslant i \leqslant N. \tag{4.3}$$

Hence, for each $\varepsilon > 0$, $|T^{(i)}_{0,k+1}| \leqslant |\lambda_i||T^{(i)}_{0,k}| + \varepsilon$, on K for k large enough. Therefore $\limsup_{k\to\infty} \|T^{(i)}_{0,k}\| = 0$, concluding the induction. □

5. Existence of discrete dilation Loewner chains: nearly-triangular case

We are going to prove the existence of Loewner chains associated to a given discrete dilation evolution family by conjugating it to a triangular evolution family by means of a time dependent intertwining map. In this perspective, we shall see that normal Loewner chains correspond to time dependent linearizations of the evolution family.

Definition 5.1. Let $D \subset \mathbb{C}^N$ be a domain containing 0. Given two discrete dilation evolution families $(\varphi_{n,m}; t\mathbb{B})$, $(\psi_{n,m}; D)$ suppose there exists, in a ball $r\mathbb{B} \subset t\mathbb{B}$, a normal family of univalent mappings $h_n : r\mathbb{B} \to D$ in $\mathrm{Tang}_1(\mathbb{C}^N, 0)$, such that

$$h_m \circ \varphi_{n,m} = \psi_{n,m} \circ h_n, \quad 0 \leqslant n \leqslant m, \tag{5.1}$$

then we shall say that (h_n) *conjugates* $(\varphi_{n,m})$ to $(\psi_{n,m})$.

Notice that if $(\varphi_{n,m})$ is conjugate to $(\psi_{n,m})$ then necessarily $d_0\varphi_{n,m} = d_0\psi_{n,m}$. Let $(\varphi_{n,m}; t\mathbb{B})$ be a discrete dilation evolution family, and let $r\mathbb{B} \subset t\mathbb{B}$. Then $r\mathbb{B}$ is invariant for every $\varphi_{n,n+1}$, and hence $(\varphi_{n,m}; t\mathbb{B})$ restricts to an evolution family $(\varphi_{n,m}; r\mathbb{B})$.

Lemma 5.2. *Let $0 < r < t$. Let $(\varphi_{n,m}; t\mathbb{B})$ be a discrete dilation evolution family. Suppose there exists a Loewner chain (f_n) associated to the evolution family $(\varphi_{n,m}; r\mathbb{B})$. Then there exists a Loewner chain (f_n^{e}) associated to $(\varphi_{n,m}; t\mathbb{B})$ which extends (f_n) in the following sense:*

$$f_n^{\mathsf{e}}(z) = f_n(z), \quad z \in r\mathbb{B},\ n \geqslant 0.$$

Proof. Fix $n \geqslant 0$. Let $0 < r \leqslant s < t$. Let $k(s) \geqslant n$ be the least integer such that $\varphi_{n,k(s)}(s\mathbb{B}) \subset r\mathbb{B}$ (which exists by Lemma 2.4). Define

$$f_n^{\mathsf{e}}(z) = f_{k(s)}\big(\varphi_{n,k(s)}(z)\big), \quad z \in s\mathbb{B}.$$

A priori the value $f_n^{\mathsf{e}}(z)$ depends on s. However if $0 < r \leqslant s < u < t$, then $k(u) \geqslant k(s)$, thus

$$f_{k(u)}\big(\varphi_{n,k(u)}(z)\big) = f_{k(u)}\big(\varphi_{k(s),k(u)}\big(\varphi_{n,k(s)}(z)\big)\big) = f_{k(s)}\big(\varphi_{n,k(s)}(z)\big), \quad \text{for all } z \in s\mathbb{B}.$$

Therefore f_n^{e} is well defined on $t\mathbb{B}$. Notice that since $k(r) = n$,

$$f_n^{\mathsf{e}}\big|_{r\mathbb{B}} = f_n\big(\varphi_{n,n}(z)\big) = f_n.$$

It is easy to see that f_n^{e} is holomorphic and injective and that (f_n^{e}) is a Loewner chain associated to $(\varphi_{n,m}; t\mathbb{B})$. $\square$

Notice that the extended chain (f_n^{e}) can also be defined by

$$f_n^{\mathsf{e}}(z) = \lim_{m\to\infty} f_m \circ \varphi_{n,m}(z), \quad z \in t\mathbb{B}. \tag{5.2}$$

Now we can show how conjugations allow us to pull-back Loewner chains:

Remark 5.3. Suppose that (h_n), defined on $r\mathbb{B}$, conjugates $(\varphi_{n,m}; t\mathbb{B})$ to $(\psi_{n,m}; D)$, and assume that (f_n) is a Loewner chain associated to $(\psi_{n,m})$. The *pull-back chain* $(f_n \circ h_n)$ on $r\mathbb{B}$ is easily seen to be associated to $(\varphi_{n,m}; r\mathbb{B})$. By Lemma 5.2 one can extend $(f_n \circ h_n)$ to all of $t\mathbb{B}$ obtaining a Loewner chain associated to $(\varphi_{n,m}; t\mathbb{B})$ and defined by

$$(f_n \circ h_n)^{\mathsf{e}}(z) = \lim_{m\to\infty} f_m \circ h_m \circ \varphi_{n,m}(z), \quad z \in t\mathbb{B}. \tag{5.3}$$

If (h_n) conjugates an evolution family $(\varphi_{n,m}; t\mathbb{B})$ to a triangular evolution family $(T_{n,m})$, a Loewner chain associated to $(\varphi_{n,m})$ is given by the functions $(T_{n,0} \circ h_n)^{\mathsf{e}}$. If in particular (h_n) linearizes the given evolution family, that is $T_{n,m}(z) = A^{m-n}z$, we obtain this way a normal Loewner chain $(A^{-n}h_n)$. Hence one has the following

Proposition 5.4. *A discrete dilation evolution family $(\varphi_{n,m})$ admits a normal Loewner chain if and only if there exists a normal family (h_n) of univalent mappings in* $\mathrm{Tang}_1(\mathbb{C}^N, 0)$ *which conjugates it to its linear part*:

$$h_m \circ \varphi_{n,m} = A^{m-n}h_n, \quad 0 \leqslant n \leqslant m.$$

Next we show how to find conjugations, provided we start with a discrete dilation evolution family close enough to a triangular evolution family.

Proposition 5.5. *Suppose that $(\varphi_{n,m}; t\mathbb{B})$ is a discrete dilation evolution family, and that $(T_{n,m})$ is a triangular evolution family with bounded degree and bounded coefficients. Let β be the constant given by Corollary* 4.4 *for $(T_{n,m})$, and let k be an integer such that*

$$|\lambda_1|^k < \frac{1}{\beta}.$$

If for each $n \geqslant 0$ we have

$$\varphi_{n,n+1}(z) - T_{n,n+1}(z) = O\big(|z|^k\big),$$

then $(\varphi_{n,m})$ is conjugate to $(T_{n,m})$.

Proof. Choose $|\lambda_1| < c < 1$ such that $c^k < 1/\beta$. Lemma 2.3 gives us $r > 0$ (we can assume $0 < r < \min\{1/2, t\}$) such that on $r\mathbb{B}$ we have $|\varphi_{n,n+1}(z)| \leqslant c|z|$ and $|T_{n,n+1}(z)| \leqslant c|z|$ for all $n \geqslant 0$. Thus for $z \in r\mathbb{B}$ we have $|\varphi_{0,n}(z)| < rc^n$. Thanks to Lemma 2.2 we have

$$\big|\varphi_{n,n+1}(\zeta) - T_{n,n+1}(\zeta)\big| \leqslant C|\zeta|^k,$$

hence

$$\begin{aligned}\big|\varphi_{0,n+1}(z) - T_{n,n+1} \circ \varphi_{0,n}(z)\big| &= \big|\varphi_{n,n+1} \circ \varphi_{0,n}(z) - T_{n,n+1} \circ \varphi_{0,n}(z)\big| \\ &\leqslant C\big|\varphi_{0,n}(z)\big|^k \leqslant Cr^k c^{kn}.\end{aligned}$$

The sequence $T_{n,0} \circ \varphi_{0,n}(z)$ verifies

$$\begin{aligned}\big|T_{n+1,0} \circ \varphi_{0,n+1}(z) - T_{n,0} \circ \varphi_{0,n}(z)\big| &= \big|T_{n+1,0} \circ \varphi_{0,n+1}(z) - (T_{n+1,0} \circ T_{n,n+1}) \circ \varphi_{0,n}(z)\big| \\ &\leqslant \beta^{n+1}\big|\varphi_{0,n+1}(z) - T_{n,n+1} \circ \varphi_{0,n}(z)\big| \\ &\leqslant \beta^{n+1}Cr^k c^{kn} \\ &= \big(c^k\beta\big)^n Cr^k\beta,\end{aligned}$$

where we used Corollary 4.4 (notice that since $r < 1/2$, we have $|\varphi_{0,n+1}(z)| < c^{n+1}/2$ and $|T_{n,n+1} \circ \varphi_{0,n}(z)| < c^{n+1}/2$, hence both $\varphi_{0,n+1}(z)$ and $T_{n,n+1} \circ \varphi_{0,n}(z)$ are in $\frac{1}{2}\Delta$).

Hence the holomorphic mappings $T_{j,0} \circ \varphi_{0,j}(z)$ converge uniformly on $r\mathbb{B}$ to a holomorphic function $h_0 \in \mathrm{Tang}_1(\mathbb{C}^N, 0)$ (and univalent for the Hurwitz Theorem in several variables). Likewise,

$$T_{j,n} \circ \varphi_{n,j}(z) \to h_n(z).$$

Each h_n is bounded by

$$1 + \sum_{n=0}^{\infty} Cr^k \beta \big(c^k \beta\big)^n,$$

hence they form a normal family. Moreover

$$h_m \circ \varphi_{n,m} = \lim_{j\to\infty} T_{j,m} \circ \varphi_{m,j} \circ \varphi_{n,m} = \lim_{j\to\infty} T_{n,m} \circ T_{j,n} \circ \varphi_{n,j} = T_{n,m} \circ h_n. \qquad \square$$

By (5.3), a Loewner chain on $t\mathbb{B}$ associated to $(\varphi_{n,m})$ is given by

$$(T_{n,0} \circ h_n)^{\mathsf{e}}(z) = \lim_{m\to\infty} T_{m,0} \circ \Big(\lim_{j\to\infty} T_{j,m} \circ \varphi_{m,j}\Big) \circ \varphi_{n,m}(z) = \lim_{m\to\infty} T_{m,0} \circ \varphi_{n,m}(z). \tag{5.4}$$

6. Existence of discrete dilation Loewner chains: general case

In this section we show how to conjugate a given discrete dilation evolution family $(\varphi_{n,m}; t\mathbb{B})$ to a nearly-triangular evolution family, by removing all non-resonant terms applying a parametric version of the Poincaré–Dulac method. This will give as a consequence the existence of Loewner chains for every discrete dilation evolution family.

Definition 6.1. A *real resonance* for a matrix A with eigenvalues λ_i is an identity

$$|\lambda_j| = \big|\lambda_1^{i_1} \cdots \lambda_N^{i_N}\big|,$$

where $i_j \geqslant 0$, and $\sum_j i_j \geqslant 2$. If $|\lambda_j| < 1$ for all $1 \leqslant j \leqslant N$, real resonances can occur only in a finite number. Moreover, if $0 < |\lambda_N| \leqslant \cdots \leqslant |\lambda_1| < 1$ then the equality $|\lambda_j| = |\lambda_1^{i_1} \cdots \lambda_N^{i_N}|$, implies $i_j = i_{j+1} = \cdots = i_N = 0$. Let $\varphi : t\mathbb{B} \to \mathbb{C}^N$ be a univalent mapping such that $\varphi(z) = Az + O(|z|^2)$, and denote its j-th component as

$$\varphi^{(j)}(z) = \lambda_j z_j + \sum_{|I| \geqslant 2} a_I^{(j)} z^I,$$

where as usual, $z^I = z_1^{i_1} \cdots z_N^{i_N}$ for $I = (i_1, \dots, i_N)$. We call a monomial $a_I^{(j)} z^I$ *resonant* if a real resonance $|\lambda_j| = |\lambda^I|$ occurs. A mapping with only resonant monomials is necessarily triangular.

Proposition 6.2. *Let $(\varphi_{n,m}; t\mathbb{B})$ be a discrete dilation evolution family. For each $i \geqslant 2$ there exist*

(1) *an evolution family $(\varphi^i_{n,m})$ defined on a ball $\mathbb{B}_i$,*

(2) *a uniformly bounded family* (k_n^i) *of univalent maps defined on a ball* $\mathbb{B}_i' \subset t\mathbb{B}$ *which conjugates* $(\varphi_{n,m})$ *to* $(\varphi_{n,m}^i)$,
(3) *a triangular evolution family* $(T_{n,m}^i)$ *with* $\deg T_{n,n+1}^i \leqslant i-1$ *for all* $n \geqslant 0$ *and bounded coefficients such that for all* $n \geqslant 0$,

$$\varphi_{n,n+1}^i = T_{n,n+1}^i + O\big(|z|^i\big).$$

Proof. We proceed by induction. For $i=2$ it suffices to set $\varphi_{n,n+1}^2 = \varphi_{n,n+1}$, $k_n^2 = \mathsf{id}$ and $T_{n,n+1}^2 = A$. Assume the proposition holds for $i \geqslant 2$. Thus there exist $(\varphi_{n,m}^i; \mathbb{B}_i)$ and $(T_{n,m}^i)$ such that

$$\varphi_{n,n+1}^i - T_{n,n+1}^i = O\big(|z|^i\big).$$

Since $\deg T_{n,n+1}^i \leqslant i-1$, we have

$$\varphi_{n,n+1}^i - T_{n,n+1}^i - P_{n,n+1}^i = O\big(|z|^{i+1}\big),$$

where $P_{n,n+1}^i$ is the homogeneous term of $\varphi_{n,n+1}^i$ of degree i. Let $R_{n,n+1}^i$ be the polynomial mapping obtained deleting every non-resonant term from $P_{n,n+1}^i$. Define the triangular evolution family $(T_{n,m}^{i+1})$ by $T_{n,n+1}^{i+1} = T_{n,n+1}^i + R_{n,n+1}^i$. First we prove that there exists a family (k_n) of polynomial mappings in $\mathrm{Tang}_1(\mathbb{C}^N, 0)$ with uniformly bounded degrees and uniformly bounded coefficients satisfying

$$k_{n+1} \circ \varphi_{n,n+1}^i - T_{n,n+1}^{i+1} \circ k_n = O\big(|z|^{i+1}\big). \tag{6.1}$$

Let I be a multi-index, $|I| = i$, and let j be an integer $1 \leqslant j \leqslant N$. Define $k_{I,j,n}$ as the polynomial mapping whose l-th component is

$$k_{I,j,n}^{(l)}(z) = z_l + \delta_{lj}\alpha_{I,n}^{(j)} z^I,$$

where δ_{lj} is the Kronecker delta and $\alpha_{I,n}^{(j)} \in \mathbb{C}$ is to be chosen. Denote the j-th component of $\varphi_{n,n+1}$ as $\lambda_j z_j + \sum_{|I|\geqslant 2} a_{I,n,n+1}^{(j)} z^I$. In the case $|\lambda_j| = |\lambda^I|$, that is when every $a_{I,n,n+1}^{(j)} z^I$ with $a_{I,n,n+1}^{(j)} \neq 0$ is resonant, set $\alpha_{I,n}^{(j)} = 0$ for each n. In the case $|\lambda_j| \neq |\lambda^I|$, by imposing the vanishing of terms in z^I in the left-hand side of Eq. (6.1) we obtain the *homological equation*:

$$\lambda^I \alpha_{I,n+1}^{(j)} + a_{I,n,n+1}^{(j)} = \lambda_j \alpha_{I,n}^{(j)}. \tag{6.2}$$

We have thus a recursive formula for $\alpha_{I,n}^{(j)}$ in the non-resonant case:

$$\alpha_{I,n+1}^{(j)} = \frac{\lambda_j \alpha_{I,n}^{(j)} - a_{I,n,n+1}^{(j)}}{\lambda^I},$$

which gives

$$\alpha^{(j)}_{I,n} = \alpha^{(j)}_{I,0}\left(\frac{\lambda_j}{\lambda^I}\right)^n - \frac{a^{(j)}_{I,0,1}}{\lambda^I}\left(\frac{\lambda_j}{\lambda^I}\right)^{n-1} - \frac{a^{(j)}_{I,1,2}}{\lambda^I}\left(\frac{\lambda_j}{\lambda^I}\right)^{n-2} - \cdots - \frac{a^{(j)}_{I,n-1,n}}{\lambda^I}. \tag{6.3}$$

Since by Cauchy estimates $(a^{(j)}_{I,n,n+1})$ is a bounded sequence, if $|\lambda_j| < |\lambda^I|$ then $(\alpha^{(j)}_{I,n})$ is bounded regardless of our choice for $\alpha^{(j)}_{I,0} \in \mathbb{C}$, so we can set $\alpha^{(j)}_{I,0} = 0$. In the case $|\lambda_j| > |\lambda^I|$ we have to choose $\alpha^{(j)}_{I,0}$ suitably in order to obtain a bounded sequence. Divide (6.3) by $(\lambda_j/\lambda^I)^n$:

$$\alpha^{(j)}_{I,n}\left(\frac{\lambda^I}{\lambda_j}\right)^n = \alpha^{(j)}_{I,0} - \frac{a^{(j)}_{I,0,1}}{\lambda^I}\left(\frac{\lambda^I}{\lambda_j}\right) - \frac{a^{(j)}_{I,1,2}}{\lambda^I}\left(\frac{\lambda^I}{\lambda_j}\right)^2 - \cdots - \frac{a^{(j)}_{I,n-1,n}}{\lambda^I}\left(\frac{\lambda^I}{\lambda_j}\right)^n, \tag{6.4}$$

and set

$$\alpha^{(j)}_{I,0} = \sum_{m=1}^{\infty} \frac{a^{(j)}_{I,m-1,m}}{\lambda^I}\left(\frac{\lambda^I}{\lambda_j}\right)^m,$$

which converges since $(a^{(j)}_{I,n,n+1})$ is a bounded sequence. With this choice,

$$\alpha^{(j)}_{I,n}\left(\frac{\lambda^I}{\lambda_j}\right)^n = \sum_{m=n+1}^{\infty} \frac{a^{(j)}_{I,m-1,m}}{\lambda^I}\left(\frac{\lambda^I}{\lambda_j}\right)^m,$$

so that

$$\alpha^{(j)}_{I,n} = \sum_{m=n+1}^{\infty} \frac{a^{(j)}_{I,m-1,m}}{\lambda^I}\left(\frac{\lambda^I}{\lambda_j}\right)^{m-n},$$

hence $(\alpha^{(j)}_{I,n})$ is also bounded. Fix an order on the set $\{(I,j)\colon |I| = i,\ 1 \leqslant j \leqslant N\}$ and define the mapping k_n as the ordered composition of all $k_{I,j,n}$ with $|I| = i$, $1 \leqslant j \leqslant N$. It is easy to check that (k_n) is a family of polynomial mappings in $\mathrm{Tang}_1(\mathbb{C}^N, 0)$ with uniformly bounded degree and uniformly bounded coefficients satisfying (6.1).

Lemma 2.5 yields a ball $r\mathbb{B} \subset D$ such that every k_n is univalent on $r\mathbb{B}$, and a ball $s\mathbb{B}$ such that $s\mathbb{B} \subset k_n(r\mathbb{B})$ for all $n \geqslant 0$. On $s\mathbb{B}$ we can define a family of holomorphic mappings

$$\varphi^{i+1}_{n,n+1} = k_{n+1} \circ \varphi^i_{n,n+1} \circ k_n^{-1}.$$

By Lemma 2.3 there exists a ball $\mathbb{B}_{i+1}$ invariant for each $\varphi^{i+1}_{n,n+1}$. Hence $(\varphi^{i+1}_{n,n+1}; \mathbb{B}_{i+1})$ is a discrete evolution family. Since (k_n) is an equicontinuous family, there exists a ball $u\mathbb{B}$ such that $k_n(u\mathbb{B}) \subset \mathbb{B}_{i+1}$ for all $n \geqslant 0$, so that (k_n) conjugates $(\varphi^i_{n,n+1}; \mathbb{B}_i)$ to $(\varphi^{i+1}_{n,n+1}; \mathbb{B}_{i+1})$:

$$k_{n+1} \circ \varphi^i_{n,n+1} = \varphi^{i+1}_{n,n+1} \circ k_n.$$

Since (6.1) holds by construction, we have

$$\varphi_{n,n+1}^{i+1} \circ k_n - T_{n,n+1}^{i+1} \circ k_n = O\big(|z|^{i+1}\big),$$

that is,

$$\varphi_{n,n+1}^{i+1} - T_{n,n+1}^{i+1} = O\big(|z|^{i+1}\big).$$

The family (k_n^{i+1}) conjugating $(\varphi_{n,m})$ to $(\varphi_{n,m}^{i+1})$ is obtained by composing for each $n \geqslant 0$ the conjugation mappings k_n^i given by inductive hypothesis with the conjugation mappings k_n. Indeed since the family (k_n^i) is equicontinuous by inductive hypothesis, there exists a ball $\mathbb{B}'_{i+1}$ such that $k_n^i(\mathbb{B}'_{i+1}) \subset u\mathbb{B}$ for all $n \geqslant 0$. Let (k_n^{i+1}) be the family of mappings defined on $\mathbb{B}'_{i+1}$ by $(k_n \circ k_n^i)$, then

$$\begin{aligned} k_{n+1}^{i+1} \circ \varphi_{n,n+1} &= \big(k_{n+1} \circ k_{n+1}^i\big) \circ \varphi_{n,n+1} \\ &= k_{n+1} \circ \varphi_{n,n+1}^i \circ k_n^i \\ &= \varphi_{n,n+1}^{i+1} \circ \big(k_n \circ k_n^i\big) \\ &= \varphi_{n,n+1}^{i+1} \circ k_n^{i+1}. \qquad \square \end{aligned}$$

Remark 6.3. Let q be the smallest integer such that $|\lambda_1^q| < |\lambda_N|$. Then no term of $P_{n,n+1}^q$ can be resonant. Hence $T_{n,n+1}^i = T_{n,n+1}^q$ for any $i \geqslant q$.

Proposition 6.4. *A discrete dilation evolution family $(\varphi_{n,m}; t\mathbb{B})$ admits an associated normalized Loewner chain (f_n) such that $\bigcup_n f_n(t\mathbb{B}) = \mathbb{C}^N$. If no real resonances occur (f_n) is a normal chain.*

Proof. Denote $(T_{n,m}) = (T_{n,m}^q)$, where q is the smallest integer such that $|\lambda_1^q| < |\lambda_N|$. Let β be the constant given by Lemma 4.2 for $(T_{n,m})$, and let l be an integer such that $|\lambda_1|^l < \frac{1}{\beta}$. Let $(\varphi_{n,m}^l; \mathbb{B}_l)$ be the evolution family given by Proposition 6.2. We can apply Proposition 5.5 obtaining a uniformly bounded family (h_n) given by

$$h_n = \lim_{m\to\infty} T_{m,n} \circ \varphi_{n,m}^l,$$

defined on a ball $r\mathbb{B} \subset \mathbb{B}_l$, which conjugates $(\varphi_{n,m}^l; \mathbb{B}_l)$ to $(T_{n,m})$.

Thus a Loewner chain associated to $(\varphi_{n,m}^l; \mathbb{B}_l)$ is given by the mappings

$$(T_{n,0} \circ h_n)^{\mathsf{e}} = \lim_{m\to\infty} T_{m,0} \circ \varphi_{n,m}^l.$$

Since (k_n^l) conjugates $(\varphi_{n,m}; t\mathbb{B})$ to $(\varphi_{n,n+1}^l; \mathbb{B}_l)$, a Loewner chain associated to $(\varphi_{n,m}; t\mathbb{B})$ is given by

$$f_n = \big((T_{n,0} \circ h_n)^{\mathsf{e}} \circ k_n^l\big)^{\mathsf{e}} = \lim_{m\to\infty} T_{m,0} \circ k_m^l \circ \varphi_{n,m}.$$

Now we prove $\bigcup_n f_n(t\mathbb{B}) = \mathbb{C}^N$. Since the family (k_n^l) is equicontinuous, there exists a ball $u\mathbb{B} \subset t\mathbb{B}$ such that $k_n^l(u\mathbb{B}) \subset r\mathbb{B}$ for all $n \geqslant 0$. On $u\mathbb{B}$,

$$f_n = T_{n,0} \circ h_n \circ k_n^l.$$

The sequence $h_n \circ k_n^l$ is uniformly bounded, hence Lemma 2.5 yields the existence of a ball V contained in each $h_n \circ k_n^l(u\mathbb{B})$. Thus

$$\bigcup_n f_n(t\mathbb{B}) \supseteq \bigcup_n T_{n,0}(V) = \mathbb{C}^N.$$

If no real resonances occur then $T_{n,m}(z) = A^{m-n}z$, hence

$$f_n = \lim_{m\to\infty} A^{-m} k_m^l \circ \varphi_{n,m}.$$

As above we have that on $u\mathbb{B}$, the sequence

$$A^n f_n = \lim_{m\to\infty} \left(A^{n-m}\varphi_{n,m}^l\right) \circ k_n^l$$

is uniformly bounded. Let $s\mathbb{B}$ be a ball contained in $t\mathbb{B}$. Lemma 2.4 yields an integer j_n such that $\varphi_{n,j_n}(s\mathbb{B}) \subset u\mathbb{B}$ and $j_n - n$ does not depend on n. From

$$A^n f_n = A^{n-j_n} A^{j_n} f_{j_n} \circ \varphi_{n,j_n}$$

we see that $A^n f_n$ is uniformly bounded on $s\mathbb{B}$, hence it is a normal family. □

7. Essential uniqueness

Proposition 7.1. *Let $(\varphi_{n,m}; t\mathbb{B})$ be a discrete dilation evolution family, and let (f_n) be the Loewner chain given by Proposition 6.4. A family of holomorphic mappings (g_n) is a subordination chain associated to $(\varphi_{n,m})$ if and only if there exists an entire mapping Ψ on $\mathbb{C}^N$ such that*

$$g_n = \Psi \circ f_n.$$

Proof. Set $\Psi_n = g_n \circ f_n^{-1}$. If $m > n$,

$$\Psi_m|_{f_n(t\mathbb{B})} = \Psi_n, \tag{7.1}$$

as it is clear from the following commuting diagram

$$\begin{tikzcd}
\mathbb{C}^N \arrow[r, "\mathrm{id}"] & \mathbb{C}^N \\
t\mathbb{B} \arrow[u, "f_n"] \arrow[r, "\varphi_{n,m}"] \arrow[d, "g_n"'] & t\mathbb{B} \arrow[u, "f_m"'] \arrow[d, "g_m"] \\
\mathbb{C}^N \arrow[r, "\mathrm{id}"] & \mathbb{C}^N.
\end{tikzcd}$$

Therefore by (7.1) we can define on $\mathbb{C}^N = \bigcup_n f_n(t\mathbb{B})$ a mapping Ψ setting

$$\Psi|_{f_n(t\mathbb{B})} = \Psi_n.$$

This proves the first statement, and the converse is trivial. $\square$

8. Continuous case

Let Λ be a complex $(N \times N)$-matrix

$$\Lambda = \mathrm{Diag}(\alpha_1, \ldots, \alpha_N), \quad \text{where } \mathsf{Re}\,\alpha_N \leqslant \cdots \leqslant \mathsf{Re}\,\alpha_1 < 0. \tag{8.1}$$

Definition 8.1. We define a *dilation evolution family* as a family $(\varphi_{s,t})_{0 \leqslant s \leqslant t}$ of holomorphic self-mappings of the unit ball $\mathbb{B} \subset \mathbb{C}^N$ such that for any $0 \leqslant s \leqslant u \leqslant t$,

$$\varphi_{s,s} = \mathsf{id}_{\mathbb{B}}, \qquad \varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}, \qquad \varphi_{s,t}(z) = e^{\Lambda(t-s)}z + O\big(|z|^2\big).$$

Definition 8.2. A family $(f_s)_{s \geqslant 0}$ of holomorphic mappings $f_s : \mathbb{B} \to \mathbb{C}^N$ is called a *subordination chain* if f_s is subordinated to f_t when $s \leqslant t$. If a subordination chain admits as transition mappings a dilation evolution family $(\varphi_{s,t})$ we say that (f_s) is *associated* to $(\varphi_{s,t})$. We define a *dilation Loewner chain* as a subordination chain such that each f_s is univalent and

$$f_s(z) = e^{-\Lambda s}z + O\big(|z|^2\big).$$

A dilation Loewner chain is *normal* if $(e^{\Lambda s} f_s)$ is a normal family.

Definition 8.3. We define a *dilation Herglotz vector field* $H(z,t)$ as a function $H : \mathbb{B} \times [0, +\infty) \to \mathbb{C}^N$ such that for all $z \in \mathbb{B}$ the mapping $H(z, \cdot)$ is measurable, and such that $H(\cdot, t)$ is a holomorphic mapping for a.e. $t \geqslant 0$ satisfying

$$H(z,t) = \Lambda z + O\big(|z|^2\big), \qquad \mathsf{Re}\big\langle H(z,t), z\big\rangle \leqslant 0 \quad \forall z \in \mathbb{B}.$$

Lemma 8.4. *Let $k_{\mathbb{B}}$ be the Kobayashi metric of $\mathbb{B}$. Given a dilation evolution family $(\varphi_{s,t})$, a dilation Loewner chain (f_s) and a dilation Herglotz vector field $H(z,t)$ the following hold*: *to each $T > 0$ and to any compact set $K \subset \mathbb{B}$ there correspond positive constants $c_{T,K}$, $C_{T,K}$ and $k_{T,K}$ such that for all $z \in K$ and $0 \leqslant s \leqslant t' \leqslant t \leqslant T$,*

(1) $k_{\mathbb{B}}(\varphi_{s,t}(z), \varphi_{s,t'}(z)) \leqslant c_{T,K}(t - t')$,
(2) $|f_s(z) - f_t(z)| \leqslant k_{K,T}(t - s)$,
(3) $|H(z,t)| \leqslant C_{K,T}$.

Therefore $(\varphi_{s,t})$ is an L^∞-evolution family, (f_s) is an L^∞-Loewner chain, and $H(z,t)$ is an L^∞-Herglotz vector field, in the sense of [1,3].

Proof. Let $H(z,t)$ be a dilation Herglotz vector field. By Lemma 1.2 [10], we have on $r\mathbb{B}$

$$\left|H(z,t)\right| \leqslant \frac{4r}{(1-r)^2}\|\Lambda\|.$$

Hence $H(z,t)$ is an L^∞-Herglotz vector field. For $(\varphi_{s,t})$ and (f_s), see the proof of [10, Theorem 2.8]. □

Recall if $(\varphi_{s,t})$ is an L^∞-evolution family, each mapping $\varphi_{s,t}$ is univalent [3, Proposition 5.1]. If we restrict the time to integer values in a dilation evolution family $(\varphi_{s,t})$ we obtain the *discretized* dilation evolution family $(\varphi_{n,m})$. We have $Az = d_0\varphi_{n,n+1}(z) = (e^{\alpha_1}z_1, \dots, e^{\alpha_N}z_N) = e^{\Lambda}z$. In the continuous framework a *real resonance* is an identity

$$\mathsf{Re}\left(\sum_{j=1}^{N} k_j\alpha_j\right) = \mathsf{Re}\,\alpha_l,$$

where $k_j \geqslant 0$ and $\sum_j k_j \geqslant 2$. It is easy to see that a continuous real resonance corresponds to a real resonance for the discretized evolution family.

Lemma 8.5. *Let $(\varphi_{s,t})$ be a dilation evolution family, and let $(\varphi_{n,m})$ be its discretized evolution family. Assume there exists a discrete Loewner chain (f_n) associated to $(\varphi_{n,m})$. Then we can extend it in a unique way to a dilation Loewner chain associated to $(\varphi_{s,t})$. If (f_n) is a normal Loewner chain, then also (f_s) is normal.*

Proof. Define for $s \geqslant 0$,

$$f_s = f_j \circ \varphi_{s,j},$$

where j is an integer such that $s \leqslant j$. The mapping f_s is well defined. Indeed, let $j < k$, then

$$f_j \circ \varphi_{s,j} = f_k \circ \varphi_{j,k} \circ \varphi_{s,j} = f_k \circ \varphi_{s,k}.$$

The family (f_s) is a subordination chain: indeed if $0 \leqslant s \leqslant t \leqslant j$, then

$$f_s = f_j \circ \varphi_{s,j} = f_j \circ \varphi_{t,j} \circ \varphi_{s,t} = f_t \circ \varphi_{s,t}.$$

Moreover each f_s is univalent and $d_0 f_s(z) = e^{-\Lambda s}z$, thus Lemma 8.4 yields that (f_s) is a dilation L^∞-Loewner chain. Assume $(e^{\Lambda n}f_n)$ is a normal family. If $0 < r < 1$ this family is uniformly bounded on $r\mathbb{B}$. For each $s \geqslant 0$ define m_s as the smallest integer greater than s. We have

$$e^{\Lambda s}f_s = e^{\Lambda s}f_{m_s} \circ \varphi_{s,m_s} = e^{\Lambda(s-m_s)}e^{\Lambda m_s}f_{m_s} \circ \varphi_{s,m_s},$$

which is uniformly bounded on $r\mathbb{B}$ since $m_s - s$ is smaller than 1. Hence $(e^{\Lambda s}f_s)$ is a normal family. □

The following result generalizes Theorem 2.3 in [10] and Theorem 3.1 in [7], where the hypothesis $2\operatorname{Re}\alpha_1 < \operatorname{Re}\alpha_N$ implies that no real resonances can occur (however in such papers the authors consider non-necessarily diagonal Λ).

Theorem 8.6. *Let $(\varphi_{s,t})$ be a dilation evolution family. Then there exists a dilation Loewner chain (f_s) associated to $(\varphi_{s,t})$, such that $\bigcup_s f_s(\mathbb{B}) = \mathbb{C}^N$. If no real resonances occur then (f_s) is a normal chain. A family of holomorphic mappings (g_s) is a subordination chain associated to $(\varphi_{s,t})$ if and only if there exists an entire mapping Ψ on $\mathbb{C}^N$ such that*

$$g_s = \Psi \circ f_s.$$

Proof. The result follows by applying Proposition 6.4 to the discretized evolution family associated to $(\varphi_{s,t})$, then Lemma 8.5 and Proposition 7.1. $\square$

Remark 8.7. For (f_s) we have the expression (with notations as in Proposition 6.4)

$$f_s(z) = \lim_{m\to\infty} T_{m,0} \circ k_m^l \circ \varphi_{s,m}(z).$$

If we assume $2\operatorname{Re}\alpha_1 < \operatorname{Re}\alpha_N$ then no real resonances can occur. Thus in this case $T_{m,0}(z) = A^{-m}z = e^{-\Lambda m}z$, so that the constant β given by Corollary 4.4 can be taken equal to $\|A^{-1}\| = 1/\lambda_N$. Hence $|\lambda_1|^l < 1/\beta$ holds for $l = 2$. Therefore we can use Proposition 5.5 directly, obtaining

$$f_s(z) = \lim_{m\to\infty} e^{-\Lambda m}\varphi_{s,m}(z),$$

in agreement with Theorem 2.3 of [10].

Recall [10, Theorem 2.1] that if H is a dilation Herglotz vector field, the *Loewner ODE*

$$\begin{cases} \dot{z}(t) = H(z,t), \\ z(s) = z \end{cases} \tag{8.2}$$

has a unique solution $t \mapsto \varphi_{s,t}(z)$, and $(\varphi_{s,t})$ is a dilation L^∞-evolution family.

Definition 8.8. The partial differential equation

$$\frac{\partial f_t(z)}{\partial t} = -d_z f_t H(z,t), \quad t \geqslant 0,\ z \in \mathbb{B},$$

where $H(z,t)$ is a dilation Herglotz vector field, is called the *Loewner PDE*.

With these notations, Theorem 8.6 can be rephrased as

Theorem 8.9. *Let H be a dilation Herglotz vector field, and let $t \mapsto \varphi_{s,t}$ be the solution of the associated Loewner ODE. Then if (f_s) is the Loewner chain associated to the dilation L^∞-evolution family $(\varphi_{s,t})$ given by Theorem* 8.6, *the mapping $t \mapsto f_t$ is a solution for the*

Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = -d_z f_t H(z,t).$$

Moreover, a family (g_s) of holomorphic mappings on the ball satisfies

(1) *the mapping $t \mapsto g_t$ is locally absolutely continuous in t, uniformly on compacta with respect to $z \in \mathbb{B}$,*
(2) *the mapping $t \mapsto g_t$ solves the Loewner PDE,*

if and only if there exists an entire mapping Ψ on $\mathbb{C}^N$ such that

$$g_s = \Psi \circ f_s.$$

Proof. It suffices to recall that such a $t \mapsto g_t$ satisfies the Loewner PDE if and only if (g_s) is a subordination chain associated to $(\varphi_{s,t})$. See the proof of [7, Theorem 3.1]. $\square$

Remark 8.10. A dilation evolution family $(\varphi_{s,t})$ is called *periodic* if $\varphi_{s,t} = \varphi_{s+1,t+1}$, for all $0 \leqslant s \leqslant t$. For periodic dilation evolution families the *pure real resonances*, that is real resonances which are not complex resonances, are not obstructions to the existence of a normal Loewner chain. Namely if $(\varphi_{s,t})$ is a periodic dilation evolution family and no complex resonances occur, then there exists a normal Loewner chain (f_s) associated to $(\varphi_{s,t})$. Indeed it is easy to see that by the Poincaré Theorem [17, pp. 80–86] the discretized evolution family $(\varphi_{n,m}) = (\varphi_{0,1}^{\circ(m-n)})$ admits a discrete normal Loewner chain (f_n). Lemma 8.5 yields then a normal Loewner chain (f_s).

9. Counterexamples

1. Let $\Lambda = \mathrm{Diag}(\alpha_1, \alpha_2)$. If $(\varphi_{s,t})$ is a dilation evolution family such that $\varphi_{s,t} = e^{\Lambda(t-s)}z + O(|z|^2)$ and $2\,\mathsf{Re}\,\alpha_1 < \mathsf{Re}\,\alpha_N$, then by Lemma 2.12 in [7] there exists a unique normal Loewner chain associated to $(\varphi_{s,t})$. This is no longer true when $2\,\mathsf{Re}\,\alpha_1 \geqslant \mathsf{Re}\,\alpha_N$. Indeed, consider on $\mathbb{B} \subset \mathbb{C}^2$ the linear dilation evolution family defined by

$$\varphi_{s,t}(z) = e^{\Lambda(t-s)}z = \left(e^{\alpha_1(t-s)}z_1, e^{\alpha_2(t-s)}z_2\right).$$

The family $(e^{-\Lambda s}z)$ is trivially a normal Loewner chain associated to $(e^{\Lambda(t-s)}z)$. The univalent family

$$k_s(z) = \left(z_1, z_2 + e^{(\alpha_2 - 2\alpha_1)s} z_1^2\right),$$

satisfies $k_t \circ e^{\Lambda(t-s)}z = e^{\Lambda(t-s)}k_s(z)$. Since $\mathsf{Re}\,\alpha_2 \leqslant 2\,\mathsf{Re}\,\alpha_1$, it is a uniformly bounded family, thus $(e^{-\Lambda s}k_s)$ is another normal Loewner chain associated to $(e^{\Lambda(t-s)}z)$.

2. Let $\Lambda = \mathrm{Diag}(\alpha_1, \alpha_2)$, $\alpha_2 = 2\alpha_1$. There exists a dilation evolution family $(\varphi_{s,t})$ such that $\varphi_{s,t} = e^{\Lambda(t-s)}z + O(|z|^2)$, which does not admit any associated normal Loewner chain. Indeed, for $c \in \mathbb{C}^*$ small enough, the family (ψ_t) defined by

$$\psi_t(z) = \left(e^{\alpha_1 t} z_1, e^{\alpha_2 t}\left(z_2 + ctz_1^2\right)\right)$$

is a semigroup on $\mathbb{B} \subset \mathbb{C}^2$. Thus

$$\varphi_{s,t}(z) = \psi_{t-s}(z)$$

defines a dilation evolution family. Assume by contradiction there exists a normal Loewner chain (f_s) associated to $(\varphi_{s,t})$. The family $(h_s) = (e^{\Lambda s} f_s)$ satisfies $h_t \circ \varphi_{s,t} = e^{\Lambda(t-s)} h_s$, so in particular

$$h_t \circ \varphi_{0,t} = e^{\Lambda t} h_0. \tag{9.1}$$

Let a_s be the coefficient of the term z_1^2 in the second component of h_s. Then imposing equality of terms in z_1^2 in Eq. (9.1) we find $e^{\alpha_2 t} ct + a_t e^{2\alpha_1 t} = a_0 e^{\alpha_2 t}$, hence

$$a_t = e^{(\alpha_2 - 2\alpha_1)t}(a_0 - ct),$$

which gives $a_t = a_0 - ct$, so that (h_s) cannot be a normal family.

3. Let $A = \mathrm{Diag}(\lambda_1, \lambda_2)$, $|\lambda_1|^2 = |\lambda_2|$, $\lambda_1^2 \neq \lambda_2$. There exists a discrete dilation evolution family $(\varphi_{n,m})$ such that $\varphi_{n,n+1}(z) = Az + O(|z|^2)$, which does not admit any associated discrete normal Loewner chain. Indeed, if $r > 0$ is sufficiently small, given any sequence $(a_{n,n+1})$ in $r\mathbb{B}$ there exists a discrete dilation evolution family defined by

$$\varphi_{n,n+1}(z) = \left(\lambda_1 z_1, \lambda_2 z_2 + a_{n,n+1} z_1^2\right).$$

Assume by contradiction there exists a normal Loewner chain (f_n) associated to $(\varphi_{n,m})$. The family $(h_n) = (A^n f_n)$ satisfies

$$h_{n+1} \circ \varphi_{n,n+1} = Ah_n. \tag{9.2}$$

Let α_n be the coefficient of the term z_1^2 in the second component of h_s, and set $\zeta = \lambda_1^2/\lambda_2$. Then imposing equality of terms in z_1^2 in Eq. (9.2) we obtain as in (6.4) the recursive formula

$$\alpha_n \zeta^n \lambda_1^2 = \alpha_0 \lambda_1^2 - a_{0,1}\zeta - a_{1,2}\zeta^2 - \cdots - a_{n-1,n}\zeta^n.$$

For $1 \leqslant j \leqslant 8$ define $C_j = \{\zeta \in S^1\colon\ 2\pi(j-1)/8 \leqslant \arg z \leqslant 2\pi j/8\}$. There exists a C_j which contains the images of a subsequence (ζ^{k_n}). Set

$$a_{m-1,m} = \begin{cases} r/2, & \text{if there exists } n \text{ such that } m = k_n, \\ 0, & \text{otherwise,} \end{cases} \tag{9.3}$$

then the sequence $(\sum_{j=0}^n a_{j-1,j}\zeta^j)$ is not bounded, hence the sequence (α_n) is also not bounded. Thus for $(\varphi_{n,m})$ no normal family (h_n) can solve (9.2).

4. There exists a discrete evolution family $(\varphi_{n,m})$ on $\mathbb{B}^3 \subseteq \mathbb{C}^3$ which does not admit any associated discrete Loewner chain. Indeed, by [9] there exists a complex manifold M which is an increasing union of open sets M_n each of which biholomorphic to the ball $\mathbb{B}^3$ by means of a biholomorphism $f_n : \mathbb{B}^3 \to M_n$, with the property that M is not Stein. By [2] this implies that M cannot be embedded into $\mathbb{C}^3$ as an open set.

Define $\varphi_{n,n+1} = f_{n+1}^{-1} \circ f_n$ for all $n \geqslant 0$. Then $(\varphi_{n,m})$ is a discrete evolution family which does not admit any associated discrete Loewner chain (g_n). Indeed, if such a family existed, then M would be biholomorphic to the open subset of $\mathbb{C}^3$ given by $\bigcup_n g_n(\mathbb{B}^3)$.

This suggests the following (open) question: does such a discrete evolution family embed into some L^∞-evolution family on $\mathbb{B}^3$?

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References

[1] L. Arosio, F. Bracci, H. Hamada, G. Kohr, An abstract approach to Loewner chains, arXiv:1002.4262v1 [math.CV], 2010.
[2] H. Behnke, K. Stein, Konvergente Folgen von Regularitätsbereichen und die Meromorphiekonvexität, Math. Ann. 116 (1938) 204–216.
[3] F. Bracci, M.D. Contreras, S. Díaz-Madrigal, Evolution families and the Loewner equation II: complex hyperbolic manifolds, Math. Ann. 344 (2009) 947–962.
[4] F. Bracci, M.D. Contreras, S. Díaz-Madrigal, Evolution families and the Loewner equation I: the unit disk, arXiv:0807.1594v1 [math.CV], 2008.
[5] M.D. Contreras, S. Díaz-Madrigal, P. Gumenyuk, Loewner chains in the unit disk, Rev. Mat. Iberoamericana 26 (2010) 975–1012.
[6] L. de Branges, A proof of the Bieberbach conjecture, Acta Math. 154 (1985) 137–152.
[7] P. Duren, I. Graham, H. Hamada, G. Kohr, Solutions for the generalized Loewner differential equation in several complex variables, Math. Ann. 347 (2010) 411–435.
[8] C.H. Fitzgerald, C. Pommerenke, The de Branges theorem on univalent functions, Trans. Amer. Math. Soc. 290 (2) (1985) 683–690.
[9] J.E. Fornaess, An increasing sequence of Stein manifolds whose limit is not Stein, Math. Ann. 223 (1976) 275–277.
[10] I. Graham, H. Hamada, G. Kohr, M. Kohr, Asymptotically spirallike mappings in several complex variables, J. Anal. Math. 105 (2008) 267–302.
[11] I. Graham, G. Kohr, Geometric Function Theory in One and Higher Dimensions, Marcel Dekker Inc., New York, 2003.
[12] P.P. Kufarev, On one-parameter families of analytic functions, Mat. Sb. 13 (1943) 87–118 (in Russian, English summary).
[13] P.P. Kufarev, V.V. Sobolev, L.V. Sporysheva, A certain method of investigation of extremal problems for functions that are univalent in the half-plane, Tr. Tomsk. Gos. Univ. Ser. Meh.-Mat. 200 (1968) 142–164.
[14] C. Loewner, Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, Math. Ann. 89 (1923) 103–121.
[15] J.A. Pfaltzgraff, Subordination chains and univalence of holomorphic mappings in $\mathbb{C}^n$, Math. Ann. 210 (1974) 55–68.
[16] C. Pommerenke, Über die Subordination analytischer Funktionen, J. Reine Angew. Math. 218 (1965) 159–173.
[17] Jean-Pierre Rosay, Walter Rudin, Holomorphic maps from C^n to C^n, Trans. Amer. Math. Soc. 310 (1) (1988) 47–86.
[18] O. Schramm, Scaling limits of loop-erased random walks and uniform spanning trees, Israel J. Math. 118 (2000) 221–288.