

# Optimality of generalized Bernstein operators

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## Abstract

We show that a certain optimality property of the classical Bernstein operator also holds, when suitably reinterpreted, for generalized Bernstein operators on extended Chebyshev systems.

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## 1. Introduction

It is well known that the classical Bernstein operator  $B_n : C[0, 1] \rightarrow \text{Span}\{1, x, \dots, x^n\}$ , defined by

$$B_n f(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \quad (1)$$

has good shape preserving properties, a fact that explains its usefulness in computer assisted geometric design. But the uniform convergence of the polynomials  $B_n f$  to  $f$  can be slow. Thus, it is natural to enquire whether one can have operators from the continuous functions to the space of polynomials, with good shape preserving and fast approximation properties simultaneously. However, in a certain sense it is impossible to do better than by using Bernstein operators. Within some natural classes of polynomial valued operators, Bernstein operators approximate convex functions in an optimal way. In [5], Berens and DeVore consider operators  $L_n : C[0, 1] \rightarrow \text{Span}\{1, x, \dots, x^n\}$ , fixing 1 and  $x$ , and such that  $(L_n f)^{(j)} \geq 0$  if  $f^{(j)} \geq 0$

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for  $j = 0, 1, 2, \dots, n$ . They show that for every  $n$ ,  $L_n x^2 \geq B_n x^2$ , and if for some  $t \in (0, 1)$ ,  $L_n t^2 = B_n t^2$ , then  $L_n \equiv B_n$ . Since  $B_n x^2 \geq x^2$ ,  $L_n$  performs strictly worse than  $B_n$  (unless  $L_n \equiv B_n$ ) at least for  $x^2$ . The result of Berens and DeVore [5, p. 214] is extended to a wider class of operators  $L_n$  in [6, Theorem 1]. While the result is stated there for  $x^2$ , the argument works for an arbitrary convex function  $\phi$  (using Jensen’s inequality instead of Hölder’s inequality). The more general result, for  $\phi$  convex, can be found in [7, Theorem 2]. A further extension appears in [12, Theorem 2].

Recently, there has been an increasing interest concerning *generalized Bernstein bases* in the framework of extended Chebyshev spaces (in particular, spaces that contain functions such as  $\sin x$  and  $\cos x$ ), cf. for instance, [8–11,16–18]. The existence of generalized Bernstein bases raises the question of the possible existence of associated generalized Bernstein operators. This topic is studied in [1], for exponential polynomials, and in [2,3,15,19], for extended Chebyshev spaces. Here we show that the operators considered in [1–3,15,19], share the preceding optimality property of the classical Bernstein operator, under a suitable generalized notion of convexity.

Additionally, we give a self-contained and streamlined presentation of the generalized Bernstein operators introduced in [20] by S. Morigi and M. Neamtu. These operators are used here to exhibit a sequence of generalized Bernstein operators converging strongly to the identity, preserving  $(1, e^x)$ -convexity, and failing to preserve standard convexity.

**2. Definitions and optimality results**

For each pair  $n, k$  of nonnegative integers, let  $\mu_{n,k}$  be a positive measure. Denote by  $p_{n,k}(x) := \binom{n}{k} x^k (1 - x)^{n-k}$  the elements of the Bernstein basis for the space of polynomials. Let  $\mathcal{L}_n$  be the class of positive linear operators  $L_n : C [0, 1] \rightarrow \text{Span}\{1, x, \dots, x^n\}$  fixing 1 and  $x$  (so  $L_n$  preserves affine functions) and defined by

$$L_n f(x) := \sum_{k=0}^n \lambda_{n,k}(f) p_{n,k}(x), \quad \text{where } \lambda_{n,k}(f) := \int_0^1 f d\mu_{n,k}. \tag{2}$$

Denote by  $\delta_x$  the Dirac delta probability, which assigns mass 1 to the singleton  $\{x\}$ , and observe that  $B_n \in \mathcal{L}_n$  is obtained by taking  $\mu_{n,k} = \delta_{k/n}$  in (2), so  $\lambda_{n,k}(f) = f(k/n)$ . Recall that if  $\phi$  is convex on  $[0, 1]$ , then  $B_n \phi \geq \phi$  on that interval. From [6, Theorem 1] and the generalization in [7, Theorem 2], it is known that  $L_n \phi \geq B_n \phi$  for every convex function  $\phi$  and every  $L_n \in \mathcal{L}_n$ , and furthermore,  $L_n \equiv B_n$  if  $L_n \psi(t) = B_n \psi(t)$  for some strictly convex function  $\psi$  and some  $t \in (0, 1)$ . In this sense the operator  $B_n$  approximates convex functions optimally within the class  $\mathcal{L}_n$ . We emphasize that this is not an asymptotic optimality result, where  $n$  is required to approach  $\infty$ . Instead, best approximation holds for each fixed  $n$ .

Observe that by uniqueness of the coefficients of the basis functions, the condition  $L_n 1 \equiv 1$  entails that each  $\mu_{n,k}$  is a probability measure, while  $L_n x \equiv x$  entails that the expectation of  $x$  with respect to  $\mu_{n,k}$  is  $\int_0^1 x d\mu_{n,k}(x) = k/n$ . If the assumption  $L_n x \equiv x$  is dropped, then the resulting operator may fail to preserve convexity, and may approximate some convex functions better than  $B_n$  (while in other cases the approximation will be worse, notably for  $x$  itself).

**Example 1.** Let  $1 < j \leq n$ . In [2, Proposition 11] and in [19, Example 5.3], a generalized Bernstein operator  $B_{n,j}$  fixing 1 and  $x^j$ , is defined by

$$B_{n,j} f(x) = \sum_{k=0}^n f \left( \left[ \frac{k(k-1) \cdots (k-j+1)}{n(n-1) \cdots (n-j+1)} \right]^{1/j} \right) \binom{n}{k} x^k (1 - x)^{n-k}.$$

In particular, if  $j = 1$  this is the classical Bernstein operator. However, for  $j > 1$  and all  $k = 1, 2, \dots, \mathcal{B}_{n,j}x^k < B_nx^k$  on  $(0, 1)$ , since the operator  $\mathcal{B}_{n,j}$  samples the strictly increasing functions  $x^k$  at nodes lower than those of  $B_n$  (except the first and the last nodes, which are 0 and 1 in both cases). It is shown in [2, Example 17] that if  $1 \leq k \leq j$ , then  $\mathcal{B}_{n,j}x^k \leq x^k$ , while if  $j \leq k$ , then  $\mathcal{B}_{n,j}x^k \geq x^k$ . Hence, on  $(0, 1)$ ,  $B_nx^k > \mathcal{B}_{n,j}x^k \geq x^k$  when  $j \leq k$ . Of course,  $\mathcal{B}_{n,j}$  does not preserve convexity, since

$$\mathcal{B}_{n,j}(1 - x) = \mathcal{B}_{n,j}(1) - \mathcal{B}_{n,j}(x) > 1 - B_n(x) = 1 - x$$

whenever  $x \in (0, 1)$ . But  $\mathcal{B}_{n,j}$  does preserve  $(1, x^j)$ -convexity, defined below.

The way to recover the results from [6,7] is thus to deal with a generalized notion of convexity. We give next the relevant definitions. An *extended Chebyshev space*  $U_n$  of dimension  $n + 1$  over the interval  $[a, b]$  is an  $n + 1$  dimensional subspace of  $C^n([a, b])$  such that each function  $f \in U_n$  not vanishing identically, has at most  $n$  zeros in  $[a, b]$ , counting multiplicities. It is well-known that extended Chebyshev spaces possess *non-negative Bernstein bases*, i.e. collections of non-negative functions  $p_{n,k}, k = 0, \dots, n$ , in  $U_n$ , such that each  $p_{n,k}$  has a zero of order  $k$  at  $a$  and a zero of order  $n - k$  at  $b$ , for  $k = 0, \dots, n$ . Now let us select two functions  $f_0, f_1 \in U_n$ , such that  $f_0 > 0$  and  $f_1/f_0$  is strictly increasing (these functions play the role of 1 and  $x$  in the classical case). It is sometimes possible to define a generalized Bernstein operator  $\mathcal{B}_n : C[a, b] \rightarrow U_n$  fixing  $f_0$  and  $f_1$ , by first suitably choosing nodes  $t_{n,0}, \dots, t_{n,n} \in [a, b]$  such that  $t_{n,0} = a$  and  $t_{n,n} = b$ , second, by selecting appropriate weights  $\alpha_{n,0}, \dots, \alpha_{n,n} > 0$ , and finally, by setting

$$\mathcal{B}_n(f) = \sum_{k=0}^n f(t_{n,k}) \alpha_{n,k} p_{n,k}. \tag{3}$$

Criteria for the existence of such  $\mathcal{B}_n$  can be found in [2,3,19], and in the specific case of exponential polynomials, in [1,15]. In this section, we shall *assume* that  $\mathcal{B}_n$  exists (fixing the given functions), and will show that it has an extremal property analogous to that of  $B_n$ . For simplicity in the notation, we relabel the basis functions so that they already incorporate the weights  $\alpha_{n,k}$ . Thus, (3) becomes

$$\mathcal{B}_n(f) = \sum_{k=0}^n f(t_{n,k}) p_{n,k}. \tag{4}$$

The functions  $f_0$  and  $f_1$  are always assumed to satisfy the conditions “ $f_0 > 0$ ” and “ $f_1/f_0$  is strictly increasing”.

**Remark 2.** Given an extended Chebyshev space  $U_n$ , it is always possible to *select* some pair  $f_0$  and  $f_1$  in  $U_n$  for which a generalized Bernstein operator can be defined (cf. [2, Corollary 8]). However, for some choices of  $f_0$  and  $f_1$ ,  $\mathcal{B}_n$  may fail to exist. For instance, it is well known that if  $0 < b < 2\pi$ , then  $U_3 := \text{Span}\{1, x, \cos x, \sin x\}$  on the interval  $[0, b]$  is an extended Chebyshev space. But there is no generalized Bernstein operator  $\mathcal{B}_n$  fixing 1 and  $x$  if  $b = 5$  (for instance) while such  $\mathcal{B}_n$  does exist if  $b = 4$ , cf. [2, Theorem 25].

**Remark 3.** The condition on the zeros of  $p_{n,k}$  entails that, like the classical Bernstein operator, the operators given by (3) interpolate functions at the endpoints of the interval  $[a, b]$ : Since  $f_0$  is strictly positive, from  $f_0(a) = \mathcal{B}_n(f_0)(a) = f_0(a) \alpha_{n,0} p_{n,0}(a)$  we conclude that  $\alpha_{n,0} p_{n,0}(a) = 1$ . Likewise,  $\alpha_{n,n} p_{n,n}(b) = 1$ , so we always have  $\mathcal{B}_n(f)(a) = f(a)$  and  $\mathcal{B}_n(f)(b) = f(b)$ .

Convexity of  $\phi$  can be defined by saying that if we interpolate between  $\phi(x)$  and  $\phi(y)$  using an affine function  $h$ , then on  $[x, y]$  the graph of  $\phi$  lies below the graph of  $h$ . If instead of affine functions we use  $(f_0, f_1)$ -affine functions, that is, functions in  $\text{Span}\{f_0, f_1\}$ , we obtain the corresponding notion of  $(f_0, f_1)$ -convexity. Thus, ordinary convexity corresponds to  $(1, x)$ -convexity. According to [14], p. 376, this generalized notion of convexity was introduced in 1926 by Hopf, and was later extensively developed by Popoviciu, specially in the context of Chebyshev spaces. Strict  $(f_0, f_1)$ -convexity is defined analogously to strict convexity. It is shown in [2, Theorem 22] (cf. also [19, Proposition 4.14]) that a generalized Bernstein operator  $\mathcal{B}_n$  fixing  $f_0$  and  $f_1$  preserves  $(f_0, f_1)$ -convexity.

We would like to stress the point that  $\mathcal{B}_n$  will in general *fail* to preserve standard convexity (recall Example 1). It will be shown in Section 3 that likewise, the standard Bernstein operator  $B_n$  does not in general preserve strengthened forms of convexity, such as  $(1, e^x)$ -convexity for increasing functions.

The following characterization of  $(f_0, f_1)$ -convexity, due to M. Bessenyei and Z. Páles (cf. [4, Theorem 5]), helps to understand its meaning.

**Theorem 4.** *Let  $I := (f_1/f_0)([a, b])$ . Then  $\phi \in C[a, b]$  is  $(f_0, f_1)$ -convex if and only if  $(\phi/f_0) \circ (f_1/f_0)^{-1} \in C(I)$  is convex in the standard sense.*

In particular, if  $f_0 = 1$ , the relationship between convexity and  $(1, f_1)$ -convexity is given by a simple change of variables (determined by  $f_1$ ). For instance, returning to Example 1, and to the functions considered there, we see that if  $1 \leq k \leq j$ , then  $x^k$  is  $(1, x^j)$ -concave, while if  $j \leq k \leq n$ , then  $x^k$  is  $(1, x^j)$ -convex.

Before we prove the announced optimality property of generalized Bernstein operators, we need the following lemma. It is entirely analogous to Theorem 4, save that strict convexity replaces convexity throughout. The proof is also essentially the same, and thus we omit it. Basically, all one needs to do is to use strict inequalities  $>$  instead of  $\geq$ , at the appropriate places.

**Lemma 5.** *Let  $I := (f_1/f_0)([a, b])$ . Then  $\psi \in C[a, b]$  is strictly  $(f_0, f_1)$ -convex if and only if  $(\psi/f_0) \circ (f_1/f_0)^{-1} \in C(I)$  is strictly convex.*

**Theorem 6.** *Let  $U_n \subset C^n[a, b]$  be an extended Chebyshev space containing the functions  $f_0$  and  $f_1$ , with  $f_0 > 0$  and  $f_1/f_0$  strictly increasing. Suppose there exists a generalized Bernstein operator  $\mathcal{B}_n : C[a, b] \rightarrow U_n$  fixing  $f_0$  and  $f_1$ . Let the operator  $L_n : C[a, b] \rightarrow U_n$ , subject to  $L_n f_0 = f_0$  and  $L_n f_1 = f_1$ , be defined by*

$$L_n(f) = \sum_{k=0}^n \lambda_{n,k}(f) p_{n,k}, \tag{5}$$

where  $p_{n,k}$  has the same meaning as in (4), and  $\lambda_{n,k}(f)$  is obtained from  $f$  and the positive measure  $\mu_{n,k}$  via

$$\lambda_{n,k}(f) := f_0(t_{n,k}) \int_a^b f d\mu_{n,k}. \tag{6}$$

Then, for every  $(f_0, f_1)$ -convex function  $\phi$ , we have  $\phi \leq \mathcal{B}_n \phi \leq L_n \phi$ . Moreover, if for some  $t \in (a, b)$  and some strictly  $(f_0, f_1)$ -convex function  $\psi$  we have  $\mathcal{B}_n \psi(t) = L_n \psi(t)$ , then  $\mathcal{B}_n \equiv L_n$ .

**Proof.** First we make explicit what the assumptions in the theorem entail about the measures  $\mu_{n,k}$ . Since

$$L_n f_0 = \sum_{k=0}^n \lambda_{n,k}(f_0) P_{n,k} = f_0 = \mathcal{B}_n f_0 = \sum_{k=0}^n f_0(t_{n,k}) P_{n,k},$$

by uniqueness of the coefficients of the basis functions, we must have  $f_0(t_{n,k}) = \lambda_{n,k}(f_0)$  for all pairs  $n, k$ . But then  $1 = \int_a^b f_0 d\mu_{n,k}$  by (6), so admissible measures  $\mu_{n,k}$  must be such that  $dP_{n,k} := f_0 d\mu_{n,k}$  defines a probability  $P_{n,k}$ . With this notation, the second condition  $L_n(f_1) = f_1$  leads to

$$\lambda_{n,k}(f_1) = f_0(t_{n,k}) \int_a^b \frac{f_1}{f_0} dP_{n,k},$$

so again by uniqueness of the coefficients of the basis functions, we have

$$f_0(t_{n,k}) \int_a^b \frac{f_1}{f_0} dP_{n,k} = \lambda_{n,k}(f_1) = f_1(t_{n,k}) = f_0(t_{n,k}) \frac{f_1(t_{n,k})}{f_0(t_{n,k})}.$$

Thus,

$$\frac{f_1(t_{n,k})}{f_0(t_{n,k})} = \int_a^b \frac{f_1}{f_0} dP_{n,k} \tag{7}$$

for all pairs  $n, k$ .

Next, suppose  $\phi$  is  $(f_0, f_1)$ -convex on  $[a, b]$ . Given  $h : X \rightarrow Y$  and a measure  $\nu$  on  $X$ , the pushforward to  $Y$  of  $\nu$ , using  $h$ , is denoted by  $h_*\nu$ ; recall that  $h_*\nu(A) := \nu(h^{-1}(A))$ . Applying the change of variables formula twice (cf., for instance, [13, Theorem C, p. 163]), Theorem 4 above, Jensen’s inequality, and (7), we obtain

$$\lambda_{n,k}(\phi) = f_0(t_{n,k}) \int_a^b \phi(x) d\mu_{n,k}(x) = f_0(t_{n,k}) \int_a^b \frac{\phi}{f_0}(x) dP_{n,k}(x) \tag{8}$$

$$= f_0(t_{n,k}) \int_{\frac{f_1}{f_0}([a,b])} \frac{\phi}{f_0} \circ \left(\frac{f_1}{f_0}\right)^{-1}(x) d\left(\frac{f_1}{f_0}\right)_* P_{n,k}(x) \tag{9}$$

$$\geq f_0(t_{n,k}) \frac{\phi}{f_0} \circ \left(\frac{f_1}{f_0}\right)^{-1} \left( \int_{\frac{f_1}{f_0}([a,b])} x d\left(\frac{f_1}{f_0}\right)_* P_{n,k}(x) \right) \tag{10}$$

$$= f_0(t_{n,k}) \frac{\phi}{f_0} \circ \left(\frac{f_1}{f_0}\right)^{-1} \left( \int_{[a,b]} \left(\frac{f_1}{f_0}\right)(x) dP_{n,k}(x) \right) \tag{11}$$

$$= f_0(t_{n,k}) \frac{\phi}{f_0} \circ \left(\frac{f_1}{f_0}\right)^{-1} \left(\frac{f_1}{f_0}(t_{n,k})\right) = \phi(t_{n,k}). \tag{12}$$

Since the basis functions are non-negative, we obtain  $\mathcal{B}_n \phi \leq L_n \phi$  just by adding up. And the fact that  $\phi \leq \mathcal{B}_n \phi$  for all  $(f_0, f_1)$ -convex functions  $\phi$  is proven in [2, Theorem 15].

Assume next that for some  $t \in (a, b)$  and some strictly  $(f_0, f_1)$ -convex function  $\psi$  we have  $\mathcal{B}_n \psi(t) = L_n \psi(t)$ . We must show that  $P_{n,k} = \delta_{t_{n,k}}$  for each pair  $n, k$ . Set  $\phi = \psi$  in (8)–(12).

Since

$$\sum_{k=0}^n \lambda_{n,k}(\psi) p_{n,k}(t) = L_n \psi(t) = \mathcal{B}_n \psi(t) = \sum_{k=0}^n \psi(t_{n,k}) p_{n,k}(t),$$

and for all  $n, k$ , we have  $\lambda_{n,k}(\psi) \geq \psi(t_{n,k})$  (by (8)–(12)) from  $p_{n,k} > 0$  on  $(a, b)$  we conclude that  $\lambda_{n,k}(\psi) = \psi(t_{n,k})$  for all  $n, k$ . Thus, with  $\phi = \psi$  we have equality in (10). By Lemma 5, the function  $\frac{\psi}{f_0} \circ \left(\frac{f_1}{f_0}\right)^{-1}$  is strictly convex. So by the equality case in Jensen’s inequality we conclude that the function  $x$  is constant a.e. with respect to  $\left(\frac{f_1}{f_0}\right)_* P_{n,k}$ . This entails that the latter measure is a Dirac delta, and since  $f_1/f_0$  is strictly increasing,  $P_{n,k}$  must also be a Dirac delta. It now follows from (7) that  $P_{n,k} = \delta_{t_{n,k}}$ .  $\square$

### 3. On the generalized Bernstein operator of Morigi and Neamtu

There are known examples of chains of extended Chebyshev spaces, for which their associated generalized Bernstein operators exist and converge strongly to the identity. An instance is given by the operators  $\mathcal{B}_{n,j}$  described in Example 1 (cf. [2, Proposition 11] or [19, Theorem 6.1] for the convergence assertion). As was observed in Example 1, the operators  $\mathcal{B}_{n,j}$  preserve  $(1, x^j)$ -convexity, and fail to preserve ordinary convexity when  $j > 1$ .

Nevertheless, these operators are somewhat degenerate in that the sequences of nodes are not strictly increasing, due to the fact that for  $j > 1$ ,  $f_1(x) = x^j$  has a zero of order at least two at zero, and in particular,  $f'_1(0) = 0$ . Thus,  $f'_1$  is not strictly positive on  $[0, 1]$ . We shall use the generalized Bernstein operators introduced by Morigi and Neamtu in [20] to present a sequence of operators that (i) converges to the identity, (ii) the operators preserve  $(1, e^x)$ -convexity, (iii) they do not preserve ordinary convexity, iv) they are defined via strictly increasing sequences of nodes (in fact, the nodes are equidistributed), and (v) they fix  $f_0 = 1$  and  $f_1(x)/f_0(x) = f_1(x) = e^x$ . Obviously we have  $f'_1 > 0$  everywhere.

Our presentation here of the Morigi and Neamtu operators is selfcontained and simplifies some of the arguments from [20]. At the beginning we will consider a setting more general than we really need (from the shape preservation perspective) allowing complex values. We start by defining these operators.

Given two different complex numbers  $\mu_0, \mu_1$ , set  $\omega_n := \frac{1}{n}(\mu_1 - \mu_0) \neq 0$ , and for  $j = 0, \dots, n$ , let

$$\lambda_j := \mu_0 + j\omega_n. \tag{13}$$

Thus,  $\lambda_0 = \mu_0$  and  $\lambda_n = \mu_1$ . Define

$$\varphi_n(x) := \frac{e^{\omega_n(x-b)} - e^{\omega_n(a-b)}}{1 - e^{\omega_n(a-b)}} \quad \text{and} \quad \psi_n(x) := 1 - \varphi_n(x) = \frac{1 - e^{\omega_n(x-b)}}{1 - e^{\omega_n(a-b)}}. \tag{14}$$

Then  $(\varphi_n)^k$  is a linear combination of the exponential functions  $e^{\omega_n j x}$ , where  $j = 0, \dots, k$ , and  $(\psi_n)^{n-k}$  is a linear combination of the functions  $e^{\omega_n j x}$ , where  $j = 0, \dots, n - k$ . Therefore,

$$p_{n,k}(x) := e^{-\lambda_0 \frac{k}{n}(b-a)} \cdot e^{\lambda_0(x-a)} \binom{n}{k} (\varphi_n(x))^k (\psi_n(x))^{n-k} \tag{15}$$

is a linear combination of the exponential functions  $e^{\lambda_j x}$ ,  $j = 0, \dots, n$ . It follows that  $p_{n,k}$  is contained in the vector space

$$U_n := \left\{ f \in C(\mathbb{R}, \mathbb{C}) : \left( \frac{d}{dx} - \lambda_0 \right) \cdots \left( \frac{d}{dx} - \lambda_n \right) f = 0 \right\}. \tag{16}$$

Since  $\varphi_n$  has a simple zero at  $x = a$  and  $\psi_n$  a simple zero at  $b$ ,  $p_{n,k}$  has a zero order  $k$  at  $a$  and a zero of order  $n - k$  at  $b$ .

It is well known that if  $\lambda_0, \dots, \lambda_n$  are real, then  $U_n^r := \{f \in U_n : f \text{ is real-valued}\}$  is an extended Chebyshev space over any bounded closed interval  $[a, b]$  (we mention that when dealing with complex values  $\lambda_0, \dots, \lambda_n$ , restrictions must be imposed on the length of  $[a, b]$  in order to ensure that  $U_n$  is an extended Chebyshev space, cf. for instance [1, Theorem 19]). Furthermore,  $\{p_{n,k} : k = 0, \dots, n\}$  is a non-negative Bernstein basis.

Next we specify nodes and weights. Let  $t_{n,k} := a + \frac{k}{n}(b - a)$  for  $k = 0, \dots, n$ , and let

$$\mathcal{B}_n(f)(x) := \sum_{k=0}^n f(t_{n,k}) p_{n,k}(x). \tag{17}$$

**Proposition 7.** *The operator  $\mathcal{B}_n$  fixes  $e^{\mu_0 x}$  and  $e^{\mu_1 x}$ .*

**Proof.** Recall from (13) that  $\mu_0 = \lambda_0$  and  $\mu_1 = \lambda_n$ . Using (17), (15) and (14), we obtain

$$\mathcal{B}_n(e^{\lambda_0 x}) = e^{\lambda_0 x} \sum_{k=0}^n \binom{n}{k} (\varphi_n(x))^k (1 - \varphi_n(x))^{n-k} = e^{\lambda_0 x} (\varphi_n(x) + 1 - \varphi_n(x))^n = e^{\lambda_0 x}.$$

Furthermore,

$$\begin{aligned} \mathcal{B}_n(e^{\lambda_n x}) &= \sum_{k=0}^n e^{\lambda_n a} e^{(\lambda_n - \lambda_0) \frac{k}{n}(b-a)} e^{\lambda_0(x-a)} \binom{n}{k} (\varphi_n(x))^k (1 - \varphi_n(x))^{n-k} \\ &= e^{\lambda_n a} e^{\lambda_0(x-a)} \sum_{k=0}^n \binom{n}{k} \left( e^{\omega_n(b-a)} \varphi_n(x) \right)^k (1 - \varphi_n(x))^{n-k} \\ &= e^{\lambda_n a} e^{\lambda_0(x-a)} \left( e^{\omega_n(b-a)} \varphi_n(x) + 1 - \varphi_n(x) \right)^n \\ &= e^{\lambda_n a} e^{\lambda_0(x-a)} \left( \left( e^{\omega_n(b-a)} - 1 \right) \varphi_n(x) + 1 \right)^n. \end{aligned}$$

But now (14) implies that

$$\begin{aligned} \left( e^{\omega_n(b-a)} - 1 \right) \varphi_n(x) + 1 &= e^{\omega_n(b-a)} \left( 1 - e^{\omega(a-b)} \right) \varphi_n(x) + 1 \\ &= e^{\omega_n(b-a)} \left( e^{\omega_n(x-b)} - e^{\omega_n(a-b)} \right) + 1 = e^{\omega_n(x-a)}, \end{aligned}$$

so from the preceding equalities we obtain

$$\mathcal{B}_n(e^{\lambda_n x}) = e^{\lambda_n a} e^{\lambda_0(x-a)} e^{n\omega_n(x-a)} = e^{\lambda_n x}. \quad \square$$

For simplicity, next we take  $[a, b]$  to be  $[0, 1]$ ,  $\mu_0 = 0$ , and  $\mu_1 = 1$ . Then the Morigi and Neamtu operator reduces to

$$\mathcal{B}_n f(x) := \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} (\varphi_n(x))^k (1 - \varphi_n(x))^{n-k}, \tag{18}$$

where by specialization of (14) we have

$$\varphi_n(x) = \frac{e^{x/n} - 1}{e^{1/n} - 1}.$$

Observe that since  $e^{x/n}$  is convex and increasing, so is  $\varphi_n$ . Furthermore,  $\varphi_n(0) = 0$  and  $\varphi_n(1) = 1$ . Recalling that  $B_n$  denotes the classical Bernstein operator, given by (1), we see that

$$\mathcal{B}_n f(x) = B_n f(\varphi_n(x)). \tag{19}$$

From the preceding expression it is immediate that  $\mathcal{B}_n$  does not in general preserve the convexity of a (decreasing) convex function. Let  $h(x) = -x$ . Since  $B_n h = h$ , we have

$$\mathcal{B}_n h(x) = B_n h(\varphi_n(x)) = -\varphi_n(x).$$

But  $\varphi_n$  is strictly convex, so the image under  $\mathcal{B}_n$  of the affine (hence convex) function  $h$  is the strictly concave function  $-\varphi_n$ . This shows that the following assertion, contained in [19, Remark 4.15 (1)], is not correct: If  $f_0 = 1$ ,  $f_1$  is convex, and  $f'_1 > 0$  everywhere on  $[a, b]$ , then the image of a decreasing convex function under  $\mathcal{B}_n$  is convex (this is claimed to follow from [19, Proposition 4.14], but it does not; we note that Proposition 4.14 from [19] is correct, cf. [2, Theorem 22] together with Theorem 4 above for a stronger result).

It is shown in [20, Theorem 3.3] that the operators  $\mathcal{B}_n$  converge strongly to the identity (more general or related results can be found in [1, Theorem 23] and [19, Theorem 6.1]). However, for the special case we are considering, it is easy to give a direct argument, so in order to make this section as self-contained as possible, we do this next.

Let  $g(x) := x$ . Since  $\mathcal{B}_n$  fixes 1 and  $e^x$ , by Korovkin’s Theorem it suffices to prove that  $\mathcal{B}_n g$  converges uniformly to  $g$  on  $[0, 1]$ . By (19),  $\mathcal{B}_n g = \varphi_n$ , so it is enough to show that  $\varphi_n$  converges uniformly to  $g$ . Since  $\varphi_n$  is convex and increasing, its derivative achieves its maximum value over  $[0, 1]$  at 1. Now let  $g_n$  be the affine function with slope  $\varphi'_n(1)$  passing through the point  $(1, 1)$ . By convexity,  $g_n \leq \varphi_n \leq g$ , so to obtain the uniform convergence it suffices to show that  $\lim_{n \rightarrow \infty} \varphi'_n(1) = 1$ . But this follows immediately by writing explicitly  $\varphi'_n(1)$ , and then using L’Hospital rule, or the Taylor expansion for  $e^{1/n}$ .

We have seen that  $\mathcal{B}_n$  does not preserve convexity. It is easy to check that  $B_n$  does not preserve  $(1, e^x)$ -convexity either. Actually, both of these facts can be derived from the slightly more general result given next.

**Proposition 8.** *Let  $f_0 = 1$  and  $f_1$  be functions in an Extended Chebyshev Space  $E$  over the interval  $[a, b]$ , where  $f_1$  is assumed to be strictly increasing and strictly convex. Suppose there exists a generalized Bernstein operator  $\mathcal{B}_n : C[a, b] \rightarrow E$  fixing  $f_0$  and  $f_1$ . Then the classical Bernstein operator  $B_n$  does not preserve  $(1, f_1)$ -convexity, and the generalized operator  $\mathcal{B}_n$  does not preserve convexity.*

**Proof.** For the first assertion, observe that while  $f_1$  is trivially  $(1, f_1)$ -convex,  $B_n f_1$  is not. To see why, note that since  $B_n f_1(a) = f_1(a)$  and  $B_n f_1(b) = f_1(b)$ , the unique function in  $\text{Span}\{1, f_1\}$  interpolating  $B_n f_1$  at  $a$  and  $b$  is  $f_1$  itself. If  $B_n f_1$  were  $(1, f_1)$ -convex, we would have  $B_n f_1 \leq f_1$  on  $[a, b]$ . But  $f_1$  is strictly convex, so by Jensen’s inequality (including the equality condition) we have  $B_n f_1(x) > f_1(x)$  for all  $x \in (a, b)$ .

An analogous argument yields the second assertion. Set  $h(x) = x$ , so both  $h$  and  $-h$  are convex. Towards a contradiction, suppose that  $\mathcal{B}_n$  preserves convexity. Then  $\mathcal{B}_n h \leq h$  and



$\mathcal{B}_n(-h) \leq -h$ , so  $\mathcal{B}_n h = h$ . Fix  $x \in (a, b)$ , and note that  $f \mapsto \mathcal{B}_n f(x)$  is a positive linear functional on  $C[a, b]$ , defined by a probability measure (since  $\mathcal{B}_n 1(x) = 1$ ). Thus we can once more apply Jensen's inequality, in this case to the strictly concave function  $f_1^{-1}$ , to conclude that

$$x = h(x) = \mathcal{B}_n h(x) = \mathcal{B}_n (f_1^{-1} \circ f_1)(x) > f_1^{-1}(\mathcal{B}_n f_1(x)) = f_1^{-1}(f_1(x)) = x.$$

Alternatively, we can reach the same conclusion without using Jensen's inequality, by observing that if in addition to fixing 1 and  $f_1$ , we have  $\mathcal{B}_n h = h$ , then  $\mathcal{B}_n$  fixes three linearly independent functions, and thus it must be the identity operator (by [19, Proposition 3.7]). So again we have a contradiction.  $\square$

We finish with an easy Proposition, which sheds some light on the relationship between the different notions of convexity considered above.

**Proposition 9.** *Let  $f_1$  be increasing and strictly convex on  $[a, b]$ . For increasing functions on  $[a, b]$ , the condition of  $(1, f_1)$ -convexity is strictly stronger than convexity. Over the decreasing functions on  $[a, b]$ , convexity is strictly stronger than  $(1, f_1)$ -convexity.*

**Proof.** Let  $f$  be increasing and  $(1, f_1)$ -convex on  $[a, b]$ . Pick  $x, y \in [a, b]$  with  $x < y$ . Since  $f(y) \geq f(x)$ , the unique interpolant  $\psi$  of  $f$  at  $x$  and  $y$  such that  $\psi \in \text{Span}\{1, f_1\}$  must be of the form  $\psi = c_0 + c_1 f_1$ , with  $c_1 \geq 0$ . Thus  $\psi$  is convex, so the unique interpolant  $\phi \in \text{Span}\{1, t\}$  of  $\psi$  (or  $f$ ) at  $x$  and  $y$  satisfies  $\phi(t) \geq \psi(t) \geq f(t)$  for all  $t \in [x, y]$ , whence  $f$  is convex. Also,  $h(t) := t$  is increasing and convex but not  $(1, f_1)$ -convex, since  $h(f_1^{-1}) = f_1^{-1}$  is strictly concave, cf. Lemma 5.

The assertion about decreasing functions is proved by the same type of argument, so we omit it.  $\square$

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