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Tau approximate solution of fractional partial differential equations

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ABSTRACT

In this study, we improve the algebraic formulation of the fractional partial differential equations (FPDEs) by using the matrix-vector multiplication representation of the problem. This representation allows us to investigate an operational approach of the Tau method for the numerical solution of FPDEs. We introduce a converter matrix for the construction of converted Chebyshev and Legendre polynomials which is applied in the operational approach of the Tau method. We present the advantages of using the method and compare it with several other methods. Some experiments are applied to solve FPDEs including linear and nonlinear terms. By comparing the numerical results obtained from the other methods, we demonstrate the high accuracy and efficiency of the proposed method.

1. Introduction

Fractional partial differential equations have received considerable interest in recent years and have been extensively investigated and applied for many real problems which are modeled in various areas. For instance, in mathematical physics [1], in fluid and continuum mechanics [2], viscoplastic and viscoelastic flows [3], biology, chemistry, acoustics and psychology [4]. Furthermore, FPDEs have been the focus of numerous studies. In addition, considerable attention has been given to the solutions of FPDEs of physical interest [1,5]. Some of FPDEs have been studied and solved, such as the fractional transport equation [6,7], the fractional Fokker–Planck equation [8], the fractional KdV equation [9], the space and time-fractional diffusion–wave equation [10], and the linear and nonlinear fractional diffusion–wave equation [11].

Most of FPDEs do not have exact analytic solutions, so approximation and numerical techniques must be used. There are several approximation and numerical methods and there has been a growing interest to develop approximate numerical techniques to solve FDEs. The most commonly used ones are the homotopy perturbation method (HPM) [12,13], Adomian decomposition method (ADM) [14,15], variational iteration method (VIM) [12], generalized differential transform method (GDTM) [16], Chebyshev spectral approximation [6] and Walsh function method [7]. From various aforementioned methods, we would like to present the operational approach of the Tau method to solve FPDEs.

The Tau method has been applied to a wide class of ordinary differential equations (ODEs), partial differential equations (PDEs), integral equations (IEs) and integro-differential equations (IDEs). In 1981, Ortiz and Samara [17] presented an operational technique for the numerical solution of nonlinear ODEs with some supplementary conditions based on the standard Tau method [18]. The theoretical analysis and numerical applications of this method have been described in a series of papers see [19–23] for linear ordinary differential eigenvalue problems. The method was developed for the numerical solution of PDEs and their related eigenvalue problems, iterated solutions of linear operator equations [24–28] and for IDEs [29], too.

The main objective of this work is the development of the operational Tau method (OTM) for the numerical solution of FPDEs. Because in the Tau method, we are dealing with a system of equations wherein the matrix of unknown coefficients

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is sparse and easily invertible. Furthermore, the integral part appearing in the equation is replaced by its operational Tau representation and then we obtain a system of algebraic equations wherein its solution is easy.

The paper has been organized as follows. Section 2 gives notations and basic definitions of the fractional calculus. Section 3 is devoted to introduce the OTM and its application to the FPDEs. Some illustrative numerical experiments are given in Section 4, followed by the discussions and conclusions presented in Section 5.

2. Basic definitions of the fractional calculus

In this section, we state the definition and preliminaries of fractional calculus [1].

Definition 1. For m to be the smallest integer that exceeds α, Caputo's space-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_x^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \begin{cases} J_x^{m-\alpha} D_x^m u(x,t), & \text{if } m-1 < \alpha < m, \\ D_x^m u(x,t), & \text{if } \alpha = m, m \in \mathbb{N}, \end{cases} \tag{1}$$

where

$$J_x^\alpha u(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\sigma)^{\alpha-1} u(\sigma,t) \mathrm{d}\sigma, \quad \alpha > 0,\ x > 0, \tag{2}$$

and the time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} J_t^{m-\alpha} D_t^m u(x,t), & \text{if } m-1 < \alpha < m, \\ D_t^m u(x,t), & \text{if } \alpha = m, m \in \mathbb{N}, \end{cases} \tag{3}$$

where

$$J_t^\alpha u(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(x,\tau) \mathrm{d}\tau, \quad \alpha > 0,\ t > 0. \tag{4}$$

Some of the most important properties of operators J_x^α and J_t^α are as follows:

i. $J_x^\alpha(x^\gamma) = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$,
ii. $J_t^\alpha(t^\gamma) = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$,
iii. $J_x^\alpha(x^\gamma t^\beta) = t^\beta J_x^\alpha(x^\gamma)$,
iv. $J_t^\alpha(x^\gamma t^\beta) = x^\gamma J_t^\alpha(t^\beta)$.

Caputo's fractional differentiations are the linear operators given as follows:

$$D_x^\alpha(\lambda u(x,t) + \eta v(x,t)) = \lambda D_x^\alpha u(x,t) + \eta D_x^\alpha v(x,t),$$
$$D_t^\alpha(\lambda u(x,t) + \eta v(x,t)) = \lambda D_t^\alpha u(x,t) + \eta D_t^\alpha v(x,t),$$

where λ and η are constants.

3. Implementation of the OTM on FPDEs

In this section, we state the required preliminaries of the Tau method to apply on fractional order PDEs. The main idea of the method is seeking a polynomial solution to solve an FPDE. Let us consider a PFDE in the form:

$$D^\alpha u + Lu = f, \quad \text{in } \Omega \subset \mathbb{R}^d, \tag{5}$$
$$Bu = g, \quad \text{on } \partial\Omega, \tag{6}$$

where d is the dimension, $\partial\Omega$ denotes the boundary of the domain Ω, L is the differential operator, D^α is the fractional differential operator of order α that operates on the interior, and B is an operator that specifies the boundary conditions. Both the f and $g : \mathbb{R}^d \to \mathbb{R}$ are known functions.

If we suppose that $u = \underline{u}\Phi X$, where ΦX is a base for polynomials in $\mathbb{R}^d$ and $\underline{u}$ is an unknown coefficient vector, then our aim is to convert $D^\alpha u$, Lu, Bu, f and g into algebraic forms based on the base of ΦX. Suppose that there exist $\underline{D}$, $\underline{L}$, $\underline{B}$, $\underline{f}$ and $\underline{g}$ such that:

$$D^\alpha u = \underline{u}\,\underline{D}\Phi X, \qquad Lu = \underline{u}\,\underline{L}\Phi X, \qquad Bu = \underline{u}\,\underline{B}\Phi X, \qquad f = \underline{f}\Phi X, \qquad g = \underline{g}\Phi X.$$

So, Eqs. (5) and (6) are replaced by

$$\underline{u}\,\underline{D}\Phi X + \underline{u}\,\underline{L}\Phi X = \underline{f}\Phi X, \quad \text{in } \Omega \subset \mathbb{R}^d,$$
$$\underline{u}\,\underline{B}\Phi X = \underline{g}\Phi X, \quad \text{on } \partial\Omega.$$

Therefore, we obtain the system $\underline{u}(\underline{D} + \underline{L}, \underline{B}) = (\underline{f}, \underline{g})$. Solving this system yields the unknown coefficients $\underline{u}$.

In this study we consider two dimensional FPDEs. At first, we assert some lemmas to apply the Tau method on FPDEs. Let us suppose that the function $u(x,t)$ is defined on $[a,b]\times[c,d]$ and is infinitely differentiable. Assume that

$$\underline{X_x}=[1,x,x^2,\ldots,x^n,\ldots]^T,$$
$$\underline{X_t}=[1,t,t^2,\ldots,t^n,\ldots]^T,$$

are standard base polynomials in $\mathbb{R}$ and

$$\underline{X_{x,t}}=\underline{X_x}\otimes\underline{X_t},$$

is the standard base polynomials in $\mathbb{R}^2$; also $\otimes$ denotes the Kronecker product.

We seek a polynomial in $\mathbb{R}^2$ to approximate $u(x,t)$ such that

$$u(x,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{i,j}\phi_i(x)\psi_j(t),$$

where $\underline{\phi_x}=\{\phi_i(x)\}_{i=0}^{\infty}$ and $\underline{\psi_t}=\{\psi_j(t)\}_{j=0}^{\infty}$ are the sets of arbitrary orthogonal polynomial bases defined on a one dimensional space. Since, we use finite series in the Tau method, we suppose that the approximation is as

$$u(x,t)=\sum_{i=0}^{n}\sum_{j=0}^{n}u_{i(n+1)+j}\phi_i(x)\psi_j(t). \tag{7}$$

By multiplying $\phi_i(x)$ and $\psi_j(t)$ in the above summation and extracting its coefficient matrix as Φ with respect to base $\underline{X_{x,t}}$, we can assume that the approximate solution of $u(x,t)$ is as

$$u(x,t)=\underline{u}\Phi\underline{X_{x,t}}. \tag{8}$$

In continuation, we consider the following lemmas.

Lemma 1. *Suppose that* $\underline{u}=[u_0,u_1,u_2,\ldots,u_{(n+1)^2-1},0,0,\ldots]$, $\Phi=[\phi_0|\phi_1|\phi_2|\ldots]$, ϕ_i *are infinite columns of* Φ, $\underline{X_{x,t}}=\underline{X_x}\otimes\underline{X_t}$ *and* $u(x,t)=\underline{u}\Phi\underline{X_{x,t}}$ *is a polynomial. Then, we have:*

$$\text{i. } D_x^r u(x,t)=\frac{\partial^r}{\partial x^r}u(x,t)=\underline{u}\Phi\eta_x^r\underline{X_{x,t}},\quad r=1,2,\ldots, \tag{9}$$

where

$$(\eta_x)_{i,j}=\begin{cases}k, & k(n+1)+1\le i=j\le(k+1)(n+1),\ k=0,1,\ldots,n,\\ 0, & \textit{otherwise}.\end{cases}$$

$$\text{ii. } D_t^r u(x,t)=\frac{\partial^r}{\partial t^r}u(x,t)=\underline{u}\Phi\eta_t^r\underline{X_{x,t}},\quad r=1,2,\ldots, \tag{10}$$

where

$$(\eta_t)_{i,j}=\begin{cases}A_0, & i=j,\ i=1,2,\ldots,n,\\ 0, & \textit{otherwise},\end{cases}\quad \textit{and}\quad (A_0)_{p,q}=\begin{cases}q, & p=q+1,\ q=1,2,\ldots,n,\\ 0, & \textit{otherwise}.\end{cases}$$

$$\text{iii. } t^s u(x,t)=\underline{u}\Phi\mu_t^s\underline{X_{x,t}},\quad s=1,2,\ldots \tag{11}$$

where

$$(\mu_t)_{i,j}=\begin{cases}B_0, & i=j,\ i=1,2,\ldots,n,\\ 0, & \textit{otherwise},\end{cases}\quad \textit{and}\quad (B_0)_{p,q}=\begin{cases}1, & q=p+1,\ p=1,2,\ldots,n,\\ 0, & \textit{otherwise}.\end{cases}$$

$$\text{iv. } x^s u(x,t)=\underline{u}\Phi\mu_x^s\underline{X_{x,t}},\quad s=1,2,\ldots \tag{12}$$

where

$$(\mu_x)_{i,j}=\begin{cases}1, & j=i+n+1,\ i=1,2,\ldots,n(n+1),\\ 0, & \textit{otherwise}.\end{cases}$$

Proof. The proof of these properties can be obtained by simple computations, which we neglect here due to the increased volume of the paper. □

Lemma 2. *By all assumptions in Lemma 1, we have*

$$\underline{X_{x,t}}\,\underline{u}\Phi\underline{X_{x,t}}=U\underline{X_{x,t}}, \tag{13}$$

where U is an upper triangular matrix as

$$U_{i,j}=\begin{cases}\sum_{k=0}^{\infty}u_k\phi_{k,j-i}, & i=1,2,\dots,\ j\geq i,\\ 0, & j<i,\end{cases}$$

and for any positive integer s, the relation

$$u^s(x,t)=\underline{u}\Phi\, U^{s-1}\underline{X_{x,t}}, \tag{14}$$

is valid.

Proof. We obtain

$$\begin{aligned}\underline{X_{x,t}}\,\underline{u}&=[\{1,t,t^2,\dots\},\{1,t,t^2,\dots\}x,\{1,t,t^2,\dots\}x^2,\dots]^T[u_0,u_1,u_2,\dots]\\&=[\{1,t,t^2,\dots\}\underline{u},\{1,t,t^2,\dots\}x\underline{u},\{1,t,t^2,\dots\}x^2\underline{u},\dots]^T,\end{aligned}$$

and

$$\begin{aligned}\underline{X_{x,t}}\,\underline{u}\Phi&=[\{1,t,t^2,\dots\}\underline{u},\{1,t,t^2,\dots\}x\underline{u},\{1,t,t^2,\dots\}x^2\underline{u},\dots]^T[\phi_0|\phi_1|\phi_2|\dots]\\&=[(\underline{u}\phi_0)\underline{X_{x,t}}|(\underline{u}\phi_1)\underline{X_{x,t}}|(\underline{u}\phi_2)\underline{X_{x,t}}|\dots]^T,\end{aligned}$$

so

$$\begin{aligned}\underline{X_{x,t}}\,\underline{u}\Phi\underline{X_{x,t}}&=[(\underline{u}\phi_0)\underline{X_{x,t}}|(\underline{u}\phi_1)\underline{X_{x,t}}|(\underline{u}\phi_2)\underline{X_{x,t}}|\dots]^T[\{1,t,t^2,\dots\},\{1,t,t^2,\dots\}x,\{1,t,t^2,\dots\}x^2,\dots]^T\\&=\begin{bmatrix}\underline{u}\phi_0 & \underline{u}\phi_1 & \underline{u}\phi_2 & \underline{u}\phi_3 & \cdots\\ 0 & \underline{u}\phi_0 & \underline{u}\phi_1 & \underline{u}\phi_2 & \cdots\\ 0 & 0 & \underline{u}\phi_0 & \underline{u}\phi_1 & \cdots\\ 0 & 0 & 0 & \underline{u}\phi_0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}[\{1,t,t^2,\dots\},\{1,t,t^2,\dots\}x,\{1,t,t^2,\dots\}x^2,\dots]^T.\end{aligned}$$

If we let U to be the last coefficient matrix, then we obtain

$$\underline{X_{x,t}}\,\underline{u}\Phi\underline{X_{x,t}}=U\underline{X_{x,t}},$$

where

$$U_{i,j}=\begin{cases}\underline{u}\phi_{j-i}, & i=1,2,\dots,\ j\geq i,\\ 0, & j<i,\end{cases}=\begin{cases}\sum_{k=0}^{\infty}u_k\phi_{k,j-i}, & i=1,2,\dots,\ j\geq i,\\ 0, & j<i.\end{cases}$$

To prove the second part of this lemma, we start with induction. For $s=1$, it is obvious that $u(x,t)=\underline{u}\Phi\underline{X_{x,t}}$. For $s=2$, we rewrite $u^2(x,t)=\underline{u}\Phi\underline{X_{x,t}}\underline{u}\Phi\underline{X_{x,t}}$ and using Eq. (13), we obtain

$$u^2(x,t)=\underline{u}\Phi U\underline{X_{x,t}},$$

therefore Eq. (14) holds for $s=2$. Now, suppose that Eq. (14) holds for $s=k$, then we must prove that for $s=k+1$ the relation is valid. Thus,

$$u^{k+1}(x,t)=u^k(x,t)u(x,t)=(\underline{u}\Phi U^{k-1}\underline{X_{x,t}})(\underline{u}\Phi\underline{X_{x,t}})=\underline{u}\Phi U^k\underline{X_{x,t}},$$

hence, Eq. (14) is proved. □

Now, our aim is to compute $D_x^{\alpha}u(x,t)$ and $D_t^{\alpha}u(x,t)$ in finite operational forms. So, we restrict the computations for finite vectors and matrices.

For computing $D_x^{\alpha}u(x,t)$, substituting Eq. (8) in Eq. (1), when $\alpha=m,\ m\in\mathbb{N}$, using Eq. (9) implies that:

$$D_x^{\alpha}u(x,t)=D_x^m(\underline{u}\Phi\underline{X_{x,t}})=\underline{u}\Phi\eta_x^m\underline{X_{x,t}}, \tag{15}$$

but when $m-1<\alpha<m$, using Eq. (9), we obtain

$$D_x^{\alpha}u(x,t)=J_x^{m-\alpha}D_x^m(\underline{u}\Phi\underline{X_{x,t}})=\underline{u}\Phi\eta_x^mJ_x^{m-\alpha}(\underline{X_{x,t}}). \tag{16}$$

Using Eq. (2) and property iii. of Section 2, we have

$$\begin{aligned}J_x^{m-\alpha}(\underline{X_{x,t}})=&[\{1,t,t^2,\dots,t^n\}J_x^{m-\alpha}(1),\{1,t,t^2,\dots,t^n\}J_x^{m-\alpha}(x),\\&\{1,t,t^2,\dots,t^n\}J_x^{m-\alpha}(x^2),\dots,\{1,t,t^2,\dots,t^n\}J_x^{m-\alpha}(x^n)]^T,\end{aligned} \tag{17}$$

but for $p, q = 0, 1, \ldots, n$ and using property i. of Section 2, we have

$$t^q J_x^{m-\alpha}(x^p) = t^q \Gamma_p x^{m-\alpha+p}, \quad \Gamma_p = \frac{\Gamma(p+1)}{\Gamma(m-\alpha+p+1)}. \tag{18}$$

By approximating $x^{m-\alpha+p}$, $p = 0, 1, 2, \ldots, n$; as follows:

$$x^{m-\alpha+p} = \sum_{i=0}^{n} a_{p,i} x^i = \underline{a_p}\, \underline{X_x}, \quad \underline{a_p} = [a_{p,0}, a_{p,1}, a_{p,2}, \ldots, a_{p,n}],$$

we obtain

$$t^q J_x^{m-\alpha}(x^p) = \Gamma_p\, t^q \underline{a_p}\, \underline{X_x}. \tag{19}$$

A simple computation shows that for $q = 0, 1, \ldots, n$, we can find a row matrix $V^{(p,q)}_{1\times(n+1)^2}$ such that:

$$\Gamma_p\, t^q \underline{a_p}\underline{X_x} = V^{(p,q)}\underline{X_{x,t}}, \qquad v_j^{(p,q)} = \begin{cases} \Gamma_p\, a_{p,r}, & j = r(n+1)+q+1,\ r = 0, 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

Therefore,

$$t^q J_x^{m-\alpha}(x^p) = V^{(p,q)}\underline{X_{x,t}}. \tag{21}$$

Substituting Eq. (21) into Eq. (17), we obtain

$$J_x^{m-\alpha}(\underline{X_{x,t}}) = V\underline{X_{x,t}} \tag{22}$$

where

$$V = [V^{(0,0)}, V^{(0,1)}, \ldots, V^{(0,n)}, V^{(1,0)}, V^{(1,1)}, \ldots, V^{(1,n)}, \ldots, \ldots, V^{(n,0)}, V^{(n,1)}, \ldots, V^{(n,n)}]^T.$$

Substituting Eq. (22) into Eq. (16), we have

$$D_x^\alpha u(x,t) = \underline{u}\Phi\eta_x^m V\underline{X_{x,t}}. \tag{23}$$

In the same way, for evaluating $D_t^\alpha u(x,t)$, we have: when $\alpha = m$, $m \in \mathbb{N}$, using Eq. (10) implies that

$$D_t^\alpha u(x,t) = D_t^m(\underline{u}\Phi\underline{X_{x,t}}) = \underline{u}\Phi\eta_t^m\underline{X_{x,t}}, \tag{24}$$

but when $m-1 < \alpha < m$, using Eq. (10), we obtain

$$D_t^\alpha u(x,t) = J_t^{m-\alpha}D_t^m(\underline{u}\Phi\underline{X_{x,t}}) = \underline{u}\Phi\eta_t^m J_t^{m-\alpha}(\underline{X_{x,t}}). \tag{25}$$

Using Eq. (4) and property iv. of Section 2, implies that:

$$\begin{aligned} J_t^{m-\alpha}(\underline{X_{x,t}}) = {} & [\{J_t^{m-\alpha}(1), J_t^{m-\alpha}(t), \ldots, J_t^{m-\alpha}(t^n)\}, \{J_t^{m-\alpha}(1), J_t^{m-\alpha}(t), \ldots, J_t^{m-\alpha}(t^n)\}x, \\ & \{J_t^{m-\alpha}(1), J_t^{m-\alpha}(t), \ldots, J_t^{m-\alpha}(t^n)\}x^2, \ldots, \{J_t^{m-\alpha}(1), J_t^{m-\alpha}(t), \ldots, J_t^{m-\alpha}(t^n)\}x^n]^T, \end{aligned} \tag{26}$$

but for $q = 0, 1, \ldots, n$ and using property ii. of Section 2, we have

$$J_t^{m-\alpha}(t^q) = \Gamma_q t^{m-\alpha+q}, \quad \Gamma_q = \frac{\Gamma(q+1)}{\Gamma(m-\alpha+q+1)}. \tag{27}$$

By approximating $t^{m-\alpha+q}$, $q = 0, 1, 2, \ldots, n$ as follows:

$$t^{m-\alpha+q} = \sum_{i=0}^{n} b_{q,i} t^i = \underline{b_q}\, \underline{X_t}, \quad \underline{b_q} = [b_{q,0}, b_{q,1}, b_{q,2}, \ldots, b_{q,n}].$$

If we let $B = [\Gamma_0\underline{b_0}, \Gamma_1\underline{b_1}, \ldots, \Gamma_n\underline{b_n}]^T$ and $W = [w_{i,j}]_{i,j=0}^n$, $w_{i,j} = \delta_{i,j}B$, then

$$J_t^{m-\alpha}(\underline{X_{x,t}}) = W\underline{X_{x,t}}. \tag{28}$$

Substituting Eq. (28) into Eq. (25), we obtain

$$D_t^\alpha u(x,t) = \underline{u}\Phi\eta_t^m W\underline{X_{x,t}}. \tag{29}$$

Therefore, for $m-1 < \alpha < m$, we have

$$D_x^\alpha u(x,t) = \underline{u}\Phi\eta_x^m V\underline{X_{x,t}} = \underline{u}\Upsilon\underline{X_{x,t}}, \quad \text{where } \Upsilon = \Phi\eta_x^m V, \tag{30}$$

and

$$D_t^\alpha u(x,t) = \underline{u}\Phi\eta_t^m W\underline{X_{x,t}} = \underline{u}\Omega\underline{X_{x,t}}, \quad \text{where } \Omega = \Phi\eta_t^m W. \tag{31}$$

Furthermore, these relations can be expressed in arbitrary polynomial bases as $D_x^\alpha u(x,t) = \underline{u}\Upsilon\Phi^{-1}\Phi\underline{X_{x,t}}$ and $D_t^\alpha u(x,t) = \underline{u}\Omega\Phi^{-1}\Phi\underline{X_{x,t}}$, where $\Phi\underline{X_{x,t}}$ is the polynomial base.

Lemma 3. *Suppose that $\underline{X_{x,t}} = \underline{X_x} \otimes \underline{X_t}$, $\underline{X_x} = [1, x, x^2, \dots, x^n]^T$ and $\underline{X_t} = [1, t, t^2, \dots, t^n]^T$. Then for any positive integers s and υ, the following relations are valid.*

$$(D_x^\alpha u(x,t))^s = \underline{u}\Upsilon K^{s-1}\underline{X_{x,t}}, \quad \text{where } K_{i,j} = \begin{cases} \sum_{k=0}^{n} u_k \Upsilon_{k,j-i}, & i = 1,2,\dots,n,\ j \ge i, \\ 0, & j < i, \end{cases} \tag{32}$$

and

$$(D_t^\alpha u(x,t))^\upsilon = \underline{u}\Omega P^{\upsilon-1}\underline{X_{x,t}}, \quad \text{where } P_{i,j} = \begin{cases} \sum_{k=0}^{n} u_k \Omega_{k,j-i}, & i = 1,2,\dots,n,\ j \ge i, \\ 0, & j < i. \end{cases} \tag{33}$$

Proof. The proof of this lemma is the same as Lemma 2. □

By using these advantages of the Tau method most of nonlinear problems can be solved and expressed as polynomials, easily. In summary, for any positive integers r, s and υ, we can write $u^r(x,t)$, $(D_x^\alpha u(x,t))^s$ and $(D_t^\alpha u(x,t))^\upsilon$ with respect to any arbitrary polynomial basis as follows:

$$u^r(x,t) = \underline{u}\Phi\, U^{r-1}\Phi^{-1}\Phi\underline{X_{x,t}}, \tag{34}$$

$$(D_x^\alpha u(x,t))^s = \underline{u}\Upsilon K^{s-1}\Phi^{-1}\Phi\underline{X_{x,t}}, \tag{35}$$

$$(D_t^\alpha u(x,t))^\upsilon = \underline{u}\Omega P^{\upsilon-1}\Phi^{-1}\Phi\underline{X_{x,t}}, \tag{36}$$

where, $\Phi\underline{X_{x,t}}$ is a polynomial base. It is obvious that any FPDEs and their supplementary conditions can be replaced with an algebraic expression based on polynomial base. Finally, if we choose finite algebraic equations gained in the problem and impose the supplementary conditions, then we can obtain a square system wherein its solution gives an unknown vector $\{u_k\}_{k=0}^{(n+1)^2}$.

We summarize the algorithm of the method as follows.

Algorithm of the Method. *Step* 1. Choose the set of Chebyshev polynomials $\{\phi_i(x)\}_{i=0}^n$ and $\{\psi_j(t)\}_{j=0}^n$ and let the approximate solution be $u(x,t) = \sum_{i=0}^n \sum_{j=0}^n u_{i(n+1)+j}\phi_i(x)\psi_j(t)$.

Step 2. Find the coefficient matrix Φ with respect to $\underline{X_{x,t}}$ such that $u(x,t) = \underline{u}\Phi\underline{X_{x,t}}$.

Step 3. Use Eqs. (34)–(36) and Lemma 1 to convert problem (5) to an algebraic system.

Step 4. Linearize the supplementary condition (6) in the same way as mentioned in Step 3. Let m be the number of equations obtained from supplementary conditions.

Step 5. Choose $(n+1)^2 - m$ equations from the first equations in the obtained system of Step 3.

Step 6. Solve the system obtained from Steps 4 and 5 to find the unknown coefficients $\underline{u} = \{u_k\}_{k=0}^{(n+1)^2-1}$.

4. Illustrative numerical experiments

In this section, some experiments of linear and nonlinear FPDEs are given to illustrate our results. In all experiments, we consider the shifted Chebyshev polynomials ($\underline{T}^* = \{T_i^*(x)\}_{i=0}^\infty = T^*\underline{X_x}$) as basis functions. It is noticeable that the results in shifted Legendre polynomials ($\underline{P}^* = \{P_i^*(x)\}_{i=0}^\infty = P^*\underline{X_x}$) has no more difference. Since converted Chebyshev (Legendre) polynomials are orthogonal, and reduce the run time of the source programs and the computation time, we can use them for a wide range of problems. They are orthogonal functions and they can be used to obtain a good approximation for transcendental functions. The computations associated with these experiments were performed in Maple 14 on a PC, CPU 2.4 GHz.

Experiment 4.1. Consider the following linear time-fractional wave equation,

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < t \le 1,\ 0 \le x \le 1,\ 1 < \alpha \le 2,$$

subject to the initial conditions

$$u(x,0) = x, \quad \text{and} \quad \frac{\partial u(x,0)}{\partial t} = x^2.$$

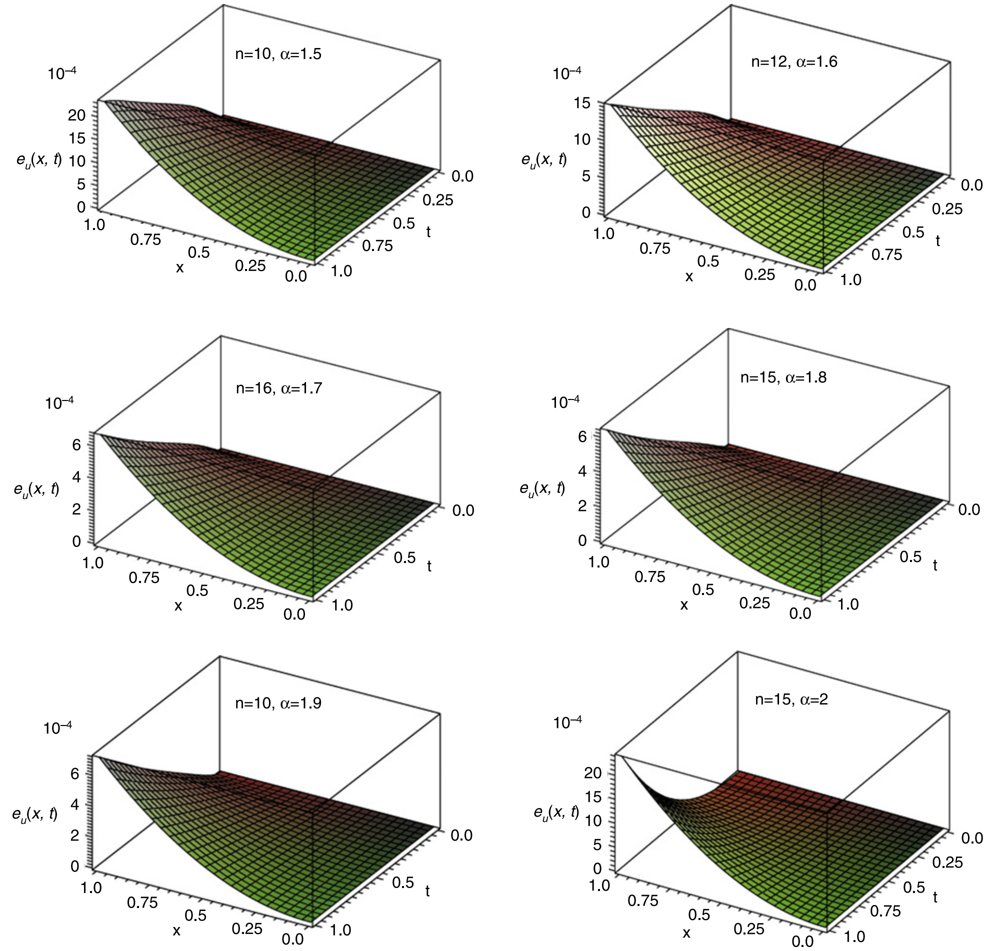

Fig. 1. The graphs of the maximum absolute error functions for different n and α of Experiment 4.1.

The exact solution of this initial value problem for $\alpha = 2$, is $u(x, t) = x + x^2 \sin h(t)$. The analytical solution using the VIM in [12,30] is given as

$$u(x, t) = x + x^2 \sum_{k=0}^{\infty} \frac{t^{k\alpha+1}}{\Gamma(k\alpha + 2)}.$$

We have solved this problem for different n and α and have compared it with the analytical solution. The graphs of the maximum absolute error functions are shown in Fig. 1.

The figure shows good agreement with the approximate solutions obtained by the VIM. When α is an integer (here $\alpha = 2$ and maximum absolute error is 10^{-14}), the approximate solution is in good agreement with the exact solution.

Experiment 4.2. Consider the following linear time-fractional convective–diffusion equation,

$$\frac{\partial^\alpha u}{\partial t^\alpha} + x\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} = 2(1 + t + x^2), \quad 0 < t \le 1,\ 0 \le x \le 1,\ 0 < \alpha \le 1,$$

with the initial condition $u(x, 0) = x^2$.

The exact solution is not known. We have solved this problem for $n = 8$ and $\alpha = 0.75, 0.85, 0.95$ and have compared it with the HPM [12] to show the efficiency of the OTM. The results are given in Table 1.

The obtained results in Table 1, demonstrate that the approximate solution obtained using the OTM, is in good agreement with the approximate solution obtained using the HPM for all values of t and x.

Experiment 4.3. Consider the following nonlinear space-fractional Fisher's equation,

$$\frac{\partial u}{\partial t} - \frac{\partial^{1.5} u}{\partial x^{1.5}} - u(x, t)(1 - u(x, t)) = x^2, \quad 0 < t \le 1,\ 0 \le x \le 1,$$

with the initial condition $u(x, 0) = x$.

Table 1
Comparison of the HPM and OTM for different t and x of Experiment 4.2.

t/x	u_{HPM}			u_{OTM}		
	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.95$	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.95$
$t = 0.1$						
0.00	0.017325	0.014486	0.011452	0.025199	0.017920	0.012299
0.25	0.079825	0.076986	0.073952	0.087699	0.080420	0.074799
0.50	0.267326	0.264486	0.261452	0.275199	0.267920	0.262291
0.75	0.579826	0.576986	0.573952	0.587699	0.580420	0.574799
1.00	1.017326	1.014486	1.011453	1.025199	1.017920	1.012299
$t = 0.2$						
0.00	0.067432	0.056444	0.045243	0.077557	0.060178	0.046034
0.25	0.129932	0.118944	0.107743	0.140057	0.122678	0.108534
0.50	0.317432	0.306444	0.295243	0.327557	0.310178	0.296034
0.75	0.629932	0.618944	0.607743	0.640057	0.622678	0.608534
1.00	1.067432	1.056444	1.045243	1.077558	1.060179	1.046035
$t = 0.3$						
0.00	0.142801	0.121285	0.099954	0.154136	0.125161	0.100719
0.25	0.205301	0.183785	0.162454	0.216636	0.187661	0.163219
0.50	0.392801	0.371285	0.349954	0.404136	0.375161	0.350719
0.75	0.705301	0.683785	0.662454	0.716636	0.687661	0.663219
1.00	1.142801	1.121285	1.099955	1.154136	1.125162	1.100719
$t = 0.4$						
0.00	0.241393	0.207880	0.175242	0.252852	0.211689	0.175989
0.25	0.303893	0.270380	0.237742	0.315352	0.274189	0.238489
0.50	0.491393	0.457880	0.425242	0.502852	0.461689	0.425989
0.75	0.803893	0.770380	0.737742	0.815352	0.774189	0.738489
1.00	1.241394	1.207880	1.175243	1.252852	1.211688	1.175989
$t = 0.5$						
0.00	0.361591	0.315341	0.270835	0.372193	0.318879	0.271561
0.25	0.424091	0.377841	0.333335	0.434693	0.381379	0.334061
0.50	0.611591	0.565341	0.520835	0.622193	0.568879	0.521561
0.75	0.924091	0.877841	0.833335	0.934693	0.881379	0.834061
1.00	1.361591	1.315341	1.270835	1.372194	1.318879	1.271561

Table 2
Comparison of the solutions of GDTM, VIM and OTM for different t and x of Experiment 4.3.

x	$t = 0.1$			$t = 0.2$		
	u_{GDTM}	u_{VIM}	u_{OTM}	u_{GDTM}	u_{VIM}	u_{OTM}
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.110197	0.110123	0.110123	0.119817	0.119386	0.119302
0.2	0.220333	0.220243	0.220239	0.240073	0.239548	0.239453
0.3	0.330252	0.330158	0.330153	0.359392	0.588663	0.358809
0.4	0.439954	0.439870	0.439869	0.477771	0.477364	0.477393
0.5	0.549439	0.549383	0.549389	0.595204	0.595069	0.595229
0.6	0.658706	0.658701	0.658716	0.711688	0.712008	0.712343
0.7	0.767755	0.767825	0.767854	0.827220	0.828207	0.828759
0.8	0.876585	0.876761	0.876806	0.941795	0.943694	0.944504
0.9	0.985196	0.985512	0.985576	1.055411	1.058498	1.059601
1.0	1.093587	1.094081	1.094166	1.168066	1.172648	1.174077
x	$t = 0.3$			$t = 0.4$		
	u_{GDTM}	u_{VIM}	u_{OTM}	u_{GDTM}	u_{VIM}	u_{OTM}
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.127035	0.126129	0.125536	0.129536	0.128692	0.126286
0.2	0.256895	0.255777	0.255291	0.267899	0.266790	0.265133
0.3	0.384480	0.383473	0.383307	0.401893	0.401330	0.401004
0.4	0.509772	0.509305	0.509667	0.531477	0.532516	0.534090
0.5	0.632755	0.633359	0.634452	0.656611	0.660556	0.664574
0.6	0.753415	0.755727	0.757743	0.777260	0.785664	0.792632
0.7	0.871740	0.876498	0.879621	0.893392	0.908056	0.918431
0.8	0.987717	0.995769	1.000160	1.004976	1.027956	1.042129
0.9	1.101336	1.113633	1.119438	1.111987	1.145590	1.163878
1.0	1.212587	1.230189	1.237524	1.214400	1.261189	1.283822

The exact solution is not known. We have obtained the approximate solution for $n = 7$ with different t and x. Comparison of the OTM with GDTM [16] and VIM [12] are given in Table 2.

From the above comparison, a good agreement of OTM with the other methods can be observed.

5. Conclusions

The OTM for solving FPDEs based on arbitrary polynomial bases is presented in this work. The results obtained using this method, and comparing it with several other methods, agree well with the numerical results presented elsewhere. The main advantage of the present method (OTM) is its simplicity, and is more convenient for computer algorithms. Furthermore, this method of solution yields the desired accuracy only in certain conditions. Some experiments of FPDEs were numerically solved, and were compared with the most powerful methods such as HPM, VIM and GDTM. Some useful results have been obtained to demonstrate the capability of the OTM. The main idea of the proposed method is to convert the problem including linear and nonlinear terms to an algebraic system in order to simplify the computations. Orthogonal bases such as Chebyshev and Legendre polynomials are used to reduce the run time of the source programs and the computation time due to their orthogonal properties. All these advantages of the OTM to solve nonlinear problems assert the method as a convenient, reliable, effective and powerful tool.

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