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Well-posedness and gradient blow-up estimate near the boundary for a Hamilton–Jacobi equation with degenerate diffusion

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ABSTRACT

This paper is concerned with weak solutions of the degenerate diffusive Hamilton–Jacobi equation

$$\partial_t u - \Delta_p u = |\nabla u|^q,$$

with Dirichlet boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^N$, where $p > 2$ and $q > p - 1$. With the goal of studying the gradient blow-up phenomenon for this problem, we first establish local well-posedness with blow-up alternative in $W^{1,\infty}$ norm. We then obtain a precise gradient estimate involving the distance to the boundary. It shows in particular that the gradient blow-up can take place only on the boundary. A regularizing effect for $\partial_t u$ is also obtained.

1. Introduction and main results

This article is concerned with the existence and qualitative properties of weak solutions of the initial boundary value problem of the p-Laplacian with a nonlinear gradient source term

$$\begin{cases} \partial_t u - \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = |\nabla u|^q, & x \in \Omega,\ t > 0, \\ u(x,t) = g(x), & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain in $\mathbb{R}^N$ of class $C^{2+\alpha}$ for some $\alpha > 0$, $p > 2$ and $q > p - 1$. Throughout the paper we assume that the boundary data $g \geqslant 0$ is the trace on $\partial\Omega$ of a regular function in $C^2(\overline{\Omega})$, also denoted g, and the initial data u_0 satisfies

$$u_0 \in W^{1,\infty}(\Omega), \qquad u_0 \geqslant 0, \qquad u_0(x) = g(x) \quad \text{for } x \in \partial\Omega. \tag{1.2}$$

We note that, as far as bounded solutions are concerned, there is no loss of generality in assuming $g, u_0 \geqslant 0$, since the partial differential equation in (1.1) is unchanged when adding a constant to u.

When $p = 2$, the differential equation of (1.1) is the so-called viscous Hamilton–Jacobi equation and it appears in the physical theory of growth and roughening of surfaces, where it is known as the Kardar–Parisi–Zhang equation ($q = 2$), and has been studied by many authors (see for example [7,27] and the references therein). It is known that, under certain conditions, $|\nabla u|$ blows up in a finite time $t = T_{\max}$ while, by the maximum principle, all solutions are uniformly bounded (cf. [30,17,32]). We shall call such phenomenon gradient blow-up (GBU). This is different from the usual blow-up in which the L^∞ norm of the solution tends to infinity as $t \to T_{\max}$ (cf. [27]). Sharp results on gradient blow-up analysis, including blow-up rate, blow-up set, blow-up profile and continuation after blow-up have been recently obtained, see e.g. [24,16,17,27,3,31] and the references therein.

When $p > 2$, Eq. (1.1) is a degenerate parabolic equation for $|\nabla u| = 0$ and one cannot expect the existence of classical solutions. Weak solutions can be obtained by approximation with solutions of regularized problems. This was done in [34] when the right-hand side in (1.1) is replaced with a general nonlinearity $f(u, \nabla u, x, t)$. In the case where f depends on ∇u, typically for problem (1.1), the results in [34] require the assumption $q \leqslant p - 1$, in which case a global solution is directly constructed for any initial data. Local-in-time existence results are also given in [34] but they require that f actually does not depend on ∇u. In [10], the existence of a global weak solution for $q > p - 1$ was proved for small data, under the assumption that the mean curvature of $\partial\Omega$ is nonpositive. In the articles [22,5], problem (1.1) was studied in the framework of viscosity solutions, but only in situations where global existence of a $W^{1,\infty}$ solution is guaranteed, namely for $q \leqslant p$ or for suitably small initial data when $q > p$. On the other hand, when $q > p$, global existence is not expected in general for large initial data. A result in this direction was given in [22, Theorem 5.2], where it was proved that problem (1.1) (with $g = 0$) cannot admit a global, Lipschitz continuous, weak solution for large initial data. See [25,13,15] and the references therein for earlier counter-examples concerning related quasilinear equations.

Our first goal will be to complete the above results by constructing a unique, maximal in time, $W^{1,\infty}$ solution, without size restriction on the initial data and to establish the blow up alternative in $W^{1,\infty}$ norm. This will enable us to interpret the above mentioned global nonexistence result from [22] appropriately as a gradient blow-up (GBU) result (see Theorem 1.4 and Remark 4.1 below), and will provide the grounds for the subsequent analysis of the asymptotic behavior of GBU solutions. For the local existence part, we will follow and suitably modify the approximation procedure used in [34].

The main difficulty is to get relevant estimates on the first order derivatives of the approximate solutions in order to pass to the limit in the nonlinear source term. To deal with this difficulty, our main new ingredient with respect to [34] is the construction of suitable barrier functions, in order to get uniform pointwise estimates on the gradients near the boundary for small time. We then use a strong result of DiBenedetto and Friedman [12] on the Hölder regularity of gradients of weak solutions of degenerate parabolic equations and consequently we will use the framework of weak rather than viscosity solutions.

First, let us state the precise definition of solution. Let $Q_T = \Omega \times (0, T)$ and $\partial_p Q_T = \{\partial\Omega \times [0, T]\} \cup \{\overline{\Omega} \times \{0\}\}$, $T > 0$. Throughout this paper, we will use the following definition of weak solution for (1.1).

Definition 1.1. Set $m = \max(p, q)$. A function $u(x, t)$ is called a weak super- (sub-)solution of problem (1.1) on Q_T if

$$u \in C\big(\overline{\Omega} \times [0, T)\big) \cap L^m\big((0, T); W^{1,m}(\Omega)\big),$$

$$\partial_t u \in L^2\big((0,T); L^2(\Omega)\big),$$
$$u(x,0) \geqslant (\leqslant) u_0(x), \qquad u \geqslant (\leqslant) g \quad \text{on } \partial\Omega, \quad \text{and}$$
$$\iint\limits_{Q_T} \partial_t u \psi + |\nabla u|^{p-2} \nabla u \cdot \nabla \psi \, dx\, dt \geqslant (\leqslant) \iint\limits_{Q_T} |\nabla u|^q \psi \, dx\, dt \tag{1.3}$$

holds for all $\psi \in C^0(\overline{Q_T}) \cap L^p((0,T); W^{1,p}(\Omega))$ such that $\psi \geqslant 0$, $\psi = 0$ on $\partial\Omega \times (0,T)$. A function u is a weak solution of (1.1) if it is a super-solution and a sub-solution.

Our first result concerns local existence and uniqueness of weak solutions (see also Section 2 for a comparison principle).

Theorem 1.1. *Assume that $q > p-1 > 1$. Let $M > 0$ and let u_0 satisfy* (1.2) *and $\|\nabla u_0\|_{L^\infty} \leqslant M$. Then*

(i) *There exist a time $T = T(M, p, q, N, \|g\|_{C^2}) > 0$ and a weak solution u of* (1.1) *on $[0,T)$, which moreover satisfies $u \in L^\infty_{\mathrm{loc}}([0,T); W^{1,\infty}(\Omega))$.*
(ii) *For any $\mathcal{T} > 0$ the problem* (1.1) *has at most one weak solution u such that $u \in L^\infty_{\mathrm{loc}}([0,\mathcal{T}); W^{1,\infty}(\Omega))$.*
(iii) *There exists a* (*unique*) *maximal, weak solution of* (1.1)*, still denoted by u. Let $T_{\max}(u_0)$ be its existence time.*

Then

$$\min_{\Omega} u_0 \leqslant u \leqslant \max_{\Omega} u_0 \quad \text{in } \Omega \times \big(0, T_{\max}(u_0)\big) \tag{1.4}$$

and

$$\text{if} \quad T_{\max}(u_0) < \infty, \quad \text{then} \quad \lim_{t \to T_{\max}(u_0)} \big\|\nabla u(t)\big\|_{L^\infty} = \infty \quad (\text{gradient blow up GBU}).$$

Remark 1.1. Concerning Definition 1.1, we note that if $0 < T_1 < T_2 < \infty$ and u is a weak solution on Q_{T_2}, then the restriction of u to Q_{T_1} is a weak solution on Q_{T_1} (this can be easily checked, taking any test function ψ on Q_{T_1}, by extending ψ as $\tilde{\psi}_n(x,t) = \psi(x,T_1)[1 - n(t-T_1)]_+$ for $t \in (T_1, T_2]$ and letting $n \to \infty$). Then, in Theorem 1.1(iii), by u being the maximal weak solution of (1.1), we mean that u is a weak solution on Q_τ for any $\tau \in (0, T_{\max}(u_0))$ but cannot be extended to a weak solution on $Q_{T'}$ for any $T' > T_{\max}(u_0)$.

We next establish a precise gradient estimate involving the distance to the boundary. Here and in the rest of the paper we denote $\delta(x) = \mathrm{dist}(x, \partial\Omega)$.

Theorem 1.2. *Let $q > p-1 > 1$. Let $M > 0$ and let u_0 satisfy* (1.2) *and $\|\nabla u_0\|_{L^\infty} \leqslant M$. Let u be the unique weak solution of* (1.1) *in $L^\infty_{\mathrm{loc}}([0, T_{\max}(u_0)); W^{1,\infty}(\Omega))$. Then*

$$|\nabla u| \leqslant C_1 \delta^{-1/(q-p+1)}(x) + C_2 \quad \text{in } \Omega \times \big(0, T_{\max}(u_0)\big), \tag{1.5}$$

where $C_1 = C_1(q,p,N) > 0$ and $C_2 = C_2(q,p,\Omega,M,\|g\|_{C^2}) > 0$.

This estimate in particular implies that $|\nabla u|$ remains bounded away from the boundary. Therefore, when $T_{\max}(u_0) < \infty$, the blow-up may only take place on the boundary and (1.5) provides information on the blow-up profile near $\partial\Omega$. Estimate (1.5) is sharp in one space dimension, see [4]. Similar results are already available for $p = 2$ and have been established in [32,3]. For $p > 2$, only global-in-space gradient estimates were available up to now (i.e. for $\Omega = \mathbb{R}^N$, see [6]). The proof of estimate (1.5) is based on similar arguments as for the case $p = 2$, namely Bernstein type arguments, but they are much more technical. Moreover, the proof of (1.5) also relies on a regularizing effect for solutions to (1.1) which seems to be new and which is stated below.

Theorem 1.3. *Assume that $q > p - 1 > 1$ and let u be the unique weak solution of problem (1.1) in $L^\infty_{loc}([0, T_{\max}(u_0)); W^{1,\infty}(\Omega))$. Then*

$$\partial_t u \leqslant \frac{1}{p-2} \frac{\sup(\|u_0\|_{L^\infty}, \|g\|_{L^\infty})}{t} \quad \text{in } \mathcal{D}'(\Omega) \text{ a.e. } t > 0. \tag{1.6}$$

Let us note that due to the positivity of the source term, this inequality implies the semi-concavity estimate

$$\Delta_p(u) = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) \leqslant \frac{C}{t}, \tag{1.7}$$

which was obtained in the case $\Omega = \mathbb{R}^N$ by a different method for $q < p$ in [21] and for $q = p$ in [14].

Finally we give the following blow-up result, which is a variant of a global nonexistence result in [22], reinterpreted in terms of GBU in the light of Theorem 1.1. Let φ_1 be the first eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary conditions

Theorem 1.4. *Assume that $q > p > 2$ and let u be the unique weak solution of (1.1) in $L^\infty_{loc}([0, T_{\max}(u_0));$ $W^{1,\infty}(\Omega))$. Let $\alpha \geqslant 1$ satisfy $\frac{p-1}{q-p+1} < \alpha < q - 1$, then there exists a constant $C = C(q, p, \alpha, \Omega, \|g\|_{L^\infty}) > 0$ such that if $\int_\Omega u_0 \varphi_1^\alpha \, dx \geqslant C$, then $T_{\max}(u_0) < \infty$, i.e. gradient blow-up occurs.*

For results concerning other aspects of Eq. (1.1) and the corresponding Cauchy problem, see e.g. [11,28,10,34,6] and the references therein. Asymptotic behavior of global solution is investigated in [33,5,22,21,23,18,2,1,8] and references therein.

The rest of the paper is organized as follows: In Section 2 we prove the well-posedness of (1.1) in $W^{1,\infty}(\Omega)$, as well as the regularizing effect. Section 3 is devoted to the proof of Theorem 1.2. Finally in Section 4 we prove the sufficient blow-up criterion of Theorem 1.4.

2. Proof of Theorem 1.1 and Theorem 1.3

2.1. Local existence

Consider the following approximate problems for (1.1):

$$\begin{cases} \partial_t u_n - \operatorname{div}\left(\left(|\nabla u_n|^2 + \dfrac{1}{n}\right)^{(p-2)/2} \nabla u_n\right) = \left(|\nabla u_n|^2 + \dfrac{1}{n}\right)^{q/2} - \dfrac{1}{n^{q/2}}, & x \in \Omega,\ t > 0, \\ u_n(x,t) = g(x), & x \in \partial\Omega,\ t > 0, \\ u_n(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

For each fixed $n \in \mathbb{N}$, problem (2.1) is no longer degenerate and the regularity theory of quasilinear parabolic equations [20] provides local-in-time solutions u_n, which are smooth for $t > 0$ and continuous up to $t = 0$.

To find the limit function $u(x,t)$ of the sequence $\{u_n(x,t)\}$, we divide our proof into 5 steps. Recall that there exists $\eta_0 > 0$ small such that, for any $x \in \overline{\Omega}$ with $\delta(x) \leqslant \eta_0$, the point $\tilde{x} := \operatorname{proj}_{\partial\Omega}(x)$ (the projection of x onto the boundary) is well defined and unique.

Step 1. Let $Q_T := \Omega \times (0,T)$. There exist a small time $T_0 > 0$, $\eta \in (0,\eta_0)$ and $M_2 > 0$, all independent of n and depending on u_0 through M only, such that

$$\|u_n\|_{L^\infty(Q_{T_0})} \leqslant M_1 := \sup\big(\|u_0\|_{L^\infty}, \|g\|_{L^\infty}\big), \tag{2.2}$$

and

$$\sup_{\substack{x\in\Omega \\ \delta(x)\leqslant\eta}} \frac{|u_n(x,t)-u_n(\tilde{x},t)|}{\delta(x)} \leqslant M_2, \quad 0<t\leqslant T_0. \tag{2.3}$$

Estimate (2.2) is a direct consequence of the maximum principle since M_1 is a super-solution for any n.

In order to prove estimate (2.3), we are going to construct a local barrier function under the exterior sphere condition satisfied by the domain Ω, i.e. for any x near $\partial\Omega$, a super-solution in a neighborhood of x.

Let $\rho > 0$ be such that for all $x \in \partial\Omega$, $\overline{B_\rho(x+\rho\nu_x)} \cap \overline{\Omega} = \{x\}$, where ν_x is the unit outward normal vector on $\partial\Omega$ at x. Fix an arbitrary $x_0 \in \Omega$ such that $\delta(x_0) \leqslant \eta$ where $\eta \in (0,\eta_0)$ will be chosen later. Define $x_1 = \widetilde{x_0} + \rho\nu_{\widetilde{x_0}}$, where $\widetilde{x_0} := \mathrm{proj}_{\partial\Omega}(x_0)$. Without loss of generality we may assume that $x_1 = 0$ and we write $r = |x|$. Let us denote, for $s \geqslant 0$,

$$a(s) = \left(s+\frac{1}{n}\right)^{(p-2)/2} \quad \text{and} \quad \kappa(s) = \frac{2a'(s)s}{a(s)} \in [0, p-2]. \tag{2.4}$$

We recall that for a function $\phi(x) = \phi(|x|)$, we have:

$$\begin{aligned}
\nabla\phi(x) &= \phi'(r)\frac{x}{r}, \\
D^2\phi(x) &= \phi''(r)\frac{x\otimes x}{r^2} + \frac{\phi'(r)\,\mathrm{Id}}{r} - \phi'(r)\frac{x\otimes x}{r^3}, \\
\Delta\phi(x) &= \phi''(r) + \frac{(N-1)\phi'(r)}{r},
\end{aligned} \tag{2.5}$$

where Id is the unit matrix and $(x\otimes x)_{ij} = x_i x_j$.

Define for $x \in \Omega$

$$\bar{v}(x,t) = \phi(r-\rho) + g(x),$$

where ϕ is a smooth function of one variable which is increasing and *concave*. First let us write

$$\begin{aligned}
\mathrm{div}\left(\left(|\nabla\bar{v}|^2+\frac{1}{n}\right)^{(p-2)/2}\nabla\bar{v}\right) &= a\big(|\nabla\bar{v}|^2\big)\Delta\bar{v} + 2a'\big(|\nabla\bar{v}|^2\big)(\nabla\bar{v})^t D^2\bar{v}\nabla\bar{v}, \\
&= a\big(|\nabla\bar{v}|^2\big)\left(\Delta\bar{v} + \kappa\big(|\nabla\bar{v}|^2\big)\frac{(\nabla\bar{v})^t D^2\bar{v}\nabla\bar{v}}{|\nabla\bar{v}|^2}\right).
\end{aligned} \tag{2.6}$$

Using (2.5), we have

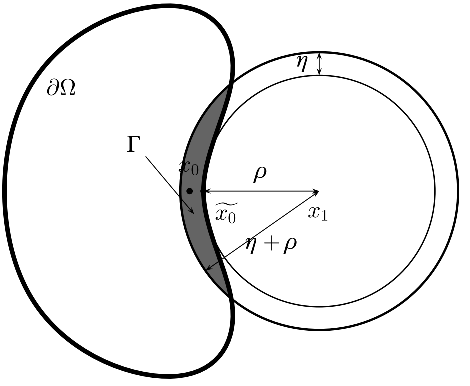

Fig. 1. Local barrier function.

$$\left[\Delta\bar{v}+\kappa\big(|\nabla\bar{v}|^2\big)\frac{(\nabla\bar{v})^tD^2\bar{v}\nabla\bar{v}}{|\nabla\bar{v}|^2}\right]=\phi''(r-\rho)+\frac{(N-1)\phi'(r-\rho)}{r}+\Delta g$$
$$+\kappa\big(|\nabla\bar{v}|^2\big)\frac{\phi''(r-\rho)(\nabla\bar{v}\cdot x)^2}{r^2|\nabla\bar{v}|^2}+\kappa\big(|\nabla\bar{v}|^2\big)\frac{\phi'(r-\rho)}{r}$$
$$-\kappa\big(|\nabla\bar{v}|^2\big)\frac{\phi'(r-\rho)(\nabla\bar{v}\cdot x)^2}{r^3|\nabla\bar{v}|^2}+\kappa\big(|\nabla\bar{v}|^2\big)\frac{(\nabla\bar{v})^tD^2g\nabla\bar{v}}{|\nabla\bar{v}|^2}.$$

Since $\phi'(r-\rho)\geqslant 0$, $r\geqslant\rho$, $\kappa(|\nabla\bar{v}|^2)\geqslant 0$ and $0\geqslant\phi''(r-\rho)$, we have

$$-\left[\Delta\bar{v}+\kappa\big(|\nabla\bar{v}|^2\big)\frac{(\nabla\bar{v})^tD^2\bar{v}\nabla\bar{v}}{|\nabla\bar{v}|^2}\right]\geqslant-\phi''(r-\rho)-\left(\frac{N-1+\kappa(|\nabla\bar{v}|^2)}{\rho}\right)\phi'(r-\rho)$$
$$-\|\Delta g\|_{L^\infty}-\kappa\big(|\nabla\bar{v}|^2\big)\big\|D^2g\big\|_{L^\infty}. \tag{2.7}$$

On the other hand $|\nabla\bar{v}|=|\phi'(r-\rho)\frac{x}{r}+\nabla g|\leqslant\phi'(r-\rho)+|\nabla g|\leqslant 2\phi'(r-\rho)$ provided that

$$\phi'(r-\rho)\geqslant\|\nabla g\|_{L^\infty}. \tag{2.8}$$

In this case we have

$$\left(|\nabla\bar{v}|^2+\frac{1}{n}\right)^{(q-p+2)/2}\leqslant\big[4\big(\phi'(r-\rho)\big)^2+1\big]^{(q-p+2)/2}. \tag{2.9}$$

We take

$$\phi(s)=s(s+\mu)^{-\beta},\quad s\geqslant 0,$$

where $\beta=\beta(q,p)\in(0,1)$ is to be chosen later and $\mu>0$. We denote $\Gamma:=B(x_1,\rho+\eta)\cap\Omega$ (see Fig. 1). Our aim is to show that, for some $T_0>0$ sufficiently small $\bar{v}$ is a super-solution in $\Gamma\times(0,T_0)$ where $\mu>0$ and $\eta\in(0,\eta_0)$ are small enough. In the rest of the proof, the constants T_0,η,μ and C will be independent of x_0, n and will depend on the initial data u_0 through M only (and they will depend on the other data p,q,N,Ω and $\|g\|_{C^2}$ without other mention). We calculate

$$\phi'(s) = \big[(1-\beta)s + \mu\big](s+\mu)^{-\beta-1},$$
$$\phi''(s) = -\beta\big[(1-\beta)s + 2\mu\big](s+\mu)^{-\beta-2}.$$

We are looking for condition on β and μ such that

$$-\mathrm{div}\left(\left(|\nabla\bar{v}|^2 + \frac{1}{n}\right)\nabla\bar{v}\right) \geqslant \left(|\nabla\bar{v}|^2 + \frac{1}{n}\right)^{q/2} - \left(\frac{1}{n}\right)^{q/2}. \tag{2.10}$$

Due to (2.6), it suffices to have

$$-\left[\Delta\bar{v} + \kappa\big(|\nabla\bar{v}|^2\big)\frac{(\nabla\bar{v})^t D^2\bar{v}\nabla\bar{v}}{|\nabla\bar{v}|^2}\right] \geqslant \left(|\nabla\bar{v}|^2 + \frac{1}{n}\right)^{\frac{q-p+2}{2}}, \tag{2.11}$$

which, by (2.4), (2.7), (2.9) reduces to

$$\begin{aligned} &-\phi''(r-\rho) + \left(\frac{3-N-p}{\rho}\right)\phi'(r-\rho) \\ &\quad\geqslant \big[4\big(\phi'(r-\rho)\big)^2 + 1\big]^{(q-p+2)/2} + (p-2+\sqrt{N})\big\|D^2 g\big\|_{L^\infty}. \end{aligned} \tag{2.12}$$

Using that $\rho < r < \rho + \eta$ and $(3-N-p) < 0$, then (2.10) holds if

$$\begin{aligned} &(r-\rho+\mu)^{-\beta-2}\left[2\beta\mu + \frac{(3-N-p)}{\rho}(\eta+\mu)^2\right] \\ &\quad\geqslant \big[4(r-\rho+\mu)^{-2\beta} + 1\big]^{(q-p+2)/2} + (p-2+\sqrt{N})\big\|D^2 g\big\|_{L^\infty}. \end{aligned}$$

Assume that η and μ are such that

$$\begin{cases} 4(r-\rho+\mu)^{-2\beta} \geqslant 4(\eta+\mu)^{-2\beta} \geqslant 1, \\ 2\beta\mu + \dfrac{(3-N-p)}{\rho}(\eta+\mu)^2 \geqslant \beta\mu, \end{cases} \tag{2.13}$$

then to get (2.10) it is sufficient to have

$$\beta\mu(r-\rho+\mu)^{-\beta-2} \geqslant (r-\rho+\mu)^{-\beta(q-p+2)}4^{(q-p+3)}, \tag{2.14}$$

and

$$\beta\mu(r-\rho+\mu)^{-\beta-2} \geqslant 4(p-2+\sqrt{N})\big\|D^2 g\big\|_{L^\infty}. \tag{2.15}$$

Inequality (2.14) holds if we choose $\eta = \mu$, $\beta = \frac{1}{2(q-p+2)}$, and μ satisfying

$$4^{p-q-4}\beta \geqslant \mu^{(q-p+3)/(2q-2p+4)}.$$

Inequalities (2.14)–(2.15) and (2.8) hold if we choose $\eta = \mu$ small enough. We have thus shown that if $\eta = \mu$ is small, then $\bar{v}$ is a super-solution on $\Gamma \times (0, T)$ for any $T > 0$.

Now we need to have a control on the parabolic boundary of $\Gamma \times (0,T)$ for $T>0$ small. For this purpose, we introduce another comparison function

$$\bar{u}(x,t) = \left(2C^2K^2 + 2\|\nabla g\|_{L^\infty}^2 + 1\right)^{q/2} t + C\left(1 - e^{-K(|x|-\rho)}\right) + g(x).$$

It is easy to check that if we fix $K>0$ large enough, then we can find a constant $C = C(p,N,M,\Omega,\|g\|_{C^2}) > 0$ sufficiently large such that

$$-\mathrm{div}\left(\left(|\nabla \bar{u}|^2 + \frac{1}{n}\right)^{(p-2)/2} \nabla \bar{u}\right) \geqslant 0 \quad \text{in } \Omega.$$

Indeed, since Ω is bounded, there exists $R(\Omega)>0$ such that $\Omega \subset B(x_1, R(\Omega))$ and hence $r-\rho \leqslant R(\Omega)$. Now once $(K > \frac{2(N+p-3)}{\rho})$ is fixed, using (2.7) it is sufficient to require that

$$CKe^{-K(r-\rho)}\left[K - \frac{N+p-3}{\rho}\right] \geqslant (p-2+\sqrt{N})\left\|D^2 g\right\|_{L^\infty},$$

which is satisfied if

$$C \geqslant \frac{2e^{KR(\Omega)}(p-2+\sqrt{N})\|D^2 g\|_{L^\infty}}{K^2}.$$

Thus

$$\partial_t \bar{u} - \mathrm{div}\left(\left(|\nabla \bar{u}|^2 + \frac{1}{n}\right)^{(p-2)/2} \nabla \bar{u}\right) \geqslant \left(|\nabla \bar{u}|^2 + \frac{1}{n}\right)^{q/2} - \left(\frac{1}{n}\right)^{q/2}.$$

We can also choose C such that $C(1-e^{-K(r-\rho)}) + g(x) \geqslant u_0(x)$. Since $\bar{u} \geqslant g$ on $\partial\Omega \subset \{x \in \mathbb{R}^N, |x| \geqslant \rho\}$, by the comparison principle we get that for any n, $u_n \leqslant \bar{u}$ in Q_T for any $T>0$. Thus

$$\begin{aligned} u_n(x,t) &\leqslant \left(2C^2K^2 + 2\|\nabla g\|_{L^\infty}^2 + 1\right)^{q/2} t + C\left(1 - e^{-K\eta}\right) + g(x) \\ &\leqslant 2^{-\beta}\eta^{1-\beta} + g(x) = \bar{v}(x,t) \end{aligned}$$

on $\{x \in \Omega,\ |x| = \rho + \eta\} \times [0, T_0]$, provided T_0 and $\eta = \mu$ are small enough (depending only on $M, p, q, \Omega, \|g\|_{C^2}$). Next we also choose $\eta = \mu$ small enough so that

$$\begin{aligned} u_0(x) &\leqslant g(\tilde{x}) + M|x - \tilde{x}| \leqslant g(\tilde{x}) + M(r-\rho) \\ &\leqslant g(\tilde{x}) + (r-\rho)\left[(2\eta)^{-\beta} - \|\nabla g\|_{L^\infty}\right] \leqslant \bar{v}(x,0). \end{aligned}$$

On the other hand $u_n = g \leqslant \bar{v}$ on $\partial\Omega \times [0,T_0]$. We conclude that $\bar{v}$ is a super-solution on $\Gamma \times (0,T_0)$. Similarly $\underline{v} := g - \phi(r-\rho)$ is a sub-solution. Applying the comparison principle we get $\underline{v} \leqslant u_n \leqslant \bar{v}$ on $\Gamma \times [0,T_0]$, and hence in particular

$$\frac{|u_n(x_0,t) - u_n(\widetilde{x_0},t)|}{|x_0 - \widetilde{x_0}|} \leqslant \sup_{0 \leqslant s \leqslant \eta}\left|\phi'(s)\right| + \|\nabla g\|_{L^\infty} \leqslant \eta^{-\beta} + \|\nabla g\|_{L^\infty} =: M_2, \quad 0 < t \leqslant T_0,$$

which yields (2.3).

Step 2. There holds

$$\|\nabla u_n\|_{L^\infty(Q_{T_0})} \leqslant M_3 := \sup\bigl(M, M_2 + \|\nabla g\|_{L^\infty}\bigr). \tag{2.16}$$

We use a similar argument as in [19, Theorem 5]. Let $h \in \mathbb{R}^N$ satisfy $|h| \leqslant \eta$. Due to the translation invariance of (2.1), if u_n is a classical solution of (2.1) in Ω, then the function $u_n^h(x,t) := u_n(x-h,t)$ is a classical solution of (2.1) in $\Omega_h \times (0,T_0)$ where $\Omega_h := \{x \in \mathbb{R}^N \mid x-h \in \Omega\}$. By the regularity of the initial data ($u_0 \in W^{1,\infty}(\Omega)$), we have $|u_n(x,0) - u_n^h(x,0)| \leqslant M|h|$ on $\Omega_h \cap \Omega$. Let $t \in [0,T_0]$ and $x \in \partial(\Omega \cap \Omega_h)$. We may assume for instance $x \in \partial\Omega$, the case $x+h \in \partial\Omega$ being similar. Then using $|\tilde{y} - \tilde{z}| \leqslant |y-z|$ and (2.3), we get

$$\begin{aligned}
\bigl|u_n(x,t) - u_n(x+h,t)\bigr| &= \bigl|u_n(\tilde{x},t) - u_n(\widetilde{x+h},t) + u_n(\widetilde{x+h},t) - u_n(x+h,t)\bigr| \\
&\leqslant \|\nabla g\|_{L^\infty}|\tilde{x} - \widetilde{x+h}| + M_2\delta(x+h) \\
&\leqslant \bigl(\|\nabla g\|_{L^\infty} + M_2\bigr)|h| \leqslant M_3|h|.
\end{aligned}$$

In particular $u_n(x,t) \leqslant u_n^h(x,t) + M_3|h|$ on $\partial(\Omega \cap \Omega_h) \times [0,T_0]$. Applying the comparison principle, we have $u_n(x,t) \leqslant u_n^h(x,t) + M_3|h|$ on $(\Omega \cap \Omega_h) \times [0,T_0]$. By the same argument $u_n^h(x,t) - M_3|h| \leqslant u_n(x,t)$ on $(\Omega \cap \Omega_h) \times [0,T_0]$, hence $|u_n(x,t) - u_n^h(x,t)| \leqslant M_3|h|$. Since $|h| \leqslant \eta$ is arbitrary, the conclusion follows.

Step 3. Let $\epsilon > 0$ and set $Q_{T_0,\epsilon} = \{x \in \Omega, \delta(x) > \epsilon\} \times (\epsilon, T_0 - \epsilon)$. There exists a constant $M_4 > 0$ independent of n, such that

$$\bigl|\nabla u_n(x_1,t_1) - \nabla u_n(x_2,t_2)\bigr| \leqslant M_4\bigl(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}}\bigr) \tag{2.17}$$

for any pair of points $(x_i,t_i) \in Q_{T_0,\epsilon}$, where M_4 and α are positive constants depending only on T_0, M_3 and ϵ. Indeed we know from a result of DiBenedetto and Friedman [12] that if $f \in L^r(\Omega_T)$ for some $r > \frac{pN}{p-1}$ then weak solutions of degenerate parabolic equation of the form

$$\partial_t v - \operatorname{div}\bigl(|\nabla v|^{p-2}\nabla v\bigr) = f(x,t) \tag{2.18}$$

are of class $C^{1,\alpha}_{\mathrm{loc}}(Q_T)$ with Hölder norm depending only on $\|f\|_{L^r}$, $\|\nabla v\|_{L^p}$ and $\|v\|_{L^\infty_t, L^2_x}$.

Step 4. There exists a constant $M_5 > 0$ independent of n, such that

$$\|\partial_t u_n\|_{L^2(Q_{T_0})} \leqslant M_5. \tag{2.19}$$

To see this, multiplying (2.1) by $\partial_t u_n$ and integrating over Q_{T_0}, we have

$$\begin{aligned}
\int_0^{T_0}\int_\Omega (\partial_t u_n)^2\,dx\,dt = &-\int_0^{T_0}\int_\Omega \left(|\nabla u_n|^2 + \frac{1}{n}\right)^{(p-2)/2} \nabla u_n \cdot \nabla(\partial_t u_n)\,dx\,dt \\
&+ \int_0^{T_0}\int_\Omega \left(|\nabla u_n|^2 + \frac{1}{n}\right)^{q/2} \partial_t u_n\,dx\,dt.
\end{aligned}$$

By Hölder's inequality and

$$\int_0^{T_0}\int_\Omega \left(|\nabla u_n|^2+\frac{1}{n}\right)^{(p-2)/2}\nabla u_n\cdot\nabla(\partial_t u_n)\,dx\,dt$$
$$=\frac{1}{p}\int_\Omega\left(\left|\nabla u_n(x,T_0)\right|^2+\frac{1}{n}\right)^{p/2}-\frac{1}{p}\int_\Omega\left(\left|\nabla u_n(x,0)\right|^2+\frac{1}{n}\right)^{p/2},$$

we get

$$\int_0^{T_0}\int_\Omega(\partial_t u_n)^2\,dx\,dt\leqslant\frac{2}{p}\int_\Omega\left(\left|\nabla u_n(x,0)\right|^2+\frac{1}{n}\right)^{p/2}dx+2\int_0^{T_0}\int_\Omega\left(|\nabla u_n|^2+\frac{1}{n}\right)^q dx\,dt$$
$$\leqslant M',$$

for some $M'=M'(|\Omega|,M_3,T_0,p,q)>0$.

Step 5. We recall that by the Arzelà–Ascoli theorem we have

$$W^{1,\infty}(\Omega)\overset{c}{\hookrightarrow}C(\overline{\Omega})\hookrightarrow L^2(\Omega). \tag{2.20}$$

Using (2.2), (2.16), (2.19)–(2.20) and the compactness theorem in [29, Corollary 4], we have that $\{u_n\}$ is relatively compact in $C([0,T_0];C(\overline{\Omega}))=C(\overline{\Omega}\times[0,T_0])$. By virtue of (2.16)–(2.17), (2.19), the Ascoli–Arzelà theorem and the relative compactness of $\{u_n\}$ in $C(\overline{\Omega}\times[0,T_0])$, we can find a subsequence, still denoted by $\{u_n\}$ for convenience, such that, for each $\epsilon>0$,

$$\left.\begin{array}{ll} u_n\to u & \text{in } C\big(\overline{\Omega}\times[0,T_0]\big),\\ \nabla u_n\to\nabla u & \text{in } C(Q_{T_0,\epsilon}),\\ \partial_t u_n\to\partial_t u & \text{weakly in } L^2(Q_{T_0}).\end{array}\right\} \tag{2.21}$$

We multiply (2.1) by a test function and integrate. Then by the Lebesgue's dominated convergence theorem and (2.21) we can pass to the limit and check that u is a weak solution of (1.1).

2.2. The blow-up alternative

Let us temporarily assume the uniqueness result which will be proved in the next section. The construction of the weak solution as a limit of classical solutions implies the blow-up alternative.

Indeed suppose that the maximal existence time $T_{\max}(u_0)<\infty$ and that there exist $\mathcal{M}>0$ and $t_k\to T_{\max}(u_0)$ such that for all k

$$\left\|\nabla u(t_k)\right\|_{L^\infty}\leqslant\mathcal{M}. \tag{2.22}$$

Then we can find $\tau=\tau(\mathcal{M})>0$ independent of k, such that the problem

$$\begin{cases}\partial_t u-\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)=|\nabla u|^q, & x\in\Omega,\ t>0,\\ u(x,t)=g(x), & x\in\partial\Omega,\ t>0,\\ u(x,0)=u(x,t_k), & x\in\Omega,\end{cases} \tag{2.23}$$

admits a unique weak solution v_k on $[0,\tau)$. Setting

$$\tilde{u}(t)=\begin{cases} u(t) & \text{for } t\in[0,t_k),\\ v_k(t-t_k) & \text{for } t\in[t_k,t_k+\tau),\end{cases}$$

it is easy to see that we get a weak solution defined on $[0,t_k+\tau)$.

Since for k large enough $t_k+\tau>T_{\max}(u_0)$, this contradicts the definition of $T_{\max}(u_0)$. Hence $T_{\max}(u_0)<\infty \;\Rightarrow\; \lim_{t\to T_{\max}(u_0)}\|\nabla u(t)\|_{L^\infty}=\infty$.

2.3. Uniqueness

In this section we prove the uniqueness of the weak solution. This result will be a consequence of the following comparison principle which, in turns, also guarantees (1.4).

Proposition 2.1. *Let u,v be respectively, sub-, super-solutions of (1.1). Assume that $u,v\in L^\infty((0,T);W^{1,\infty}(\Omega))$. Then $u\leqslant v$ on $\Omega\times(0,T)$.*

The proof of Proposition 2.1 is mostly based on the following algebraic lemma from which we can show that the source term can be counterbalanced by the diffusion effect (c.f. [9] and [26] for useful inequalities for the p-Laplacian).

Lemma 2.1 *(Monotonicity Property). Let $\sigma>1$. For all a and $b\in\mathbb{R}^N$:*

$$\left\langle |a|^{\sigma-2}a-|b|^{\sigma-2}b,a-b\right\rangle \geqslant \frac{4}{\sigma^2}\left||a|^{(\sigma-2)/2}a-|b|^{(\sigma-2)/2}b\right|^2.$$

Proof of Proposition 2.1. We set $w=(u-v)^+$. By definition we have $w=0$ on $\partial\Omega$. By Remark 1.1, for any $\tau\in(0,T)$, using $\psi=w$ as test-function, we have

$$\int_0^\tau\int_\Omega \partial_t w\, w\,dx\,dt$$
$$\leqslant \underbrace{\int_0^\tau\int_{\{w(\cdot,t)>0\}}\left[|\nabla u|^q-|\nabla v|^q\right]w\,dx\,dt}_{\mathcal{B}}-\underbrace{\int_0^\tau\int_{\{w(\cdot,t)>0\}}\left[|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\right]\cdot\nabla w\,dx\,dt}_{\mathcal{H}}.$$

We set $a=\nabla u$ and $b=\nabla v$. We get by Lemma 2.1,

$$\mathcal{H}\geqslant c(p)\int_0^\tau\int_{\{w(\cdot,t)>0\}}\left||\nabla u|^{(p-2)/2}\nabla u-|\nabla v|^{(p-2)/2}\nabla v\right|^2dx\,dt. \tag{2.24}$$

Let's consider the term $\mathcal{B}$. We put $h(s)=s^{\frac{2q}{p}}$ for $s\geqslant 0$. Given that $q\geqslant p-1\geqslant\frac{p}{2}$, we have $h'(s)=\frac{2q}{p}s^{\frac{2q-p}{p}}$. The mean value theorem yields

$$\left||\nabla u|^q-|\nabla v|^q\right|^2\leqslant Ch'(\theta)^2\left||\nabla u|^{p/2}-|\nabla v|^{p/2}\right|^2,$$

for some $0\leqslant\theta\leqslant\max(|\nabla u|^{\frac{p}{2}},|\nabla v|^{\frac{p}{2}})$.

Now a direct computation shows that

$$\left||\nabla u|^{p/2}-|\nabla v|^{p/2}\right|^2\leqslant\left||\nabla u|^{(p-2)/2}\nabla u-|\nabla v|^{(p-2)/2}\nabla v\right|^2.$$

Since we assumed $u, v\in L^\infty((0,T);W^{1,\infty}(\Omega))$, it follows that

$$\left||\nabla u|^{q}-|\nabla v|^{q}\right|^2\leqslant C\left||\nabla u|^{(p-2)/2}\nabla u-|\nabla v|^{(p-2)/2}\nabla v\right|^2.$$

On the other hand, the Young's inequality implies

$$\mathcal{B}\leqslant\epsilon\int_0^\tau\int_{\{w(\cdot,t)>0\}}\left||\nabla u|^{q}-|\nabla v|^{q}\right|^2dx\,dt+C(\epsilon)\int_0^\tau\int_{\{w(\cdot,t)>0\}}w^2\,dx\,dt.$$

Combining these two inequalities, we arrive at

$$\mathcal{B}\leqslant C\epsilon\int_0^\tau\int_{\{w(\cdot,t)>0\}}\left||\nabla u|^{(p-2)/2}\nabla u-|\nabla v|^{(p-2)/2}\nabla v\right|^2dx\,dt+C(\epsilon)\int_0^\tau\int_{\{w(\cdot,t)>0\}}w^2\,dx\,dt. \tag{2.25}$$

Choosing ϵ small enough, we get

$$\int_\Omega w^2(\tau)\,dx\leqslant\int_\Omega w^2(0)\,dx+C(\epsilon)\int_0^\tau\int_\Omega w^2\,dx\,dt,\quad 0<\tau<T. \tag{2.26}$$

The Gronwall lemma implies that for any $t\in(0,T)$,

$$\int_\Omega w^2(x,t)\,dx\leqslant e^{Ct}\int_\Omega w(x,0)^2\,dx.$$

We conclude that $w\equiv 0$ almost everywhere. □

Remark 2.1.

(a) The question of uniqueness was partially open in [33]. The preceding result can be applied to show uniqueness in the case $p-1\geqslant q\geqslant\frac{p}{2}$ with $p\geqslant 2$.
(b) In [12] we have a weaker inequality for $p\in(1,2)$ but it is sufficient to prove uniqueness for the case $q>1$:

$$\left\langle |a|^{p-2}a-|b|^{p-2}b,a-b\right\rangle\geqslant(p-1)|a-b|^2\left(|a|^p+|b|^p\right)^{\frac{p-2}{p}}.$$

2.4. Regularizing effect

We use a technique similar to that used by Zhao for the p-Laplace equation without source term [35]. The idea is to compare $\lambda^\gamma u(x,\lambda t)$ and $u(x,t)$. Let u be a weak solution of (1.1) in $L^\infty_{\mathrm{loc}}([0,T);W^{1,\infty}(\Omega))$. Set

$$u_\lambda(x,t)=\lambda^\gamma u(x,\lambda t),\quad \lambda>1,\ \gamma=\frac{1}{p-2}.$$

Then u_λ is a weak solution of

$$\begin{cases} \partial_t u_\lambda - \operatorname{div}\big(|\nabla u_\lambda|^{p-2}\nabla u_\lambda\big) = \lambda^{-(q-p+1)\gamma}|\nabla u_\lambda|^q, & x \in \Omega,\ t \in \left(0, \dfrac{T}{\lambda}\right), \\ u_\lambda(x,t) = \lambda^\gamma g(x), & x \in \partial\Omega,\ t \in \left(0, \dfrac{T}{\lambda}\right), \\ u_\lambda(x,0) = \lambda^\gamma u_0(x), & x \in \Omega. \end{cases}$$

Set $v(x,t) = u(x,t) + k$ where $k := (\lambda^\gamma - 1)\sup(\|u_0\|_{L^\infty}, \|g\|_{L^\infty})$, then v satisfies the same equation (1.1) as u with $v(x,0) = u_0(x) + k$ and $v(x,t) = g(x) + k$ on $\partial\Omega \times (0,T)$. Given that $\lambda^\gamma u_0(x) = u_0(x) + (\lambda^\gamma - 1)u_0(x) \leqslant u_0(x) + (\lambda^\gamma - 1)\|u_0\|_{L^\infty}$ and $\lambda^\gamma g(x) \leqslant g(x) + (\lambda^\gamma - 1)\|g\|_{L^\infty}$, we have $u_\lambda(x,0) \leqslant v(x,0)$ in Ω and $u_\lambda \leqslant v$ in $\partial\Omega \times (0, \frac{T}{\lambda})$. Since $\lambda > 1$ and $q > p-1$, we have $\lambda^{-(q-p+1)\gamma}|\nabla u_\lambda|^q \leqslant |\nabla u_\lambda|^q$ and hence u_λ is a sub-solution of Eq. (1.1). Using Proposition 2.1, we have $u_\lambda(x,t) \leqslant v(x,t)$ in $\Omega \times (0, \frac{T}{\lambda})$ that is

$$\lambda^\gamma u(x,\lambda t) - u(x,t) \leqslant \big(\lambda^\gamma - 1\big)\sup\big(\|u_0\|_{L^\infty}, \|g\|_{L^\infty}\big). \tag{2.27}$$

Dividing (2.27) by $(\lambda - 1)$ and letting $\lambda \to 1^+$, we get

$$\gamma u(x,t) + t\,\partial_t u(x,t) \leqslant \gamma \sup\big(\|u_0\|_{L^\infty}, \|g\|_{L^\infty}\big).$$

We conclude using the positivity of u.

Remark 2.2. The homogeneity of the operator and the boundedness of u are essential.

3. Gradient estimate: proof of Theorem 1.2

The proof of (1.5) relies on a modification of the Bernstein technique and the use of a suitable cut-off function. It requires the study of the partial differential equation satisfied by $|\nabla u|^2$. We follow the ideas used in [32] and [6]. Let $x_0 \in \Omega$ be fixed, $0 < t_0 < T < T_{\max}(u_0)$, $R > 0$ such that $B(x_0,R) \subset \Omega$ and write $Q^{t_0}_{T,R} = B(x_0,R) \times (t_0,T)$

Let $\alpha \in (0,1)$ and set $R' = \frac{3R}{4}$. We select a cut-off function $\eta \in C^2(\overline{B}(x_0,R'))$, $0 < \eta < 1$, with $\eta(x_0) = 1$ and $\eta = 0$ for $|x - x_0| = R'$, such that

$$\left.\begin{aligned} &|\nabla\eta| \leqslant CR^{-1}\eta^\alpha \\ &\big|D^2\eta\big| + \eta^{-1}|\nabla\eta|^2 \leqslant CR^{-2}\eta^\alpha \end{aligned}\right\} \quad \text{for } |x - x_0| < R' \tag{3.1}$$

with $C = C(\alpha) > 0$ (see [32] for an example of such function).

First let us state the following lemma.

Lemma 3.1. *Let u_0, u be as in Theorem 1.2. We denote $w = |\nabla u|^2$ and $z = \eta w$. Then at any point $(x_1,t_1) \in Q^{t_0}_{T,R'}$ such that $|\nabla u(x_1,t_1)| > 0$, z is smooth and satisfies the following differential inequality*

$$\mathcal{L}z + Cz^{\frac{2q-p+2}{2}} \leqslant C\left(\frac{\sup(\|u_0\|_{L^\infty}, \|g\|_{L^\infty})}{t_0}\right)^{\frac{2q-p+2}{q}} + CR^{-\frac{2q-p+2}{q-p+1}},$$

where

$$\mathcal{L}z = \partial_t z - \mathcal{A}z - H \cdot \nabla z, \tag{3.2}$$

$$\mathcal{A}z = |\nabla u|^{p-2}\Delta z + (p-2)|\nabla u|^{p-4}(\nabla u)^t D^2 z \nabla u, \tag{3.3}$$

H is defined by (3.6) *and $C = C(p, q, N) > 0$.*

Proof of Lemma 3.1. At points where $|\nabla u| > 0$ Eq. (1.1) is uniformly parabolic and weak solutions are smooth at these points [20]. More precisely, we know that $\nabla u \in C^{2,1}$ in a neighborhood of such points and hence we can differentiate the equation. As observed in [6], $w = |\nabla u|^2$ satisfies the following differential equation:

$$\partial_t w - \mathcal{A}w = -2|\nabla u|^{p-2}|D^2 u|^2 + H \cdot \nabla w.$$

Indeed, for $i = 1, \dots, N$, put $u_i = \frac{\partial u}{\partial x_i}$ and $w_i = \frac{\partial w}{\partial x_i}$. Differentiating (1.1) in x_i, we have

$$\begin{aligned} &\partial_t u_i - |\nabla u|^{p-2}\Delta u_i - \frac{p-2}{2}|\nabla u|^{p-4}\sum_{j=1}^N \frac{\partial w_i}{\partial x_j}u_j - \frac{p-2}{2}|\nabla u|^{p-4}\sum_{j=1}^N w_j \frac{\partial u_i}{\partial x_j} \\ &\quad = \frac{q}{2}w^{\frac{q-2}{2}}w_i + \frac{p-2}{2}w^{\frac{p-4}{2}}w_i \Delta u + \frac{(p-2)(p-4)}{4}w^{\frac{p-6}{2}}(\nabla u \cdot \nabla w)w_i. \end{aligned} \tag{3.4}$$

Multiplying (3.4) by $2u_i$, summing up, and using $\Delta w = 2\nabla u \cdot \nabla(\Delta u) + 2|D^2 u|^2$, we deduce that

$$\mathcal{L}w = -2w^{\frac{p-2}{2}}|D^2 u|^2, \tag{3.5}$$

where

$$\begin{aligned} H := &\left[(p-2)w^{\frac{p-4}{2}}\Delta u + \frac{(p-2)(p-4)}{2}w^{\frac{p-6}{2}}\nabla u \cdot \nabla w + q w^{\frac{q-2}{2}}\right]\nabla u \\ &+ \frac{p-2}{2}w^{\frac{p-4}{2}}\nabla w. \end{aligned} \tag{3.6}$$

Setting $z = \eta w$, we get

$$\mathcal{L}z = \eta\mathcal{L}w + w\mathcal{L}\eta - 2w^{\frac{p-2}{2}}\nabla\eta \cdot \nabla w - 2(p-2)w^{\frac{p-4}{2}}(\nabla\eta \cdot \nabla u)(\nabla w \cdot \nabla u).$$

Now we shall estimate the different terms. In what follows $\delta_i > 0$ can be chosen arbitrarily small.

- Estimate of $|2w^{\frac{p-2}{2}}\nabla\eta \cdot \nabla w|$.
  Using Young's inequality, we have

$$\left|2w^{\frac{p-2}{2}}\nabla\eta \cdot \nabla w\right| \leqslant w^{\frac{p-2}{2}}\left[C\eta^{-1}|\nabla\eta|^2 w + \delta_1 \eta \left|D^2 u\right|^2\right], \tag{3.7}$$

  where we used the fact that $\nabla w = 2D^2 u \nabla u$.
- Estimate of $|2(p-2)w^{\frac{p-4}{2}}(\nabla\eta \cdot \nabla u)(\nabla w \cdot \nabla u)|$.

$$\left|2(p-2)w^{\frac{p-4}{2}}(\nabla\eta \cdot \nabla u)(\nabla w \cdot \nabla u)\right| \leqslant w^{\frac{p-2}{2}}\left[C\eta^{-1}|\nabla\eta|^2 w + \delta_2 \eta \left|D^2 u\right|^2\right]. \tag{3.8}$$

- Estimate of $|wH\cdot\nabla\eta|$.

$$|wH\cdot\nabla\eta|\leqslant \underbrace{w^{\frac{p-2}{2}}\left(C\eta^{-1}|\nabla\eta|^2w+\delta_3\left|D^2u\right|^2\eta\right)}_{(1)}+\underbrace{w^{\frac{p-2}{2}}\left(C\eta^{-1}|\nabla\eta|^2w+\delta_4\left|D^2u\right|^2\eta\right)}_{(2)}$$
$$+\underbrace{w^{\frac{p-2}{2}}\left(C\eta^{-1}|\nabla\eta|^2w+\delta_5\left|D^2u\right|^2\eta\right)}_{(3)}+Cw^{\frac{q+1}{2}}|\nabla\eta|. \tag{3.9}$$

(1) comes from an estimate based on Young's inequality of $w^{\frac{p-2}{2}}\Delta u(\nabla u\cdot\nabla\eta)$, (2) comes from (3.8) and (3) comes from an estimate of $w^{\frac{p-2}{2}}\nabla w\cdot\nabla\eta$.

Finally choosing δ_i such that $-2+\delta_1+\delta_2+\delta_3+\delta_4+\delta_5=-1$, we arrive at

$$\mathcal{L}z+\eta w^{\frac{p-2}{2}}\left|D^2u\right|^2\leqslant C(p,q,N)w^{\frac{p}{2}}\left[\left|D^2\eta\right|+|\Delta\eta|+\eta^{-1}|\nabla\eta|^2\right]+C|\nabla\eta|w^{\frac{q+1}{2}}.$$

Using the properties of the cut-off function η, we get

$$\mathcal{L}z+\eta w^{\frac{p-2}{2}}\left|D^2u\right|^2\leqslant C(p,q,N)R^{-2}\eta^{\alpha}w^{\frac{p}{2}}+C(p,q,N)R^{-1}\eta^{\alpha}w^{\frac{q+1}{2}}. \tag{3.10}$$

Using the result of Theorem 1.3, we shall estimate $|\nabla u|^{p-2}|D^2u|^2$ in terms of a power of w. For $(x_1,t_1)\in Q^{t_0}_{T,R'}$ such that $|\nabla u(x_1,t_1)|>0$, we have

$$\begin{aligned}\left|\nabla u(x_1,t_1)\right|^q&=\partial_t u(x_1,t_1)-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u(x_1,t_1)\right)\\&\leqslant\frac{\sup(\|u_0\|_{L^\infty},\|g\|_{L^\infty})}{(p-2)t_0}+(p-2+\sqrt{N})|\nabla u|^{p-2}\left|D^2u(x_1,t_1)\right|.\end{aligned}$$

Hence

$$\frac{1}{2(p-2+\sqrt{N})^2}\left|\nabla u(x_1,t_1)\right|^{2q}\leqslant\left(\frac{\sup(\|u_0\|_{L^\infty},\|g\|_{L^\infty})}{(p-2)(p-2+\sqrt{N})t_0}\right)^2+|\nabla u|^{2p-4}\left|D^2u(x_1,t_1)\right|^2.$$

There are two cases:

$$\begin{aligned}\text{either}\quad&\frac{1}{2(p-2+\sqrt{N})^2}\left|\nabla u(x_1,t_1)\right|^{2q}\leqslant 2\left(\frac{\sup(\|u_0\|_{L^\infty}\|g\|_{L^\infty})}{(p-2)(p-2+\sqrt{N})t_0}\right)^2,\\ \text{or}\quad&\frac{1}{2(p-2+\sqrt{N})^2}\left|\nabla u(x_1,t_1)\right|^{2q-p+2}\leqslant 2|\nabla u|^{p-2}\left|D^2u(x_1,t_1)\right|^2.\end{aligned}$$

In both cases we arrive at

$$\begin{aligned}\frac{1}{C(N,p)}\left|\nabla u(x_1,t_1)\right|^{2q-p+2}\leqslant{}& C(p,q,N)\left(\frac{\sup(\|u_0\|_{L^\infty},\|g\|_{L^\infty})}{t_0}\right)^{\frac{2q-p+2}{q}}\\&+|\nabla u|^{p-2}\left|D^2u(x_1,t_1)\right|^2.\end{aligned}$$

Using this inequality, it follows from (3.10) that, at (x_1, t_1),

$$\mathcal{L}z + \frac{1}{C(N,p)}\eta w^{\frac{2q-p+2}{2}} \leqslant C(p,q,N)\left(\frac{\sup(\|u_0\|_{L^\infty}, \|g\|_{L^\infty})}{t_0}\right)^{\frac{2q-p+2}{q}} + CR^{-2}\eta^\alpha w^{\frac{p}{2}} + CR^{-1}\eta^\alpha w^{\frac{q+1}{2}}.$$

We take $\alpha = \frac{q+1}{2q-p+2} \in (0,1)$ (since $q > p-1$). Using Young's inequality, we have

$$CR^{-1}\eta^\alpha w^{\frac{q+1}{2}} \leqslant CR^{-\frac{2q-p+2}{q-p+1}} + \frac{1}{4C(N,p)}\eta w^{\frac{2q-p+2}{2}},$$
$$CR^{-2}\eta^\alpha w^{\frac{p}{2}} \leqslant CR^{-\frac{2q-p+2}{q-p+1}} + \frac{1}{4C(N,p)}\eta^{\frac{q+1}{p}} w^{\frac{2q-p+2}{2}}.$$

Using that $\eta \leqslant 1$, we get

$$\mathcal{L}z + \frac{1}{C(N,p)}\eta|\nabla u|^{2q-p+2} \leqslant C(p,q,N)\left(\frac{\sup(\|u_0\|_{L^\infty}, \|g\|_{L^\infty})}{t_0}\right)^{\frac{2q-p+2}{q}} + CR^{-\frac{2q-p+2}{q-p+1}} + \frac{1}{2C(N,p)}\eta|\nabla u|^{2q-p+2}.$$

Hence

$$\mathcal{L}z + \frac{1}{2C(N,p)}z^{\frac{2q-p+2}{2}} \leqslant C(p,q,N)\left(\frac{\sup(\|u_0\|_{L^\infty}, \|g\|_{L^\infty})}{t_0}\right)^{\frac{2q-p+2}{q}} + CR^{-\frac{2q-p+2}{q-p+1}}. \quad \square \tag{3.11}$$

Proof of Theorem 1.2. First let us note that by the proof of the local existence there exists $t_0 \in (0, T_{\max}(u_0))$ with $t_0 = t_0(M, p, q, N, \|g\|_{C^2})$, such that

$$\sup_{0\leqslant t\leqslant t_0} \|\nabla u\|_{L^\infty} \leqslant C\big(p, q, \Omega, M, \|g\|_{C^2}\big). \tag{3.12}$$

We also know that ∇u is a locally Hölder continuous function and thus z is a continuous function on $\overline{B(x_0, R')} \times [t_0, T] = \overline{Q}$, for any $T < T_{\max}(u_0)$. Therefore, unless $z \equiv 0$ in $\overline{Q}$, z must reach a positive maximum at some point $(x_1, t_1) \in \overline{B(x_0, R')} \times [t_0, T]$. Since $z = 0$ on $\partial B_{R'} \times [t_0, T]$, we deduce that $x_1 \in B_{R'}$. Therefore $\nabla z(x_1, t_1) = 0$ and $D^2 z(x_1, t_1) \leqslant 0$. Now we have either $t_1 = t_0$, or $t_0 < t_1 \leqslant T$. If $t_1 = t_0$, then

$$z(x_1, t_1) \leqslant \big\|\nabla u(t_0)\big\|^2_{L^\infty} \leqslant C\big(p, q, \Omega, M, \|g\|_{C^2}\big).$$

If $t_0 < t_1 \leqslant T$, we have $\partial_t z(x_1, t_1) \geqslant 0$ and therefore $\mathcal{L}z \geqslant 0$. Using (3.11) we arrive at

$$\frac{1}{2C(N,p)}z(x_1,t_1)^{\frac{2q-p+2}{2}} \leqslant C(p,q,N)\left(\frac{\sup(\|u_0\|_{L^\infty}, \|g\|_{L^\infty})}{t_0}\right)^{\frac{2q-p+2}{q}} + CR^{-\frac{2q-p+2}{q-p+1}}, \tag{3.13}$$

that is

$$\sqrt{z(x_1,t_1)} \leqslant C(p,q,N)\left(\frac{\sup(\|u_0\|_{L^\infty}, \|g\|_{L^\infty})}{t_0}\right)^{\frac{1}{q}} + C(p,q,N)R^{-\frac{1}{q-p+1}}. \tag{3.14}$$

Since $z(x_0,t)\leqslant z(x_1,t_1)$ and $\eta(x_0)=1$, we get

$$\left|\nabla u(x_0,t)\right|\leqslant C(p,q,N)\left(\frac{\sup(\|u_0\|_{L^\infty},\|g\|_{L^\infty})}{t_0}\right)^{\frac{1}{q}}+C(p,q,N)R^{-\frac{1}{q-p+1}}\quad \text{for} t\in[t_0,T].$$

The proof of (1.2) follows by taking $R=\delta(x_0)$, letting $T\to T_{\max}(u_0)$ and using (3.12). □

4. Blow-up criterion: proof of Theorem 1.4

Assume that $T_{\max}(u_0)=\infty$, taking φ_1^α as test-function, we have for any $\tau>0$,

$$\int_0^\tau\int_\Omega \partial_t u\,\varphi_1^\alpha\,dx\,dt=\int_0^\tau\int_\Omega|\nabla u|^q\,\varphi_1^\alpha\,dx\,dt-\alpha\int_0^\tau\int_\Omega|\nabla u|^{p-2}\,\varphi_1^{\alpha-1}\,\nabla u\cdot\nabla\varphi_1\,dx\,dt.\tag{4.1}$$

Set $y(t)=\int_\Omega u(t)\,\varphi_1^\alpha\,dx$. Since by definition $\partial_t u\in L^2_{\mathrm{loc}}((0,\infty);L^2(\Omega))$, we have $y\in W^{1,1}_{\mathrm{loc}}(0,\infty)$ and $y'(t)=\int_\Omega\partial_t u\,\varphi_1^\alpha\,dx$. Differentiating (4.1) with respect to τ we have, for a.e. $\tau>0$,

$$y'(\tau)=\int_\Omega\left|\nabla u(\tau)\right|^q\varphi_1^\alpha\,dx-\alpha\int_\Omega\left|\nabla u(\tau)\right|^{p-2}\varphi_1^{\alpha-1}\,\nabla u(\tau)\cdot\nabla\varphi_1\,dx.\tag{4.2}$$

Assume that $\alpha>\frac{p-1}{(q-p+1)}$. Since $q>p-1>1$ and $\|\nabla\varphi_1\|_{L^\infty}\leqslant C'$, using Hölder and Young inequalities we get:

$$\begin{aligned}\alpha\int_\Omega\left|\nabla u(\tau)\right|^{p-2}\varphi_1^{\alpha-1}\nabla u(\tau)\cdot\nabla\varphi_1\,dx&\leqslant\frac{1}{2}\int_\Omega\left|\nabla u(\tau)\right|^q\varphi_1^\alpha\,dx+C\int_\Omega\varphi_1^{\alpha-q/(q-p+1)}\,dx\\&\leqslant\frac{1}{2}\int_\Omega\left|\nabla u(\tau)\right|^q\varphi_1^\alpha\,dx+C.\end{aligned}$$

Here we used the fact that $\int_\Omega\varphi_1^{-l}\,dx<\infty$ for $l<1$. Therefore

$$y'(\tau)\geqslant\frac{1}{2}\int_\Omega\left|\nabla u(\tau)\right|^q\varphi_1^\alpha\,dx-C.$$

Assuming that $\alpha<q-1$, we get

$$\begin{aligned}\int_\Omega\left|\nabla u(\tau)\right|dx=\int_\Omega\left|\nabla u(\tau)\right|\varphi_1^{\frac{\alpha}{q}}\varphi_1^{-\frac{\alpha}{q}}\,dx&\leqslant\left(\int_\Omega\left|\nabla u(\tau)\right|^q\varphi_1^\alpha\,dx\right)^{1/q}\left(\int_\Omega\varphi_1^{\frac{-\alpha}{q-1}}\,dx\right)^{\frac{q-1}{q}}\\&\leqslant C\left(\int_\Omega\left|\nabla u(\tau)\right|^q\varphi_1^\alpha\,dx\right)^{1/q}.\end{aligned}$$

On the other hand using that $\int_\Omega|u(\tau)|\,dx\leqslant C\|u\|_{L^\infty(\partial\Omega)}+C\int_\Omega|\nabla u(\tau)|\,dx$, we have

$$\int_\Omega u(\tau)\,\varphi_1^\alpha\,dx\leqslant\left\|\varphi_1^\alpha\right\|_{L^\infty}\int_\Omega u(\tau)\,dx\leqslant C+C\int_\Omega\left|\nabla u(\tau)\right|dx.$$

Combining these two inequalities we arrive at

$$\int_{\Omega} \left|\nabla u(\tau)\right|^q \varphi_1^\alpha \, dx \geqslant C\left(\int_{\Omega} u(\tau)\, \varphi_1^\alpha \, dx\right)^q - C.$$

Finally we get the blow-up inequality

$$y'(\tau) \geqslant C_1 y(\tau)^q - C_2, \quad \text{for a.e. } \tau > 0,$$

with $C_1 = C_1(p,q,\Omega) > 0$ and $C_2 = C_2(p,q,\alpha,\Omega,\|g\|_{L^\infty})$.

Remark 4.1. Instead of assuming that $\int_\Omega u_0\varphi_1^\alpha \, dx$ is large in Theorem 1.4, it would be sufficient to assume that $\|u_0\|_{L^r}$ is large for some $r \in [1,\infty)$. In fact, assuming without loss of generality $r \geqslant (2q-p)/(q-p)$ and denoting $y(t) = \int_\Omega u^r(t)\, dx$, the Poincaré and Hölder inequalities can be used in order to prove the blow-up inequality $y' \geqslant C_1 y^{(q+r-1)/r} - C_2$ (see [22]).

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References

[1] L. Amour, M. Ben-Artzi, Global existence and decay for viscous Hamilton–Jacobi equations, Nonlinear Anal. 31 (1998) 621–628.

[2] D. Andreucci, A. Tedeev, M. Ughi, The Cauchy problem for degenerate parabolic equations with source and damping, Ukr. Math. Bull. 1 (2004) 1–23.

[3] J.M. Arrieta Rodriguez-Bernal, P. Souplet, Boundedness of global solutions for nonlinear parabolic equations involving gradient blow-up phenomena, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 3 (2004) 1–15.

[4] A. Attouchi, Boundedness of global solutions of a p-Laplacian evolution equation with a nonlinear gradient term, in preparation, 2012.

[5] G. Barles, P. Laurençot, C. Stinner, Convergence to steady states for radially symmetric solutions to a quasilinear degenerate diffusive Hamilton–Jacobi equation, Asymptot. Anal. 67 (2010) 229–250.

[6] J.-Ph. Bartier, Ph. Laurençot, Gradient estimates for a degenerate parabolic equation with gradient absorption and applications, J. Funct. Anal. 254 (2008) 851–878.

[7] M. Ben-Artzi, P. Souplet, F. Weissler, The local theory for viscous Hamilton–Jacobi equations in Lebesgue spaces, J. Math. Pures Appl. 81 (2002) 343–378.

[8] S. Benachour, S. Dăbuleanu-Hapca, P. Laurençot, Decay estimates for a viscous Hamilton–Jacobi equation with homogeneous Dirichlet boundary conditions, Asymptot. Anal. 51 (2007) 209–229.

[9] B. Bojarski, T. Iwaniec, p-Harmonic equation and quasiregular mappings, in: Partial Differential Equations, Warsaw, 1984, in: Banach Center Publ., vol. 19, 1987, pp. 25–38.

[10] C. Chen, M. Nakao, Y. Ohara, Global existence and gradient estimates for quasilinear parabolic equations of the m-Laplacian type with a strong perturbation, Adv. Math. Sci. Appl. 10 (2000) 225–237.

[11] M. Chen, J. Zhao, On the Cauchy problem of evolution p-Laplacian equation with nonlinear gradient term, Chin. Ann. Math. Ser. B 30 (2009) 1–16.

[12] E. DiBenedetto, A. Friedman, Hölder estimates for nonlinear degenerate parabolic systems, J. Reine Angew. Math. (Crelles J.) 357 (1985) 1–22.

[13] T. Dlotko, Examples of parabolic problems with blowing-up derivatives, J. Math. Anal. Appl. 154 (1991) 22–237.

[14] J. Esteban, P. Marcati, Approximate solutions to first and second order quasilinear evolution equations via nonlinear viscosity, Trans. Amer. Math. Soc. 342 (1994) 501–521.

[15] Y. Giga, Interior derivative blow-up for quasilinear parabolic equations, Discrete Contin. Dyn. Syst. 1 (1995) 449–461.

[16] J. Guo, B. Hu, Blowup rate estimates for the heat equation with a nonlinear gradient source term, Discrete Contin. Dyn. Syst. 20 (2008) 927–937.

[17] M. Hesaaraki, A. Moameni, Blow-up of positive solutions for a family of nonlinear parabolic equations in general domain in $\mathbb{R}^N$, Michigan Math. J. 52 (2004) 375–389.

[18] R.G. Iagar, P. Laurençot, J.L. Vázquez, Asymptotic behaviour of a nonlinear parabolic equation with gradient absorption and critical exponent, preprint, 2010.

[19] B. Kawohl, N. Kutev, Comparison principle and Lipschitz regularity for viscosity solutions of some classes of nonlinear partial differential equations, Funkcial. Ekvac. 43 (2000) 241–253.
[20] O. Ladyzhenskaya, V. Solonnikov, N. Ural'ceva, Linear and Quasi-Linear Equations of Parabolic Type, American Mathematical Society, 1968.
[21] P. Laurençot, Non-diffusive large time behavior for a degenerate viscous Hamilton–Jacobi equation, Comm. Partial Differential Equations 34 (2009) 281–304.
[22] P. Laurençot, C. Stinner, Convergence to separate variables solutions for a degenerate parabolic equation with gradient source, J. Dynam. Differential Equations 24 (2012) 29–49.
[23] P. Laurençot, J.L. Vázquez, Localized non-diffusive asymptotic patterns for nonlinear parabolic equations with gradient absorption, J. Dynam. Differential Equations 19 (2007) 985–1005.
[24] Y. Li, P. Souplet, Single-point gradient blow-up on the boundary for diffusive Hamilton–Jacobi equations in planar domains, Comm. Math. Phys. 293 (2010) 499–517.
[25] G. Lieberman, The first initial-boundary value problem for quasilinear second order parabolic equations, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 13 (1986) 347–387.
[26] P. Lindqvist, Notes on the p-Laplace equation, http://www.math.ntnu.no/~lqvist/p-laplace.pdf, 2006.
[27] P. Quittner, P. Souplet, Superlinear Parabolic Problems: Blow-up, Global Existence and Steady States, Birkhäuser, 2007.
[28] P. Shi, Self-similar singular solution of a p-Laplacian evolution equation with gradient absorption term, J. Partial Differ. Equ. 17 (2004) 369–383.
[29] J. Simon, Compact sets in the space $L^p(0, T; B)$, Ann. Mat. Pura Appl. 146 (1987) 65–96.
[30] P. Souplet, Gradient blow-up for multidimensional nonlinear parabolic equations with general boundary conditions, Differential Integral Equations 15 (2002) 237–256.
[31] P. Souplet, J.L. Vázquez, Stabilization towards a singular steady state with gradient blow-up for a diffusion–convection problem, Discrete Contin. Dyn. Syst. 14 (2006) 221–234.
[32] P. Souplet, Q. Zhang, Global solutions of inhomogeneous Hamilton–Jacobi equations, J. Anal. Math. 99 (2006) 355–396.
[33] C. Stinner, Convergence to steady states in a viscous Hamilton–Jacobi equation with degenerate diffusion, J. Differential Equations 248 (2010) 209–228.
[34] J. Zhao, Existence and nonexistence of solutions for $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) + f(\nabla u, u, x, t)$, J. Math. Anal. Appl. 172 (1993) 130–146.
[35] J. Zhao, A note to the regularity of solutions for the evolution p-Laplacian equations, Methods Appl. Anal. 8 (2001) 595–598.