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Advances in Mathematics 216 (2007) 387–442
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Primitives and central detection numbers in group cohomology ☆

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Received 5 December 2006; accepted 22 May 2007

Available online 2 June 2007

Communicated by Michael J. Hopkins

Abstract

Fix a prime p. Given a finite group G, let $H^*(G)$ denote its mod p cohomology. In the early 1990s, Henn, Lannes, and Schwartz introduced two invariants $d_0(G)$ and $d_1(G)$ of $H^*(G)$ viewed as a module over the mod p Steenrod algebra. They showed that, in a precise sense, $H^*(G)$ is respectively detected and determined by $H^d(C_G(V))$ for $d \leqslant d_0(G)$ and $d \leqslant d_1(G)$, with V running through the elementary abelian p-subgroups of G.

The main goal of this paper is to study how to calculate these invariants. We find that a critical role is played by the image of the restriction of $H^*(G)$ to $H^*(C)$, where C is the maximal central elementary abelian p-subgroup of G. A measure of this is the top degree $e(G)$ of the finite dimensional Hopf algebra $H^*(C) \otimes_{H^*(G)} \mathbb{F}_p$, a number that tends to be quite easy to calculate.

Our results are complete when G has a p-Sylow subgroup P in which every element of order p is central. Using the Benson–Carlson duality, we show that in this case, $d_0(G) = d_0(P) = e(P)$, and a similar exact formula holds for d_1. As a bonus, we learn that $H^{e(G)}(P)$ contains nontrivial essential cohomology, reproving and sharpening a theorem of Adem and Karagueuzian.

In general, we are able to show that $d_0(G) \leqslant \max\{e(C_G(V)) \mid V < G\}$ if certain cases of Benson's Regularity Conjecture hold. In particular, this inequality holds for all groups such that the difference between the p-rank of G and the depth of $H^*(G)$ is at most 2. When we look at examples with $p = 2$, we learn that $d_0(G) \leqslant 14$ for all groups with 2-Sylow subgroup of order up to 64, with equality realized when $G = SU(3,4)$.

En route we study two objects of independent interest. If C is any central elementary abelian p-subgroup of G, then $H^*(G)$ is an $H^*(C)$-comodule, and we prove that the subalgebra of $H^*(C)$-primitives is always Noetherian of Krull dimension equal to the p-rank of G minus the p-rank of C. If the depth of $H^*(G)$ equals

☆ This research was partially supported by a grant from the National Science Foundation.
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doi:10.1016/j.aim.2007.05.015

the rank of $Z(G)$, we show that the depth essential cohomology of G is nonzero (reproving and extending a theorem of Green), and Cohen–Macauley in a certain sense, and prove related structural results.

MSC: primary 20J06; secondary 55R40

Keywords: Group cohomology; Steenrod algebra; p-central groups; Primitives

1. Introduction

Fix a prime p, and let $H^*(G)$ denote the mod p cohomology ring of a finite group G. The p-elementary abelian subgroups of G have had a featured role in the study of group cohomology since D. Quillen's famous work [34] in the late 1960s. In particular, these subgroups become the objects in a category $\mathcal{A}(G)$ having as morphisms the homomorphisms generated by the subgroup inclusion and conjugation by elements in G. The inclusions $V < G$ then induce a map

$$H^*(G) \xrightarrow{\lambda_0} \lim_{V \in \mathcal{A}(G)} H^*(V) \subseteq \prod_V H^*(V),$$

and λ_0 is shown to have kernel and cokernel that are nilpotent in an appropriate sense.

Viewing $H^*(G)$ as the mod p cohomology of the classifying space BG makes it evident that $H^*(G)$ is an object in $\mathcal{K}$ and $\mathcal{U}$, the categories of unstable algebras and modules over the mod p Steenrod algebra $\mathcal{A}$. The 1980s and 1990s saw a revolution in our understanding of these categories, and the 1995 paper of H.-W. Henn, J. Lannes, and L. Schwartz [25] revisited Quillen's approximation of $H^*(G)$ from this new perspective.

For each $d \geqslant 0$, the group homomorphisms $V \times C_G(V) \to G$ induce a map of unstable algebras

$$H^*(G) \to \prod_V H^*(V) \otimes H^{\leqslant d}\big(C_G(V)\big),$$

where $M^{\leqslant d}$ denotes the quotient of a graded module M^* by all elements of degree more than d. The image of this map lands in an evident subalgebra of 'compatible' elements which Henn, Lannes, and Schwartz show can be naturally identified with $L_d H^*(G)$, where $L_d : \mathcal{U} \to \mathcal{U}$ is localization away from the localizing subcategory generated by $(d+1)$-fold suspensions of unstable modules. Thus Quillen's map can be viewed as just the bottom of a tower of localizations of $H^*(G)$ associated to the nilpotent filtration of $\mathcal{U}$:

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & L_2H^*(G) \arrow[d, "p_2"] \\
 & L_1H^*(G) \arrow[d, "p_1"] \\
H^*(G) \arrow[uur, "\lambda_2"] \arrow[ur, "\lambda_1"] \arrow[r, "\lambda_0"] & L_0H^*(G),
\end{tikzcd}$$

where we have

$$H^*(G) \xrightarrow{\lambda_d} L_d H^*(G) \subseteq \prod_V H^*(V) \otimes H^{\leqslant d}\big(C_G(V)\big).$$

This caused the authors of [25] to introduce two new invariants of G: $d_0(G)$ and $d_1(G)$ are the smallest d's such that $H^*(G)$ is respectively detected by, and isomorphic to, $L_d H^*(G)$. Alternatively, $d_0(G)$ is the smallest d such that $H^*(G)$ contains no $(d+1)$-fold suspensions of a nontrivial unstable module, and $d_1(G)$ is the smallest d such that also $\mathrm{Ext}^1_{\mathcal{A}}(\Sigma^{d+1}N, H^*(G)) = 0$ for all $N \in \mathcal{U}$.

These invariants satisfy a few easily verified nice properties: $d_0(G \times H) = d_0(G) + d_0(H)$, $d_1(G \times H) = \max\{d_1(G) + d_0(H), d_0(G) + d_1(H)\}$, and $d_i(G) \leqslant d_i(P)$ if P is a p-Sylow subgroup of G. However, they are not well behaved under taking subgroups, quotient groups, and extensions; e.g., every G embeds in a symmetric group Σ_n and $d_0(\Sigma_n) = 0$. Rough upper bounds for $d_0(G)$ and $d_1(G)$ were found in [25]; e.g., $d_0(G)$ is bounded by n^2 if a p-Sylow subgroup of G admits a faithful n dimensional complex representation. However, in all but a few examples, these bounds seem far from optimal. Up to now, what determines these group invariants has remained mysterious, and they have not been connected to other work in the group cohomology.

A main goal of this paper is to present a way to calculate the number $d_0(G)$, and, in some cases, $d_1(G)$. Our finding is that these numbers seem to be controlled by the restriction of cohomology to maximal central p-elementary abelian subgroups. Our results are complete when G has a Sylow subgroup that is p-central, i.e. a group in which every element of order p is central. For example, when $p = 2$, we compute that

$$d_0\big(SU(3,4)\big) = 14 \quad \text{and} \quad d_1\big(SU(3,4)\big) = 18,$$

where, by contrast, the estimates from [25] yield only that

$$d_0\big(SU(3,4)\big) \leqslant 64 \quad \text{and} \quad d_1\big(SU(3,4)\big) \leqslant 120.$$

Our method is to combine $\mathcal{U}$-technology, in the spirit of [25], with duality results as in the work of D. Benson and J. Carlson [6]. We ultimately connect a conjectured upper bound for $d_0(G)$ to Benson's Regularity Conjecture [5], known to hold if the p-rank of G and the p-rank of $Z(G)$ differ by at most 2. This is the case for all 2-groups of order 64 or less, and, using cohomology calculations from [14], we have been able to verify by hand that $d_0(G) \leqslant 14$ for all such groups.

A number of side results of independent interest come up in our investigations.

We are led to study carefully the cohomology of central extensions, in particular the structure of associated algebras of primitives. One outcome of this is a new proof of A. Adem and D. Karagueuzian's theorem [1] that p-central p-groups have nonzero essential cohomology. We show that in an explicit degree there is a nonzero cohomology class that is simultaneously essential and annihilated by all Steenrod operations of positive degree.

Deriving our general estimate of $d_0(G)$ involves a careful study of the depth essential cohomology of Carlson et al. [14] in the important special case that the depth of $H^*(G)$ equals the rank of the center. We prove that then the depth essential cohomology is both nonzero—reproving the main theorem of [20] without D. Green's hypothesis that G be a p-group—and Cohen–Macauley.

In the next section we describe our results in more detail.

2. Main results

2.1. The cohomology of central extensions

Suppose we have a central extension of finite groups

$$C \xrightarrow{i} G \xrightarrow{q} Q,$$

where C is p-elementary abelian of rank c.

We define various objects associated to this situation.

The extension corresponds to an element $\tau \in H^2(Q; C)$. Since $H^2(Q; C) = \mathrm{Hom}(H_2(Q), C)$, the extension can also be considered as corresponding to a homomorphism $\tau : H_2(Q) \to C$, or, equivalently, its dual $\tau^{\#} : C^{\#} \to H^2(Q)$.

Let $\{E_r^{*,*}\}$ denote the Serre spectral sequence associated to the extension, converging to $H^*(G)$, and with $E_2^{*,*} = H^*(Q) \otimes H^*(C)$. Under the identification $C^{\#} = H^1(C)$, it is standard that $\tau^{\#}$ corresponds to $d_2 : E_2^{0,1} \to E_2^{2,0}$.

Let $I_\tau \subset H^*(Q)$ be the ideal generated by $\mathcal{A} \cdot \mathrm{im}(\tau^{\#})$, so that $H^*(Q)/I_\tau$ is an unstable algebra. It is easy to see that I_τ is contained in the kernel of inflation $q^* : H^*(Q) \to H^*(G)$.

Call a subalgebra A of $H^*(G)$ a *(G, C)-Duflot subalgebra*, if the composite $A \subseteq H^*(G) \xrightarrow{i^*} \mathrm{im}(i^*)$ is an isomorphism, where $i^* : H^*(G) \to H^*(C)$ is the restriction. As we will describe more precisely in Section 2.5, as an algebra, the Hopf algebra $\mathrm{im}(i^*) \subseteq H^*(C)$ will necessarily be free graded commutative on c polynomial generators, possibly tensored with an exterior algebra on some generators in degree 1, if p is odd. It follows that Duflot subalgebras exist and have the same form. Let $Q_A H^*(G)$ denote the graded algebra[1] of A-indecomposables $H^*(G) \otimes_A \mathbb{F}_p$, or, equivalently, the quotient of $H^*(G)$ by the ideal generated by the positive degree elements of A.

$H^*(C)$ is a Hopf algebra, and the multiplication map $m : C \times G \to G$ induces a map of unstable algebras

$$m^* : H^*(G) \to H^*(C) \otimes H^*(G)$$

making $H^*(G)$ into a $H^*(C)$-comodule. We define the associated algebra of primitives to be

$$\begin{aligned} P_C H^*(G) &= \{x \in H^*(G) \mid m^*(x) = 1 \otimes x\} \\ &= \mathrm{Eq}\Big\{H^*(G) \underset{\pi^*}{\overset{m^*}{\rightrightarrows}} H^*(C \times G)\Big\}, \end{aligned}$$

where $\pi : C \times G \to G$ is the projection. It is easy to check that $P_C H^*(G)$ is an unstable algebra that contains the image of the inflation map. Thus $q^* : H^*(Q) \to H^*(G)$ refines to a map of unstable algebras

$$q_\tau : H^*(Q)/I_\tau \to P_C H^*(G).$$

Theorem 2.1. *With the notation as above, the following are true.*

[1] $Q_A H^*(G)$ will not necessarily be an unstable algebra, as A need not be closed under Steenrod operations.

(a) *$H^*(G)$ is a free A-module. Moreover $\{E_r^{*,*}\}$ is a spectral sequence of free $E_\infty^{0,*}$-modules, and applying $Q_{E_\infty^{0,*}}$ to the spectral sequence yields a spectral sequence converging to $Q_A H^*(G)$ with E_2-term $Q_{E_\infty^{0,*}} H^*(C) \otimes H^*(Q)$.*
(b) *The composite $P_C H^*(G) \hookrightarrow H^*(G) \twoheadrightarrow Q_A H^*(G)$ is monic.*
(c) *Both $P_C H^*(G)$ and $Q_A H^*(G)$ are finitely generated $H^*(Q)$-modules.*
(d) *The map $q_\tau : H^*(Q)/I_\tau \to P_C H^*(G)$ is an F-isomorphism,[2] and the rings $H^*(Q)/I_\tau$, $\mathrm{im}(q^*)$, $P_C H^*(G)$, and $Q_A H^*(G)$, are all Noetherian of Krull dimension equal to (the p-rank of G) $-$ (the rank of C).*

Let $C(G) < G$ be the p-elementary abelian part of $Z(G)$. If $C = C(G)$, the first part of statement (a) recovers J. Duflot's result [16] that the depth of $H^*(G)$ is at least as great as the rank of $C(G)$.[3] We will call a $(G, C(G))$-Duflot subalgebra of $H^*(G)$ simply a *Duflot subalgebra*.

The p-rank of G equals the rank of C exactly when G is p-central and $C = C(G)$. We thus have the following corollary.

Corollary 2.2. *If G is p-central and $C = C(G)$, then the rings $H^*(Q)/I_\tau$, $\mathrm{im}(q^*)$, $P_C H^*(G)$, and $Q_A H^*(G)$ all have Krull dimension zero and so are finite dimensional $\mathbb{F}_p$-algebras.*

2.2. Quillen's category and functors involving primitives

Our algebras of primitives arise in two formulae associated to $H^*(G)$, viewed as an object in $\mathcal{K}$. To describe these, we need to introduce some notation.

Given a small category $\mathcal{C}$, we let $\mathcal{C}^\#$ denote the associated twisted arrow category: the objects of $\mathcal{C}^\#$ are the morphisms of $\mathcal{C}$, and a morphism $\alpha \rightsquigarrow \beta$ from $\alpha : A_1 \to A_2$ to $\beta : B_1 \to B_2$ is a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd} A_1 \arrow[r, "\alpha"] \arrow[d] & A_2 \\ B_1 \arrow[r, "\beta"] & B_2. \arrow[u] \end{tikzcd}$$

The functor assigning $H^*(V)$ to $V \in \mathcal{A}(G)$ is contravariant, while the assignment of $H^*(C_G(V))$ is covariant. Now observe that the assignment of $P_{\alpha(V_1)} H^*(C_G(V_2))$ to $\alpha : V_1 \to V_2$ can be viewed as defining a contravariant functor of $\mathcal{A}(G)^\#$.

Let $\mathcal{A}_C(G)$ denote the full subcategory of $\mathcal{A}(G)$ having as objects the V containing $C(G)$. If G is p-central, then $\mathcal{A}_C(G)$ has a single object and morphism.

2.3. A formula for the locally finite part of $H^*(G)$

If M is an unstable $\mathcal{A}$-module, we define M_{LF}, the locally finite part of M, by

$$M_{LF} = \{x \in M \mid \mathcal{A}x \subset M \text{ is finite}\}.$$

[2] In the sense of Quillen [34]: $\ker(q_\tau^*)$ is nilpotent, and for all $x \in P_C H^*(G)$, there exists a k so that $x^{p^k} \in \mathrm{im}(q_\tau^*)$.

[3] We are claiming no originality in the proof of this, which is similar to all proofs of Duflot's theorem following [8]. The spectral sequence refinement seems to be a new observation.

This is again an unstable module, and is an unstable algebra if M is.

Theorem 2.3. *There is a natural isomorphism of unstable algebras*

$$H^*(G)_{LF} \simeq \lim_{V_1 \xrightarrow{\alpha} V_2} P_{\alpha(V_1)} H^*\big(C_G(V_2)\big),$$

where the limit is over $\mathcal{A}_C(G)^\#$.

Corollary 2.4. *If G is p-central, then $H^*(G)_{LF} = P_{C(G)} H^*(G)$.*

2.4. A formula for $\bar{R}_d H^*(G)$

An unstable module $M \in \mathcal{U}$ has a canonical 'nilpotent' filtration [25,26,35]:

$$\cdots \subseteq nil_2\, M \subseteq nil_1\, M \subseteq nil_0\, M = M.$$

In general, $nil_d\, M / nil_{d+1}\, M = \Sigma^d R_d M$, where $R_d M$ is reduced, i.e. has no nontrivial submodules that are suspensions. We let $\bar{R}_d M$ denote the nilclosure $L_0 R_d M$ of $R_d M$.

The module $nil_d\, M$ identifies with the kernel of $\lambda_d : M \to L_{d-1} M$, and a bit of diagram chasing will show that $\Sigma^d \bar{R}_d M$ is isomorphic to the kernel of $L_d M \to L_{d-1} M$: see Proposition 3.1. Thus $d_0(G)$ is the length of the filtration of $H^*(G)$, and also is the biggest d such that $\bar{R}_d H^*(G) \neq 0$.

Theorem 2.5. *There is a natural isomorphism of unstable modules*

$$\bar{R}_d H^*(G) \simeq \lim_{V_1 \xrightarrow{\alpha} V_2} H^*(V_1) \otimes P_{\alpha(V_1)} H^d\big(C_G(V_2)\big),$$

where the limit is over $\mathcal{A}_C(G)^\#$.

Corollary 2.6. *If G is p-central, then there is an isomorphism of unstable modules*

$$\bar{R}_d H^*(G) \simeq H^*\big(C(G)\big) \otimes P_{C(G)} H^d(G).$$

2.5. Invariants of restriction to $C(G)$

If $i : C < G$ is a central p-elementary abelian of rank c, then

$$H^*(C) \simeq \begin{cases} \mathbb{F}_2[x_1, \ldots, x_c] & \text{if } p = 2, \\ \Lambda(x_1, \ldots, x_c) \otimes \mathbb{F}_p[y_1, \ldots, y_c] & \text{if } p \text{ is odd}, \end{cases}$$

where $|x_i| = 1$ and $y_i = \beta(x_i)$, and is a Hopf algebra in the usual way.

In Section 6, we will see that, after a change of basis for $H^1(C)$, the image of the restriction homomorphism $i^* : H^*(G) \to H^*(C)$ will be a sub-Hopf algebra of $H^*(C)$ of the form

$$\operatorname{im}(i^*) = \begin{cases} \mathbb{F}_2[x_1^{2^{j_1}}, \ldots, x_c^{2^{j_c}}] & \text{if } p = 2, \\ \mathbb{F}_p[y_1^{p^{j_1}}, \ldots, y_b^{p^{j_b}}, y_{b+1}, \ldots, y_c] \otimes \Lambda(x_{b+1}, \ldots, x_c) & \text{if } p \text{ is odd}, \end{cases}$$

with the j_i forming a sequence of nonincreasing nonnegative integers.[4]

Now suppose that $C = C(G)$. We will say that G has *type* $[a_1, \dots, a_c]$ where

$$(a_1, \dots, a_c) = \begin{cases} (2^{j_1}, \dots, 2^{j_c}) & \text{if } p = 2, \\ (2p^{j_1}, \dots, 2p^{j_b}, 1, \dots, 1) & \text{if } p \text{ is odd.} \end{cases}$$

The type of G has the form $[1, \dots, 1]$ if and only if $G = C \times H$, where $Z(H)$ has order prime to p. In all other cases, $a_1 = 2p^k$ for some $k \geqslant 0$.

Define $e(G)$ and $h(G)$ by

$$e(G) = \sum_{i=1}^{c} (a_i - 1),$$

and

$$h(G) \simeq \begin{cases} 2p^{k-1} & \text{if } a_1 = 2p^k \text{ with } k \geqslant 1, \\ 1 & \text{if } a_1 = 2, \\ 0 & \text{if } a_1 = 1. \end{cases}$$

For example, $Q_8 \times \mathbb{Z}/4$ has type $[4, 2]$ when $p = 2$, so that $e(Q_8 \times \mathbb{Z}/4) = 4$, and $h(Q_8 \times \mathbb{Z}/4) = 2$.

Remark 2.7. The careful reader will observe that the type of G is just the list of the degrees of the unstable $\mathcal{A}$-algebra generators of $\operatorname{im}(i^*)$, listed in decreasing order, $e(G)$ is the top nonzero degree of the finite dimensional Hopf algebra

$$H^*(C) \otimes_{H^*(G)} \mathbb{F}_p = H^*(C)/\big(\operatorname{im}\big(i^{*>0}\big)\big),$$

and $h(G)$ is the top nonzero degree of the module $\mathcal{A} \cdot H^1(C)$ projected into this Hopf algebra.

2.6. $P_{C(G)}H^*(G)$, $d_0(G)$, and $d_1(G)$ when G is p-central

If G is p-central with $C = C(G)$, then $P_C H^*(G)$ is a finite dimensional unstable algebra. We identify its top degree and more.

Theorem 2.8. *Let G be p-central, $C = C(G)$, and A be a Duflot subalgebra of $H^*(G)$. Then both $P_C H^*(G)$ and $Q_A H^*(G)$ are zero in degrees greater than $e(G)$, and one dimensional in degree $e(G)$. Furthermore, $P_C H^{e(G)}(G)$ is annihilated by all positive degree elements of the Steenrod algebra, and, if G is a p-group, consists of essential*[5] *cohomology classes.*

The last statement implies the main result of [1]: p-central p-groups have nonzero essential cohomology.

This theorem, combined with Corollary 2.6 and related results, leads to the following calculation.

[4] In the odd prime case, $c - b$ will be the rank of the largest subgroup of C splitting off G as a direct summand.

[5] Recall that $x \in H^*(G)$ is essential if it restricts to zero on all proper subgroups.

Theorem 2.9. *Let G be p-central. Then $d_0(G) = e(G)$ and $d_1(G) = e(G) + h(G)$. Furthermore, if G is a finite group with p-Sylow subgroup P, and P is p-central, then $d_0(G) = d_0(P)$ and $d_1(G) = d_1(P)$.*

Corollary 2.10. *If G is p-central, and $H < G$, then $d_0(H) \leqslant d_0(G)$ and $d_1(H) \leqslant d_1(G)$.*

Examples 2.11. (a) As Q_8 is 2-central of type [4], $d_0(Q_8) = 3$ and $d_1(Q_8) = 3 + 2 = 5$, in agreement with [25, (II.4.6)].

(b) The hypotheses of p-centrality are needed in the last part of the theorem: as observed in [25, II.4.7], if $G = GL_2(\mathbb{F}_3)$ and $P = SD_{16}$, then P is the 2-Sylow subgroup of G, but $d_0(G) = 0 < 2 = d_0(P)$, and $d_1(G) = 2 < 4 = d_1(P)$. Similarly, G needs to be p-central in the corollary: if $H = Z/4 < D_8 = G$, then $d_0(H) = 1 > 0 = d_0(G)$ and $d_1(H) = 2 > 0 = d_1(G)$. The example $H = \mathbb{Z}/4 < \mathbb{Z}/8 = G$ shows that the inequalities of the corollary can be equalities, even when H is a proper subgroup of a p-central p-group G.

(c) The 2-Sylow subgroup P of the simple group $SU(3,4)$ is 2-central of type [8, 8]. Thus $d_0(SU(3,4)) = d_0(P) = 14$ and $d_1(SU(3,4)) = d_1(P) = 18$. Similarly, the 2-Sylow subgroup Q of the simple group $Sz(8)$ is 2-central of type [4, 4, 4]. Thus $d_0(Sz(8)) = d_0(Q) = 9$ and $d_1(Sz(8)) = d_1(Q) = 11$. We will see that P and Q have the largest d_0 of all 2-groups of order dividing 64. For more about the $SU(3,4)$ example, see Section 9.

(d) In [3], the authors associate a 2-central Galois group $\mathcal{G}_{\mathbb{F}}$ to every field $\mathbb{F}$ of characteristic different from 2 that is not formally real. (They call this the W-group of $\mathbb{F}$ because of its connections to the Witt ring $W\mathbb{F}$ [33].) From their construction it is easy to deduce that $\mathcal{G}_{\mathbb{F}}$ has type $[2, \ldots, 2]$. Thus $d_0(\mathcal{G}_{\mathbb{F}}) = r$ and $d_1(\mathcal{G}_{\mathbb{F}}) = r + 1$, where $\mathcal{G}_{\mathbb{F}}$ has rank r. In particular, the universal W-group $W(n)$ has $d_0(W(n)) = \binom{n+1}{2}$ and $d_1(W(n)) = \binom{n+1}{2} + 1$. For more about this example, see Section 9.

2.7. Central essential cohomology

Our calculation of $d_0(G)$ when G is p-central relies on Corollary 2.6. To understand $d_0(G)$ for general G, one needs to use the more complicated formula given in Theorem 2.5. Using some analysis of this already done by us in our companion paper [28], we are led to a formula[6] for $d_0(G)$ that makes use of the following variant of essential cohomology.

We define $Cess^*(G)$, the *central essential cohomology* of G, to be the kernel of the restriction map

$$H^*(G) \to \prod_{C(G)<U} H^*\big(C_G(U)\big),$$

where the product is over p-elementary abelian subgroups U of G that are strictly bigger than $C(G)$.

This product is over the empty set if G is p-central, so the concept is really only interesting when this is not the case. Furthermore, a theorem of Carlson [12] implies that $Cess^*(G)$ is nonzero only if the rank of $C(G)$ equals the depth of $H^*(G)$: see Theorem 2.13 below for the converse. If this is the case, $Cess^*(G)$ is precisely the *depth essential* cohomology of [14].

[6] Thus far, we have not found an analogous formula for $d_1(G)$.

Note that $Cess^*(G)$ has the following structure, compatible in the usual ways: it is an ideal in $H^*(G)$, an unstable module, and an $H^*(C(G))$-comodule. We have the following general structural results.

Theorem 2.12. *If A is a Duflot subalgebra of $H^*(G)$, then the following hold.*

(a) *$Cess^*(G)$ is a finitely generated free A-module.*
(b) *The composite $P_C\, Cess^*(G) \hookrightarrow Cess^*(G) \twoheadrightarrow Q_A\, Cess^*(G)$ is monic.*
(c) *The sequence $0 \to Q_A\, Cess^*(G) \to Q_A H^*(G) \to \prod_{C(G)<U} Q_A H^*(C_G(U))$ is exact.*

Statement (a) implies that $Cess^*(G)$ is a Cohen–Macauley module, and thus can be viewed as a variant of D. Green's theorem [22] about the essential cohomology $Ess^*(G)$. Statement (b) will have application below. Statement (c) gets us most of the way towards proving the following theorem.

Theorem 2.13. *$Cess^*(G) \neq 0$ if and only if the depth of $H^*(G)$ is the rank of $C(G)$.*

The 'only if' statement here is just Carlson's theorem. The 'if' statement is a special case of Carlson's Depth Conjecture, and has been previously proved by D. Green under the extra hypothesis that G is a p-group [20].

2.8. $d_0(G)$ for general G

Thanks to Theorem 2.12, we can make the following definitions. If $Cess^*(G)$ is nonzero, define $e'(G)$ to be the largest d such that $Q_A\, Cess^d(G)$ is nonzero, and $e''(G)$ to be the largest d such that $P_C Cess^d(G)$ is nonzero. If $Cess^*(G) = \mathbf{0}$, we let $e''(G) = e'(G) = -1$. Note that Theorem 2.12(b) implies that $e''(G) \leqslant e'(G)$.

Using the formula for $\bar{R}_d H^*(G)$ given in Theorem 2.5, we will prove the following.

Theorem 2.14. $d_0(G) = \max\{e''(C_G(V)) \mid V < G\}$.

Corollary 2.15. $d_0(G) \leqslant \max\{e'(C_G(V)) \mid V < G\}$.

When computing these maxima, one can restrict to the p-elementary abelian groups V which satisfy $V = C(C_G(V))$.[7] The next proposition says that if one has some a priori computation of the depth of $H^*(G)$, one may be able to cut down even more on the V's to be checked.

Proposition 2.16. *Assuming that $V = C(C_G(V))$, $Cess^*(C_G(V)) = \mathbf{0}$ unless the rank of V is at least equal to the depth of $H^*(G)$.*

This will be proved by combining Carlson's theorem with some $\mathcal{U}$-technology. This leads to the following generalization of our calculation of $d_0(G)$ for p-central G.

[7] Since $C_G(V) = C_G(U)$, where $U = C(C_G(V))$.

Corollary 2.17. *If $H^*(G)$ is Cohen–Macauley, then*

$$d_0(G) = \max\{e(C_G(V)) \mid V < G \text{ is maximal}\}.$$

This follows from the above, as $C_G(V)$ is p-central when V is maximal. Here we have used that, by Theorem 2.8, when G is p-central, $e''(G) = e'(G) = e(G)$.

Conjecture 2.18. *$e'(G) < e(G)$ if G is not p-central.*

As will be explained in Section 8, Benson's Strong Regularity Conjecture [5] asserts that certain local cohomology groups $H^{i,j}_{\tilde{H}^*(G)}(H^*(G))$ vanish. We connect our conjecture to his.

Proposition 2.19. *For a fixed finite group G, Conjecture 2.18 is implied by the Strong Regularity Conjecture.*

Let G have p-rank r and $C(G)$ have rank c with $c < r$. Benson [5] has shown that his conjecture is true if $r - c \leqslant 2$. We deduce the next corollary.

Corollary 2.20. *Let G have p-rank r, and let d be the depth of $H^*(G)$. If $r - d \leqslant 2$, then*

$$d_0(G) \leqslant \max\{e(C_G(V)) \mid V < G\}.$$

The hypothesis of this corollary applies to all 2-groups of order dividing 64.

2.9. Calculations when $p = 2$

The appendix has various tables of values of $d_0(G)$, $d_1(G)$, $e(G)$, $e'(G)$, and $e''(G)$ for 2-groups of order dividing 64. The tables were compiled by hand using the calculations in [14]. Their calculations let one immediately determine if $Cess^*(G) \neq 0$, and, when this is the case, one can read off the values of $e(G)$ and $e'(G)$, and sometimes $e''(G)$.

From our tables, one learns the following about $d_0(G)$ when $p = 2$:

Theorem 2.21. *Let G be a finite group with 2-Sylow subgroup P of order dividing 64. Then $d_0(G) \leqslant 7$ unless P is isomorphic to either the Sylow subgroup of $SU(3,4)$, in which case $d_0(G) = 14$, or the Sylow subgroup of $Sz(8)$, in which case $d_0(G) = 9$.*

2.10. Organization of the paper

The rest of the paper is organized as follows. The nilpotent filtration of $\mathcal{U}$ is reviewed in Section 3, along with basic properties of the functors $\bar{R}_d$, and the invariants d_0 and d_1. Starting from results in [25], in Section 4 we then deduce the formulae given in Theorems 2.3 and 2.5. In Section 5, we prove Theorem 2.1 with a careful analysis of the Lyndon–Serre spectral sequence associated to the group extension $C \to G \to G/C$, heavily using that the spectral sequence is a spectral sequence of $H^*(C)$-comodules. In the p-central case, we also input Carlson and Benson's theorem that if $H^*(G)$ is Cohen–Macauley then it is Gorenstein: this leads to proofs of Theorems 2.8 and 2.9 in Section 7. Using an analysis of the formula in Theorem 2.5 done by us in [28], Theorem 2.14 is proved in Section 8, which then continues with our results about $Cess^*(G)$

and the conjectured inequality $e'(G) \leqslant e(G)$. Though short examples occur throughout, some longer examples that illustrate the general theory make up Section 9.

3. The nilpotent filtration of $\mathcal{U}$

The nilpotent filtration of $\mathcal{U}$ was introduced in [35], and its main properties were developed in [9,25,26,36]. Here we collect the results that we need.[8]

3.1. The definition of L_d, R_d, and $\bar{R}_d$

For $d \geqslant 0$, let $\mathcal{N}il_d \subset \mathcal{U}$ be the localizing subcategory generated by d-fold suspensions of unstable $\mathcal{A}$-modules, i.e. $\mathcal{N}il_d$ is the smallest full subcategory containing all d-fold suspensions of unstable modules that is closed under extensions and filtered colimits. Associated to the descending filtration

$$\cdots \subset \mathcal{N}il_2 \subset \mathcal{N}il_1 \subset \mathcal{N}il_0 = \mathcal{U},$$

there is a natural localization tower for $M \in \mathcal{U}$,

$$\begin{tikzcd} & \vdots \arrow[d] \\ & L_2 M \arrow[d] \\ & L_1 M \arrow[d] \\ M \arrow[uur, "\lambda_2"] \arrow[ur, "\lambda_1"] \arrow[r, "\lambda_0"] & L_0 M, \end{tikzcd}$$

where $L_d : \mathcal{U} \to \mathcal{U}$ is localization away from $\mathcal{N}il_{d+1}$.[9] The natural transformation $\lambda_d : M \to L_d M$ is characterized by the following properties:

(a) $L_d M$ is $\mathcal{N}il_{d+1}$-closed, i.e. $\mathrm{Ext}^s_{\mathcal{U}}(N, L_d M) = 0$ for $s = 0, 1$ and $N \in \mathcal{N}il_{d+1}$,
(b) λ_d is a $\mathcal{N}il_{d+1}$-isomorphism, i.e. $\ker \lambda_d$ and $\operatorname{coker} \lambda_d$ are both in $\mathcal{N}il_{d+1}$.

A module $M \in \mathcal{U}$ admits a natural filtration

$$\cdots \subseteq nil_2\, M \subseteq nil_1\, M \subseteq nil_0\, M = M,$$

where $nil_d\, M$ is the largest submodule in $\mathcal{N}il_d$. For $d > 0$, $nil_d\, M = \ker \lambda_{d-1}$.

[8] This section necessarily overlaps with the presentation in our recent preprint [28].
[9] What we are calling L_d here was called L_{d+1} in [25].

An unstable module M is called *reduced* if $nil_1 M = 0$. As observed in [26, Proposition 2.2], $nil_d M / nil_{d+1} M = \Sigma^d R_d M$, where $R_d M$ is a reduced unstable module. (See also [36, Lemma 6.1.4].) Then $\bar{R}_d M$ is defined to be the $\mathcal{N}il_1$-closure of $R_d M$. Thus $R_d M \subseteq L_0 R_d M = \bar{R}_d M$.

We have the following useful alternative definition of $\bar{R}_d M$. (Compare with [25, I(3.8.1)].)

Proposition 3.1. *There is a natural isomorphism*

$$\Sigma^d \bar{R}_d M \simeq \ker\{L_d M \to L_{d-1} M\}.$$

The functors L_d and L_{d-1} are left exact, as they are localizations, and thus we conclude

Corollary 3.2. *$\bar{R}_d : \mathcal{U} \to \mathcal{U}$ is left exact.*

Proof of Proposition 3.1. Let $c_{d+1} M = \operatorname{coker}\{\lambda_d : M \to L_d M\}$. Then $c_{d+1} M \in \mathcal{N}il_{d+1}$, and there is an exact sequence

$$0 \to nil_{d+1} M \to M \to L_d M \to c_{d+1} M \to 0.$$

Diagram chasing then shows that there is a natural short exact sequence

$$0 \to nil_d M / nil_{d+1} M \to \ker\{L_d M \to L_{d-1} M\} \to \ker\{c_{d+1} M \to c_d M\} \to 0.$$

As the middle module here is $\mathcal{N}il_{d+1}$-closed, and the right module is in $\mathcal{N}il_{d+1}$, we see that the left map identifies with λ_d. Recalling that $nil_d M / nil_{d+1} M = \Sigma^d R_d M$, this says that there is a natural isomorphism

$$L_d\bigl(\Sigma^d R_d M\bigr) \simeq \ker\{L_d M \to L_{d-1} M\}.$$

The proof of the proposition is then completed by observing that $L_d(\Sigma^d R_d M) \simeq \Sigma^d \bar{R}_d M$, a consequence of the next proposition. □

Proposition 3.3. *There is a natural isomorphism $L_{c+d}(\Sigma^d M) \simeq \Sigma^d L_c M$, for all $M \in \mathcal{U}$.*

Proof. We need to check that the map $\Sigma^d \lambda_c : \Sigma^d M \to \Sigma^d L_c M$ satisfies the two properties characterizing localization away from $\mathcal{N}il_{c+d+1}$.

That $\ker(\Sigma^d \lambda_c)$ and $\operatorname{coker}(\Sigma^d \lambda_c)$ are both in $\mathcal{N}il_{c+d+1}$ is clear, as $\ker(\lambda_c)$ and $\operatorname{coker}(\lambda_c)$ are both $\mathcal{N}il_{c+1}$, and the d-fold suspension of a module in $\mathcal{N}il_{c+1}$ will be in $\mathcal{N}il_{c+d+1}$.

To see that the range of $\Sigma^d \lambda_c$ is $\mathcal{N}il_{c+d+1}$-closed, we check that if $M \in \mathcal{U}$ is $\mathcal{N}il_{c+1}$-closed then $\Sigma^d M$ is $\mathcal{N}il_{c+d+1}$-closed. This follows from the following characterization of $\mathcal{N}il_{c+1}$-closed modules: $M \in \mathcal{U}$ is $\mathcal{N}il_{c+1}$-closed if and only if it fits into an exact sequence of the form

$$0 \to M \to \prod_{\alpha} H^*(V_\alpha) \otimes M_\alpha \to \prod_{\beta} H^*(W_\beta) \otimes N_\beta,$$

with all the modules M_α and N_β concentrated in degrees between 0 and c. See [10, Proposition 1.15]. □

3.2. Further properties of L_d, R_d, and $\bar{R}_d$

We need to recall some notation and terminology. If V is an elementary p-group, $T_V : \mathcal{U} \to \mathcal{U}$ is defined to be the left adjoint to $H^*(V) \otimes \_\_$, as famously studied by Lannes [30,31]. Given a Noetherian unstable algebra $K \in \mathcal{K}$, $K_{f.g.} - \mathcal{U}$ is defined to be the category studied in [25, I.4] whose objects are finitely generated K-modules M whose K-module structure map $K \otimes M \to M$ is in $\mathcal{U}$, and morphisms are K-module maps in $\mathcal{U}$.

Proposition 3.4. *The functor $L_d : \mathcal{U} \to \mathcal{U}$ satisfies the following properties.*

(a) *There are natural isomorphisms $L_0(M \otimes N) \simeq L_0 M \otimes L_0 N$.*
(b) *There are natural isomorphisms $T_V L_d M \simeq L_d T_V M$.*
(c) *If $K \in \mathcal{K}$, then $L_d K \in \mathcal{K}$, and $K \to L_d K$ is a map of unstable algebras. If K is also Noetherian, and $M \in K_{f.g.} - \mathcal{U}$, then $L_d K \in K_{f.g.} - \mathcal{U}$, and thus is Noetherian, and $L_d M \in L_d K_{f.g.} - \mathcal{U}$.*

Property (b) can be deduced from properties of T_V as follows. First, to see that $T_V L_d M$ is $\mathcal{N}il_{d+1}$-closed, we compute, for $s = 0, 1$ and $N \in \mathcal{N}il_{d+1}$:

$$\mathrm{Ext}^s_{\mathcal{U}}(N, T_V L_d M) = \mathrm{Ext}^s_{\mathcal{U}}\big(H^*(V) \otimes N, L_d M\big) = 0,$$

since $H^*(V) \otimes N$ will be in $\mathcal{N}il_{d+1}$ if N is. Second, $T_V \lambda_d : T_V M \to T_V L_d M$ is a $\mathcal{N}il_{d+1}$-isomorphism, as the kernel and cokernel are in $\mathcal{N}il_{d+1}$, since T_V is exact and sends $\mathcal{N}il_{d+1}$ to itself.

See [25, I.4] and [9] for more details about properties (a) and (c).

Proposition 3.5. *The functors $R_d : \mathcal{U} \to \mathcal{U}$ satisfy the following properties.*

(a) *There are a natural isomorphisms $R_*(M \otimes N) \simeq R_* M \otimes R_* N$ of graded objects in $\mathcal{U}$.*
(b) *There are natural isomorphisms $T_V R_d M \simeq R_d T_V M$.*
(c) *If $K \in \mathcal{K}$, then $R_0 K \in \mathcal{K}$, and $K \to R_0 K$ is a map of unstable algebras. If K is also Noetherian, and $M \in K_{f.g.} - \mathcal{U}$, then $R_0 K$ is also a Noetherian unstable algebra, and $R_d M \in R_0 K_{f.g.} - \mathcal{U}$, for all d.*

For the first two properties, see [26, §3], and the last follows easily from the first.

Proposition 3.6. *The functors $\bar{R}_d : \mathcal{U} \to \mathcal{U}$ satisfy the following properties.*

(a) *There are natural isomorphisms $\bar{R}_*(M \otimes N) \simeq \bar{R}_* M \otimes \bar{R}_* N$ of graded objects in $\mathcal{U}$.*
(b) *There are natural isomorphisms $T_V \bar{R}_d M \simeq \bar{R}_d T_V M$.*
(c) *If $K \in \mathcal{K}$, then $\bar{R}_0 K \in \mathcal{K}$, and $K \to \bar{R}_0 K$ is a map of unstable algebras. If K is also Noetherian, and $M \in K_{f.g.} - \mathcal{U}$, then $\bar{R}_0 K$ is also a Noetherian unstable algebra, and $\bar{R}_d M \in \bar{R}_0 K_{f.g.} - \mathcal{U}$, for all d.*

This, of course, follows from the previous two propositions.

A Noetherian unstable algebra K has a finite Krull dimension $\dim K$. We have an addendum to Proposition 3.4.

Proposition 3.7. *(See [28, Proposition 4.10].) If an unstable algebra K is Noetherian, then* $\dim K = \dim L_0 K$.

Another special property of L_0 that we will need goes as follows.

Proposition 3.8. *(See [30, Lemma 4.3.3].) Let $f : M \to N$ be a map in $\mathcal{K}$. Then*

$$L_0 f : L_0 M \to L_0 N$$

is an isomorphism if and only if, for all p-elementary abelian groups V, the induced map

$$f^* : \mathrm{Hom}_{\mathcal{K}}\big(N, H^*(V)\big) \to \mathrm{Hom}_{\mathcal{K}}\big(M, H^*(V)\big)$$

is a bijection.

As in the introduction, given $M \in \mathcal{U}$, M_{LF} denotes the submodule of locally finite elements: $x \in M$ such that $\mathcal{A}x \subseteq M$ is finite.

Proposition 3.9. *There is a natural isomorphism* $(R_d M)^0 = (\bar{R}_d M)^0 \simeq (M_{LF})^d$.

See [26, §3] for a proof.

Finally, Henn [24] proved the following important finiteness result.

Proposition 3.10. *Let $K \in \mathcal{K}$ be Noetherian, and $M \in K_{f.g.} - \mathcal{U}$. Then M is $\mathcal{N}il_d$-local for $d \gg 0$. In particular, the nilpotent filtration of M has finite length.*

3.3. Properties of $d_0 M$ and $d_1 M$

The authors of [25] define $d_0 M$ and $d_1 M$ as follows.

Definition 3.11. Let M be an unstable module.

(a) Let $d_0 M$ be the smallest d such that λ_d is monic, or ∞ if no such d exists. Equivalently, $d_0 M$ is the smallest d such that $\mathrm{Hom}_{\mathcal{U}}(N, M) = 0$ for all $N \in \mathcal{N}il_{d+1}$, or the smallest d such that $nil_{d+1} M = 0$. If M is nonzero, $d_0 M$ is also the largest d such that $R_d M$ is nonzero, or the largest d such that $\bar{R}_d M$ is nonzero.
(b) Let $d_1 M$ be the smallest d such that λ_d is an isomorphism, or ∞ if no such d exists. Equivalently, $d_1 M$ is the smallest d such that $\mathrm{Ext}^s_{\mathcal{U}}(N, M) = 0$ for $s = 0, 1$ and all $N \in \mathcal{N}il_{d+1}$.

As fundamental examples, we have that $d_0 H^*(V) = d_1 H^*(V) = 0$ for all elementary abelian p-groups V.

Proposition 3.12. *Let M and N be unstable modules.*

(a) *For $s = 0, 1$, $d_s(M \oplus N) = \max\{d_s M, d_s N\}$.*
(b) *If M and N are nonzero, $d_0(M \otimes N) = d_0 M + d_0 N$ and $d_1(M \otimes N) = \max\{d_1 M + d_0 N, d_0 M + d_1 N\}$.*

(c) *For* $s = 0, 1$, $d_s T_V M = d_s M$.
(d) *If* M *is nonzero, for* $s = 0, 1$, $d_s(\Sigma^n M) = d_s M + n$.

For properties (a) and (b) see [25, Proposition I.3.6]. Using the exactness of T_V, property (c) follows from Proposition 3.4(b). Property (d) follows from Proposition 3.3.

Proposition 3.13. *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be a short exact sequence in* $\mathcal{U}$.

(a) *For* $s = 0, 1$, $d_s M_2 \leqslant \max\{d_s M_1, d_s M_3\}$. *Furthermore, if* $d_s M_3 < d_s M_1$, *then* $d_s M_2 = d_s M_1$.
(b) $d_0 M_1 \leqslant d_0 M_2$ *and* $d_1 M_1 \leqslant \max\{d_1 M_2, d_0 M_3\}$. *Furthermore, if* $d_1 M_2 < d_0 M_3$, *then* $d_1 M_1 = d_0 M_3$.

This is proved with straightforward use of the long exact Ext^* sequence associated to a short exact sequence. Compare with [25, Proposition I.3.6].

Corollary 3.14. *If* $M \in \mathcal{U}$ *is reduced, then* $d_1 M = d_0(L_0 M/M)$.

This follows by applying Proposition 3.13(b) to $0 \to M \to L_0 M \to L_0 M/M \to 0$.

3.4. Basic properties of $d_0(G)$ *and* $d_1(G)$

By abuse of notation, if G is a finite group, for $s = 0, 1$, we write $d_s(G)$ for $d_s H^*(G)$. For example, $d_0(V) = d_1(V) = 0$ for all elementary abelian p-groups V.

The properties of $d_0 M$ and $d_1 M$ presented above have the following immediate consequences for $d_0(G)$ and $d_1(G)$.

Proposition 3.15. *Let* G *and* H *be finite groups.*

(a) $d_0(G \times H) = d_0(G) + d_0(H)$.
(b) $d_1(G \times H) = \max\{d_1(G) + d_0(H), d_0(G) + d_1(H)\}$.
(c) *If* P *is a* p*-Sylow subgroup of* G, *then* $d_s(G) \leqslant d_s(P)$ *for* $s = 0, 1$.
(d) *If* V *is a* p*-elementary abelian subgroup of* G, *then* $d_s(C_G(V)) \leqslant d_s(G)$ *for* $s = 0, 1$.

Properties (a) and (b) follow from Proposition 3.12(b). As the unstable module $H^*(G)$ is a direct summand of $H^*(P)$ if P is a p-Sylow subgroup, property (c) follows from Proposition 3.12(a). Similarly property (d) follows from Proposition 3.12(c), as $H^*(C_G(V))$ is a direct summand of $T_V H^*(G)$ [29].[10]

4. Formulae for $H^*(G)_{LF}$ and $\bar{R}_d(H^*(G))$

In this section we prove the formulae for $H^*(G)_{LF}$ and $\bar{R}_d H^*(G)$ given in Section 2.

[10] It is unfortunate that this much referenced elegant 1986 preprint has never been published.

4.1. *A formula for $L_d H^*(G)$*

The starting point for all of these are the following constructions. Given a morphism $\alpha : V_1 \to V_2$ in $\mathcal{A}(G)$, there are maps

$$\alpha^* : H^*(V_2) \to H^*(V_1),$$
$$\alpha_* : H^*\big(C_G(V_1)\big) \to H^*\big(C_G(V_2)\big), \quad \text{and}$$
$$m^*_\alpha : H^*\big(C_G(V_2)\big) \to H^*(V_1) \otimes H^*\big(C_G(V_2)\big).$$

Here α_* is induced by conjugation by g^{-1} where $g \in G$ is any element[11] chosen so that conjugation by g induces α, and

$$m_\alpha : V_1 \times C_G(V_2) \to C_G(V_2)$$

is the homomorphism sending (x, y) to $\alpha(x)y$. We also let $m_V : V \times V \to V$ denote multiplication in an elementary abelian group V.

To state one of the formulae from [25], we recall two other bits of notation from Section 2. Given an unstable module M, we let $M^{\leqslant d}$ denote M modulo degrees greater than d. Given a category $\mathcal{C}$, we let $\mathcal{C}^\#$ denote the associated twisted arrow category: the objects of $\mathcal{C}^\#$ are the morphisms of $\mathcal{C}$, and a morphism $\alpha \rightsquigarrow \beta$ from $\alpha : A_1 \to A_2$ to $\beta : B_1 \to B_2$ is a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd} A_1 \arrow[r, "\alpha"] \arrow[d] & A_2 \\ B_1 \arrow[r, "\beta"] & B_2 . \arrow[u] \end{tikzcd}$$

[25, Formula I(5.5.1)] now reads

Theorem 4.1. *The homomorphisms $V_1 \times C_G(V_2) \xrightarrow{m_\alpha} C_G(V_2) \subset G$ induce an isomorphism of unstable algebras from $L_d H^*(G)$ to*

$$\lim_{V_1 \xrightarrow{\alpha} V_2} \mathrm{Eq}\Big\{ H^*(V_1) \otimes H^{\leqslant d}\big(C_G(V_2)\big) \underset{\nu(\alpha)}{\overset{\mu(\alpha)}{\rightrightarrows}} H^*(V_1) \otimes \big(H^*(V_1) \otimes H^*\big(C_G(V_2)\big)\big)^{\leqslant d} \Big\},$$

where $\mu(\alpha)$ is induced by $1 \otimes m^*_\alpha$, $\nu(\alpha)$ is induced by $m^*_{V_1} \otimes 1$, and the limit is over $\mathcal{A}(G)^\#$.

4.2. *A formula for $\bar{R}_d H^*(G)$*

Recall our notation from Section 2: if W is a central elementary abelian p-subgroup of Q, then $P_W H^*(Q)$ denotes the algebra of primitives in the $H^*(W)$-comodule $H^*(Q)$.

[11] This is well defined as any two choices will differ by an element of $C_G(V_1)$, and so will agree on cohomology.

Proposition 4.2. *As unstable modules, $\bar{R}_d H^*(G)$ is naturally isomorphic to*

$$\lim_{V_1 \xrightarrow{\alpha} V_2} H^*(V_1) \otimes P_{\alpha(V_1)} H^d\big(C_G(V_2)\big),$$

where the limit is over $\mathcal{A}(G)^{\#}$.

Proof. Recall that $\Sigma^d \bar{R}_d M$ is the kernel of $L_d M \to L_{d-1} M$. As kernels commute with limits and equalizers, it follows from the previous theorem that $\bar{R}_d H^*(G)$ is naturally isomorphic to

$$\lim_{V_1 \xrightarrow{\alpha} V_2} \mathrm{Eq}\Big\{H^*(V_1) \otimes H^d\big(C_G(V_2)\big) \underset{\nu(\alpha)}{\overset{\mu(\alpha)}{\rightrightarrows}} H^*(V_1) \otimes \big(H^*(V_1) \otimes H^*\big(C_G(V_2)\big)\big)^d\Big\},$$

where $\mu(\alpha)$ is induced by $1 \otimes m_\alpha^*$ and $\nu(\alpha)$ is induced by $m_{V_1}^* \otimes 1$. But now we observe that the equalizer in this formula is precisely $H^*(V_1) \otimes P_{\alpha(V_1)} H^d(C_G(V_2))$. For $\nu(\alpha)$ is the composite

$$\begin{aligned} H^*(V_1) \otimes H^d\big(C_G(V_2)\big) &\xrightarrow{m_{V_1}^* \otimes 1} H^*(V_1) \otimes H^*(V_1) \otimes H^d\big(C_G(V_2)\big) \\ &\xrightarrow{\text{truncate}} H^*(V_1) \otimes H^0(V_1) \otimes H^d\big(C_G(V_2)\big), \end{aligned}$$

and this identifies with

$$H^*(V_1) \otimes H^d\big(C_G(V_2)\big) \xrightarrow{1 \otimes \pi^*} H^*(V_1) \otimes \big(H^*(V_1) \otimes H^*\big(C_G(V_2)\big)\big)^d,$$

where $\pi : V_1 \times C_G(V_2) \to C_G(V_2)$ is the projection. □

4.3. A formula for $H^*(G)_{LF}$

Proposition 4.3. *As unstable algebras, $H^*(G)_{LF}$ is naturally isomorphic to*

$$\lim_{V_1 \xrightarrow{\alpha} V_2} P_{\alpha(V_1)} H^*\big(C_G(V_2)\big),$$

where the limit is over $\mathcal{A}(G)^{\#}$.

Proof. As there are no nonzero locally finite elements in $\tilde{H}^*(V_1) \otimes H^*(C_G(V_2))$, the composite $H^*(G)_{LF} \subset H^*(G) \to H^*(C_G(V_2))$ has image in $P_{\alpha(V_1)} H^*(C_G(V_2))$ for any $\alpha : V_1 \to V_2$ in $\mathcal{A}(G)$. Thus one gets a natural map of unstable algebras

$$H^*(G)_{LF} \to \lim_{V_1 \xrightarrow{\alpha} V_2} P_{\alpha(V_1)} H^*\big(C_G(V_2)\big).$$

That this is an isomorphism follows from Proposition 4.2, recalling that Proposition 3.9 said that there is a natural isomorphism $(\bar{R}_d M)^0 \simeq (M_{LF})^d$. □

4.4. Replacing $\mathcal{A}(G)$ with $\mathcal{A}_C(G)$

Recall that $C(G)$ denotes the maximal central p-elementary abelian subgroup of G, and $\mathcal{A}_C(G)$ denotes the full subcategory of $\mathcal{A}(G)$ with objects $C(G) \leqslant V < G$.

Theorem 4.4. *One can take the limit over $\mathcal{A}_C(G)^\#$, rather than $\mathcal{A}(G)^\#$ in Theorem* 4.1*, Propositions* 4.2 *and* 4.3.

This will follow quite formally from the following simple observations. Let $C = C(G)$. Given $V < G$, let $CV < G$ be the subgroup generated by C and V. This induces an evident functor $C : \mathcal{A}(P) \to \mathcal{A}_C(G)$. Furthermore, the natural inclusion $V \to CV$ induces an identification $C_G(CV) = C_G(V)$.

Given $\alpha : V_1 \to V_2$, let $\alpha_C : V_1 \to CV_2$ be the evident map, and then let

$$\alpha \xleftarrow{f_\alpha} \alpha_C \xrightarrow{g_\alpha} C\alpha,$$

be morphisms in $\mathcal{A}(G)^\#$, correspond to the diagram in $\mathcal{A}(G)$

$$\begin{tikzcd} V_1 \arrow[r, equal] \arrow[d, "\alpha"] & V_1 \arrow[r] \arrow[d, "\alpha_C"] & CV_1 \arrow[d, "C\alpha"] \\ V_2 \arrow[r] & CV_2 \arrow[r, equal] & CV_2. \end{tikzcd}$$

Lemma 4.5. *Let $F : \mathcal{A}(G)^\# \to \mathbb{F}_p$-vector spaces be a contravariant functor such that for all $\alpha : V_1 \to V_2$, $F(f_\alpha) : F(\alpha) \to F(\alpha_C)$ is an isomorphism. Then the natural map*

$$\Psi : \lim_{\alpha \in \mathcal{A}(G)^\#} F(\alpha) \to \lim_{\alpha \in \mathcal{A}_C(G)^\#} F(\alpha)$$

is an isomorphism.

Note that both $F(V_1 \xrightarrow{\alpha} V_2) = H^*(V_1)$ and $F(V_1 \xrightarrow{\alpha} V_2) = H^*(C_G(V_2))$ satisfy the hypothesis of the lemma. Theorem 4.4 then follows from the lemma, as the relevant F's are built from these two examples by constructions that preserve isomorphisms.

Proof of Lemma 4.5. We define $\Phi : \lim_{\alpha \in \mathcal{A}^C(G)^\#} F(\alpha) \to \lim_{\alpha \in \mathcal{A}(P)^\#} F(\alpha)$, an inverse to Ψ, as follows. Given $x = (x_\beta) \in \lim_{\beta \in \mathcal{A}^C(G)^\#} F(\beta)$, let $\Phi(x) = (\Phi(x)_\alpha) \in \prod_{\alpha \in \mathcal{A}(G)^\#} F(\alpha)$, where $\Phi(x)_\alpha = F(f_\alpha)^{-1} F(g_\alpha)(x_{C\alpha})$. One then checks that $\Phi(x) \in \lim_{\alpha \in \mathcal{A}(G)^\#} F(\alpha)$, $\Psi \circ \Phi = 1$, and $\Phi \circ \Psi = 1$. $\square$

4.5. Rewriting the formulae

If $\mathcal{C}$ is a small category, and

$$F : \mathcal{C}^\# \to \mathbb{F}_p\text{-vector spaces}$$

is a contravariant functor, there is a canonical isomorphism

$$\lim_{\mathcal{C}^\#} F = \mathrm{Eq}\left\{ \prod_{C \in ob\,\mathcal{C}} F(1_C) \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_{\alpha \in mor\,\mathcal{C}} F(\alpha) \right\},$$

where, given $\alpha : C_1 \to C_2$, the α-component of μ and ν are induced by applying F to the canonical morphisms in $\mathcal{C}^\#$ from α to 1_{C_1} and 1_{C_2}, respectively.

Thus, for example, $\bar{R}_d H^*(G)$ will be naturally isomorphic to

$$\mathrm{Eq}\left\{ \prod_V H^*(V) \otimes P_V H^d\big(C_G(V)\big) \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_{\alpha : V_1 \to V_2} H^*(V_1) \otimes P_{\alpha(V_1)} H^d\big(C_G(V_2)\big) \right\},$$

where μ and ν are induced by

$$1 \otimes \alpha_* : H^*(V_1) \otimes P_{V_1} H^d\big(C_G(V_1)\big) \to H^*(V_1) \otimes P_{\alpha(V_1)} H^d\big(C_G(V_2)\big)$$

and

$$\alpha^* \otimes i : H^*(V_2) \otimes P_{V_2} H^d\big(C_G(V_2)\big) \to H^*(V_1) \otimes P_{\alpha(V_1)} H^d\big(C_G(V_2)\big)$$

for each α: $V_1 \to V_2$. (i is the evident inclusion.)

Morphisms in $\mathcal{A}(G)$ factor as inclusions followed by isomorphisms induced by the inner automorphism group $Inn(G)$, so this last formula rewrites as follows.

Proposition 4.6. *$\bar{R}_d H^*(G)$ is naturally isomorphic to*

$$\mathrm{Eq}\left\{ \left[\prod_V H^*(V) \otimes P_V H^d\big(C_G(V)\big)\right]^{Inn(G)} \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_{V_1 < V_2} H^*(V_1) \otimes P_{V_1} H^d\big(C_G(V_2)\big) \right\},$$

where μ and ν are induced by

$$1 \otimes \eta_* : H^*(V_1) \otimes P_{V_1} H^d\big(C_G(V_1)\big) \to H^*(V_1) \otimes P_{V_1} H^d\big(C_G(V_2)\big)$$

and

$$\eta^* \otimes i : H^*(V_2) \otimes P_{V_2} H^d\big(C_G(V_2)\big) \to H^*(V_1) \otimes P_{V_1} H^d\big(C_G(V_2)\big)$$

for each inclusion η: $V_1 < V_2$ in $\mathcal{A}_C(G)$.

Otherwise said,

$$x = (x_V) \in \left[\prod_V H^*(V) \otimes P_V H^d\big(C_G(V)\big)\right]^{Inn(G)}$$

is in $\bar{R}_d H^*(G)$ exactly when the components are related by

$$(1 \otimes \eta_*)(x_{V_1}) = (\eta^* \otimes i)(x_{V_2}),$$

for each inclusion $\eta : V_1 < V_2$ in $\mathcal{A}_C(G)$.

Similarly, we have

Proposition 4.7. *$H^*(G)_{LF}$ is naturally isomorphic to*

$$\mathrm{Eq}\left\{\left[\prod_V P_V H^*\big(C_G(V)\big)\right]^{Inn(G)} \underset{\nu}{\overset{\mu}{\rightrightarrows}} \prod_{V_1 < V_2} P_{V_1} H^*\big(C_G(V_2)\big)\right\},$$

where μ and ν are induced by

$$\eta_* : P_{V_1} H^*\big(C_G(V_1)\big) \to P_{V_1} H^*\big(C_G(V_2)\big)$$

and

$$i : P_{V_2} H^*\big(C_G(V_2)\big) \subseteq P_{V_1} H^*\big(C_G(V_2)\big)$$

for each inclusion $\eta : V_1 < V_2$ in $\mathcal{A}_C(G)$.

5. The cohomology of central extensions

Let C be a central p-elementary abelian subgroup of a finite group G, and let $Q = G/C$. This is the first of two sections in which we study the rich structure of the Lyndon–Hochschild–Serre spectral sequence $\{E_r^{*,*}(G, C)\}$ associated to the central extension:

$$C \xrightarrow{i} G \xrightarrow{q} Q.$$

En route, we will prove Theorem 2.1.

To begin, we recall the following standard facts. The spectral sequence is a spectral sequence of differential graded algebras, converging to $H^*(G)$, and with $E_2^{*,*} = H^*(Q) \otimes H^*(C)$. Furthermore, $E_r^{*,*} = E_\infty^{*,*}$ for $r \gg 0$ [17].

Recall that the extension corresponds to an element $\tau \in H^2(Q; C)$, or equivalently a homomorphism $\tau : H_2(Q) \to C$. Under the identification $C^\# = H^1(C)$, its dual $\tau^\# : C^\# \to H^2(Q)$ corresponds to $d_2 : E_2^{0,1} \to E_2^{2,0}$.

5.1. *$H^*(C)$-comodule structure of the spectral sequence*

As C is central, multiplication $m : C \times G \to G$ is a group homomorphism. The induced algebra map

$$m^* : H^*(G) \to H^*(C) \otimes H^*(G),$$

makes $H^*(G)$ into a $H^*(C)$-comodule. The restriction $i^* : H^*(G) \to H^*(C)$ is both an algebra and comodule map, and it follows that $E_\infty^{0,*} = \mathrm{im}(i^*)$ is a sub-Hopf algebra of $E_2^{0,*} = H^*(C)$.

One can strengthen these last observations to statements about the whole spectral sequence. A good functorial model for BG, say the reduced bar construction, shows that BC is an abelian topological group, BG is a BC-space equipped with proper free action via $Bm : BC \times BG \to BG$, and $BG \to BQ$ is the associated principal BC-bundle. The Serre spectral sequence arises from the pullback to BG of the skeletal filtration of BQ. This will be a filtration of BG by BC-subspaces, and we conclude the following.

Lemma 5.1. *For all k and r, $E_r^{k,*}$ is an $H^*(C)$-comodule, such that the maps*

$$d_r : E_r^{k,*} \to E_r^{k+r,*}$$

and

$$E_r^{i,*} \otimes E_r^{j,*} \to E_r^{i+j,*}$$

are maps of $H^*(C)$-comodules. In particular, $E_r^{0,*}$ is a sub-Hopf algebra of $E_2^{0,*} = H^*(C)$.

5.2. A handy Hopf algebra lemma

We now digress to state and prove a handy statement about (connected graded) Hopf algebras that we can apply to the situation of the previous subsection.

We need some notation. Let H be a graded connected Hopf algebra over a field $\mathbb{F}$. There is a canonical splitting of vector spaces $H = \mathbb{F} \oplus I(H)$, where $I(H)$ is the augmentation ideal. If M is a right H-module, let the module of indecomposables be defined by $Q_H M = M \otimes_H \mathbb{F} = M/MI(H)$. Dually, if M is a right H-comodule, let the module of primitives be defined by

$$\begin{aligned} P_H M &= \mathrm{Eq}\Big\{ M \underset{i}{\overset{\Delta}{\rightrightarrows}} M \otimes H \Big\} \\ &= \ker\big\{ \bar{\Delta} : M \to M \otimes I(H) \big\}, \end{aligned}$$

where $\Delta : M \to M \otimes H$ is the comodule structure, i is the inclusion induced by the unit $\mathbb{F} \to H$, and $\bar{\Delta}$ is the composite $M \xrightarrow{\Delta} M \otimes H \to M \otimes I(H)$.

Lemma 5.2. *Let K be a sub-Hopf algebra of a Hopf algebra H. Suppose M is simultaneously an H-comodule and K-module such that the K-module structure map $M \otimes K \to M$ is a map of H-comodules. Then*

(a) *M is a free K-module, and*
(b) *the composite $P_H M \hookrightarrow M \twoheadrightarrow Q_K M$ is monic.*

Remark 5.3. To put this in perspective, the lemma has long been known if $K = H$, and, in this case, $P_H M \simeq Q_H M$ [37, Theorem 4.1.1]. Our proof is very similar to the proofs of Proposition 1.7 and Theorem 4.4 of Milnor and Moore's classic paper [32]. Compare also to Green's lemma [22, Lemma 2.1].

Before proving the lemma, we note the following consequence. Given H and K as in the lemma, let $K - H - \mathcal{M}od$ be the category of M as in the lemma: an object is a vector space M that is simultaneously an H-comodule and K-module such that the K-module structure map $M \otimes K \to M$ is a map of H-comodules. Morphisms are linear maps that are both K-module and H-comodule maps. $K - H - \mathcal{M}od$ is an abelian category in the obvious way.

Corollary 5.4. (a) *Every short exact sequence* $0 \to M_1 \to M \to M_2 \to 0$ *in* $K - H - \mathcal{M}od$ *is split as a sequence of* K*-modules.*

(b) *The functor sending* M *to* $Q_K(M)$ *is exact on* $K - H - \mathcal{M}od$.

Proof of Lemma 5.2. Choose a section $s : Q_K M \to M$ of the quotient $\pi : M \to Q_K M$, and let $m_s : Q_K M \otimes I(K) \to MI(K)$ be the epimorphism given by $m_s(x, k) = s(x)k$. Statement (a) is asserting that m_s is an isomorphism.

Let $\Delta_K : MI(K) \to M \otimes I(H)$ be the composite

$$MI(K) \subset M \xrightarrow{\bar{\Delta}} M \otimes I(H).$$

Statement (b) asserts that $P_H M \cap MI(K) = \{0\}$, i.e. that Δ_K is monic.

Thus both statements will follow from the following claim:

$$\Delta_K \circ m_s : Q_K M \otimes I(K) \to M \otimes I(H)$$

is monic.

To prove this claim, let $F_n M$ be the K-submodule of M generated by elements of degree up to n. Given $x \in (Q_K M)^n$, and $k \in I(K)$, let $\Delta(s(x)) = \sum y' \otimes h'$, and $\Delta(k) = \sum k' \otimes k''$. Then

$$\begin{aligned}\Delta_K\big(m_s(x, k)\big) &= \bar{\Delta}\big(s(x)k\big)\\ &\equiv s(x) \otimes k\end{aligned}$$

modulo terms of the form $y'k' \otimes h'k''$ with either $|y'| < |s(x)| = n$, or $k' \in I(K)$. Otherwise said,

$$\Delta_K\big(m_s(x, k)\big) \equiv s(x) \otimes k \mod \big(F_{n-1}M + I(K)M\big) \otimes I(H).$$

Thus

$$\pi\big(\Delta_K\big(m_s(x, k)\big)\big) \equiv x \otimes k \mod (Q_K M)^{<n} \otimes I(H),$$

and so both $\pi \circ \Delta_K \circ m_s$ and $\Delta_K \circ m_s$ are monic. □

5.3. Proof of statements (a) and (b) of Theorem 2.1

Let $B_r^{*,*} \subseteq Z_r^{*,*} \subseteq E_r^{*,*}$ denote the r-boundaries and r-cycles of the spectral sequence.

We can apply Lemma 5.2 to our spectral sequence by letting $H = H^*(C)$, $K = E_\infty^{0,*}$, and M any of $E_r^{k,*}$, $Z_r^{k,*}$, $B_r^{k,*}$. We deduce

Proposition 5.5. *For all k and r, we have*

(a) $E_r^{k,*}$, $Z_r^{k,*}$, *and* $B_r^{k,*}$ *are free* $E_\infty^{0,*}$*-modules, and*
(b) *the composite* $P_{H^*(C)}E_r^{k,*} \hookrightarrow E_r^{k,*} \twoheadrightarrow Q_{E_r^{0,*}}E_r^{k,*}$ *is monic.*

It follows that the short exact sequences of $E_\infty^{0,*}$-modules

$$0 \to Z_r^{*,*} \to E_r^{*,*} \to B_r^{*,*} \to 0,$$

and

$$0 \to B_r^{*,*} \to Z_r^{*,*} \to E_{r+1}^{*,*} \to 0,$$

are all split as $E_\infty^{0,*}$-modules. Thus the spectral sequence remains a spectral sequence after applying $Q_{E_\infty^{0,*}}$.

Now let A be a (G, C)-Duflot subalgebra of $H^*(G)$ as defined in Section 2: a subalgebra such that the composite

$$A \hookrightarrow H^*(G) \overset{i^*}{\twoheadrightarrow} \operatorname{im}(i^*) = E_\infty^{0,*}$$

is an isomorphism.[12] We check the first two parts of Theorem 2.1: (a) $H^*(G)$ is a free A-module so that the spectral sequence $\{Q_{E_\infty^{0,*}}E_r^{*,*}\}$ converges to $Q_A H^*(G)$, and (b) $P_{H^*(C)}H^*(G) \to Q_A H^*(G)$ is monic.

Let $F_k BG$ be the inverse image of the k-skeleton of BQ under the projection $BG \to BQ$, and then let F^k be the image of $H^*(BG) \to H^*(F_k BG)$. Then $F^0 = E_\infty^{0,*}$, and for $k \geqslant 1$, there are short exact sequences

$$0 \to E_\infty^{k,*} \to F^k \to F^{k-1} \to 0 \tag{5.1}$$

of objects that are simultaneously A-modules and $H^*(C)$-comodules. Proposition 5.5(a) and induction on k show these sequences split as A-modules.

Now consider the induced diagram

$$\begin{tikzcd}
0 \arrow[r] & P_{H^*(C)}E_\infty^{k,*} \arrow[r] \arrow[d] & P_{H^*(C)}F^k \arrow[r] \arrow[d] & P_{H^*(C)}F^{k-1} \arrow[d] & \\
0 \arrow[r] & Q_A E_\infty^{k,*} \arrow[r] & Q_A F^k \arrow[r] & Q_A F^{k-1} \arrow[r] & 0.
\end{tikzcd}$$

Here the top sequence is exact as indicated, as is the bottom, as (5.1) is split as A-modules. The left vertical arrow is monic by Proposition 5.5(b), as is the right vertical arrow, by induction on k, and it follows the middle arrow is also.

[12] We still need to show that such subalgebras exist.

Thus we have proved that, for all k, F^k is a free A-module and $P_{H^*(C)}F^k \to Q_A F^k$ is monic. As the connectivity of the maps $H^*(G) \to F^k$ goes to infinity as k goes to infinity, we conclude that the same is true for $H^*(G)$.

Remark 5.6. The decreasing filtration of $H^*(G)$ induces a filtration on $P_C H^*(G)$ and $Q_A H^*(G)$. We have shown that $Q_{E_\infty^{0,*}} E_\infty^{*,*}$ is the bigraded algebra associated to the filtration of $Q_A H^*(G)$. By contrast, we can only conclude that the associated bigraded algebra of $P_C H^*(G)$ embeds in $P_{H^*(C)} E_\infty^{*,*}$.

5.4. *Finite generation*

Statement (c) of Theorem 2.1 says that both $P_C H^*(G)$ and $Q_A H^*(G)$ are finitely generated $H^*(Q)$-modules. Our proof of this is similar to arguments used by L. Evens in [17].

We first note that $i^*: H^*(G) \to H^*(C)$ makes $H^*(C)$ into a finitely generated $H^*(G)$-module. Otherwise put, $E_2^{0,*}$ is a finitely generated module over the ring $E_\infty^{0,*}$, which is Noetherian.

It follows that $E_2^{*,*} = H^*(Q) \otimes E_2^{0,*}$ is a finitely generated module over the Noetherian ring $H^*(Q) \otimes E_\infty^{0,*}$. By induction on r, we conclude that, for all $r \geqslant 2$, $E_r^{*,*}$ is a finitely generated $H^*(Q) \otimes E_\infty^{0,*}$-module.

Passing to $E_\infty^{0,*}$-indecomposables, it follows that $Q_{E_\infty^{0,*}} E_\infty^{*,*}$ is a finitely generated $H^*(Q)$-module, and thus the same is true for $Q_A H^*(G)$, $P_{H^*(C)} E_\infty^{*,*}$, and $P_{H^*(C)} H^*(G)$.

5.5. *The image of inflation*

The quotient map $q: G \to G/C$ induces the inflation homomorphism $q^*: H^*(G/C) \to H^*(G)$. Its image, $\operatorname{im}(q^*)$, is an unstable subalgebra of $H^*(G)$ and also identifies with $E_\infty^{*,0}$ in the spectral sequence.

One approach to understanding $\operatorname{im}(q^*)$ is to try to understand $\ker(q^*)$. Recall that the classifying homomorphism $\tau^\#: C^\# \to H^2(G/C)$ corresponds to $d_2 : E_2^{0,1} \to E_2^{2,0}$. Thus $\ker(q^*)$ is an ideal that is closed under Steenrod operations, and contains $\operatorname{im}(\tau^\#)$. As in Section 2, we let $I_\tau \subset H^*(G/C)$ be the smallest ideal with these properties. Thus there is an epimorphism of unstable algebras

$$H^*(G/C)/I_\tau \twoheadrightarrow \operatorname{im}(q^*),$$

which in many cases is an isomorphism.

As has already been said, $\operatorname{im}(q^*)$ is contained in the subalgebra $P_C H^*(G)$, but it seems worthwhile, at this point, to explicitly explain why. The diagram

$$C \times G \underset{\pi}{\overset{m}{\rightrightarrows}} G \xrightarrow{q} G/C$$

is a coequalizer diagram in the category of groups, i.e., a group homomorphism $f: G \to H$ satisfies $f \circ m = f \circ \pi$ if and only if f factors uniquely through q. Applying cohomology, we have that $q^* \circ m^* = q^* \circ \pi^*$, and so

$$\mathrm{im}(q^*) \subseteq \mathrm{Eq}\{m^*, \pi^*\} = P_C H^*(G).$$

In degree 1, inflation is as nice as possible.

Lemma 5.7. $q^* : H^1(G/C) \to P_C H^1(G)$ *is an isomorphism.*

Proof. The exact sequence arising from the corner of the spectral sequence,

$$0 \to H^1(G/C) \xrightarrow{q^*} H^1(G) \xrightarrow{i^*} H^1(C),$$

can be viewed as the degree 1 part of a sequence of $H^*(C)$-comodules, if $H^*(G/C)$ is given the trivial comodule structure. Taking primitives yields an exact sequence

$$0 \to P_C H^1(G/C) \xrightarrow{q^*} P_C H^1(G) \xrightarrow{i^*} P_C H^1(C)$$

which identifies with

$$0 \to H^1(G/C) \xrightarrow{q^*} P_C H^1(G) \to 0,$$

as $P_C H^1(G/C) = H^1(G/C)$ and $P_C H^1(C) = 0$. □

In higher degrees, the inclusion $\mathrm{im}(q^*) \subseteq P_C H^*(G)$ certainly may be proper: see Example 9.1. However, we now show that the $\mathcal{N}il_1$-closures of each of the maps

$$H^*(G/C)/I_\tau \twoheadrightarrow \mathrm{im}(q^*) \hookrightarrow P_C H^*(G)$$

is an isomorphism. Equivalently, the composite is an F-isomorphism, as asserted in statement (d) of Theorem 2.1.

We prove this in two steps.

Proposition 5.8. $L_0(\mathrm{im}(q^*)) \simeq L_0(P_C H^*(G))$.

Proof. By Proposition 3.8, we need to show that, for all U, there are bijections

$$\mathrm{Hom}_{\mathcal{K}}\big(P_C H^*(G), H^*(U)\big) \simeq \mathrm{Hom}_{\mathcal{K}}\big(\mathrm{im}\big(\mathrm{Inf}^G_{G/C}\big), H^*(U)\big).$$

Using that $H^*(U)$ is injective in $\mathcal{K}$, $\mathrm{Hom}_{\mathcal{K}}(P_C H^*(G), H^*(U))$ identifies with

$$\mathrm{Coeq}\Big\{\mathrm{Hom}_{\mathcal{K}}\big(H^*(C \times G), H^*(U)\big) \underset{\pi_*}{\overset{m_*}{\rightrightarrows}} \mathrm{Hom}_{\mathcal{K}}\big(H^*(G), H^*(U)\big)\Big\},$$

and $\mathrm{Hom}_{\mathcal{K}}(\mathrm{im}(q^*), H^*(U))$ identifies with the image of

$$\mathrm{Hom}_{\mathcal{K}}\big(H^*(G), H^*(U)\big) \xrightarrow{q^*} \mathrm{Hom}_{\mathcal{K}}\big(H^*(G/C), H^*(U)\big).$$

Lannes showed [30, Proposition 4.3.1] that $\mathrm{Rep}(U, G) \simeq \mathrm{Hom}_{\mathcal{K}}(H^*(G), H^*(U))$, where $\mathrm{Rep}(U, G)$ is set of orbits of $\mathrm{Hom}(U, G)$ under the conjugation action of G. Thus the next lemma is equivalent to the proposition. □

Lemma 5.9. *The diagram of sets*

$$\mathrm{Rep}(U, C\times G) \underset{\pi_*}{\overset{m_*}{\rightrightarrows}} \mathrm{Rep}(U,G) \xrightarrow{q_*} \mathrm{Rep}(U, G/C)$$

is exact in the following sense: *given homomorphisms* $\alpha, \beta : U \to G$, $q\circ\alpha = q\circ\beta$ *if and only if there exists* $\gamma : U \to C\times G$ *such that* $\alpha = m\circ\gamma$ *and* $\beta = \pi\circ\gamma$.

Proof. Such a γ is equivalent to a pair (δ, β), where $\delta : U \to C$ is a homomorphism satisfying $\delta(u)\beta(u) = \alpha(u)$ for all $u \in U$. Now suppose given $\alpha, \beta : U \to G$ such that $q\circ\alpha = q\circ\beta$. Then the function $\delta : U \to G$ defined by $\delta(u) = \alpha^{-1}(u)\beta(u)$ will take values in C because $q\circ\alpha = q\circ\beta$, and will be a homomorphism because C is central in G. $\square$

Proposition 5.10. $L_0(H^*(G/C)/I_\tau) \simeq L_0(\mathrm{im}(q^*))$.

Proof. In this case, $\mathrm{Hom}_{\mathcal{K}}(H^*(G/C)/I_\tau, H^*(U))$ identifies with the set

$$\bigl\{\alpha \in \mathrm{Rep}(U, G/C) \bigm| \alpha^*(\tau) = 0 \in H^2(G/C; C)\bigr\},$$

while $\mathrm{Hom}_{\mathcal{K}}(\mathrm{im}(q^*), H^*(U))$ can be viewed as the set

$$\bigl\{\alpha \in \mathrm{Rep}(U, G/C) \bigm| \alpha \text{ factors through } q : G \to G/C\bigr\}.$$

But these sets are the same, because $\alpha^*(\tau)$ represents the top extension in the pullback diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, equal] & G(\alpha) \arrow[r] \arrow[d] & U \arrow[d, "\alpha"] \\ C \arrow[r] & G \arrow[r, "q"] & G/C, \end{tikzcd}$$

and this extension is trivial if and only if α factors through q. $\square$

We have a formula for $L_0(\mathrm{im}(q^*))$ analogous to the formula

$$L_0\bigl(H^*(G)\bigr) = \lim_{V\in\mathcal{A}(G)} H^*(V).$$

Let $\mathcal{A}(G, C)$ be the full subcategory of $\mathcal{A}(G)$ with objects the $V \in \mathcal{A}(G)$ containing C, and note that $H^*(V/C) = P_C H^*(V)$ for all such V.

Proposition 5.11. $L_0(\mathrm{im}(q^*)) \simeq \lim_{V\in\mathcal{A}(G,C)} H^*(V/C)$.

Proof. As $H^*(V/C)$ is $\mathcal{N}il_1$-closed, so is $\lim_{V\in\mathcal{A}(G,C)} H^*(V/C)$. Arguing as in the previous proofs, the proposition is equivalent to the statement that, for all U, the image of $\mathrm{Rep}(U, G) \xrightarrow{q_*} \mathrm{Rep}(U, G/C)$ identifies with $\mathrm{colim}_{V\in\mathcal{A}(G,C)} \mathrm{Hom}(U, V/C)$. This is easily checked: details are left to the reader. $\square$

These propositions allow us to quickly prove the last statement of Theorem 2.1: the Krull dimension of any of $H^*(G/C)/I_\tau$, $\mathrm{im}(q^*)$, $P_C H^*(G)$, or $Q_A H^*(G)$ equals (the p-rank of G) − (the rank of C).

To begin with, Proposition 3.7 said that $\dim K = \dim L_0 K$ if K is a Noetherian unstable algebra. Thus

$$\dim H^*(G/C)/I_\tau = \dim \mathrm{im}(q^*) = \dim P_C H^*(G) = \dim \lim_{V \in \mathcal{A}(G,C)} H^*(V/C).$$

The second equality also follows from the fact that $P_C H^*(G)$ is a finitely generated $H^*(G/C)/I_\tau$-module, and similarly $\dim H^*(G/C)/I_\tau = \dim Q_A H^*(G)$ is true.

As $\prod_{V \in \mathcal{A}(G,C)} H^*(V/C)$ is a finitely generated $H^*(G/C)$-module, it certainly is also finitely generated over $\lim_{V \in \mathcal{A}(G,C)} H^*(V/C)$. Thus we conclude

$$\begin{aligned}\dim \lim_{V \in \mathcal{A}(G,C)} H^*(V/C) &= \dim \prod_{V \in \mathcal{A}(G,C)} H^*(V/C)\\ &= \max_{V \in \mathcal{A}(G,C)} \{\text{rank of } V/C\}\\ &= (\text{the } p\text{-rank of } G) - (\text{the rank of } C).\end{aligned}$$

6. Transgressions and the structure of $E_r^{0,*}$

In this section we continue our examination of the spectral sequence associated to the central extension $C \xrightarrow{i} G \xrightarrow{q} Q$, where C is p-elementary abelian of rank c. We carefully describe the form of the differentials

$$d_r : E_r^{0,r-1+*} \to E_r^{r,*},$$

and prove that for all r, the Hopf algebra $E_r^{0,*} \subset H^*(C)$ must be a free commutative algebra[13] of a standard form. In particular, $\mathrm{im}(i^*) = E_\infty^{0,*}$ is free, and so a subalgebra of $H^*(G)$ generated by any lift of a minimal set of generators of $\mathrm{im}(i^*)$ will be a (G,C)-Duflot subalgebra.

To begin our analysis, we know that $\tau^\# : C^\# \to H^2(Q)$ corresponds to the transgression $d_2 : E_2^{0,1} \to E_2^{2,0}$.

6.1. The cokernel of τ

The cokernel of τ has a group theoretic meaning. The proof of the next lemma is left to the reader.

Lemma 6.1. *Let $f : C \to C_0$ be the cokernel of $\tau : H_2(Q) \to C$. Then f factors through $C \xrightarrow{i} G$ and is the universal homomorphism from C to a p-elementary abelian group with this property.*

As f is split epic, the lemma tells us that (G,C) is isomorphic to a pair of the form $C_0 \times (G_1, C_1)$, where no factor of C_1 splits off G_1. The spectral sequences will be related by

[13] In the usual graded sense, if p is odd.

$E_r^{*,*}(G,C) \simeq H^*(C_0) \otimes E_r^{*,*}(G_1,C_1)$, and $d_2 : E_r^{0,1}(G_1,C_1) \to E_r^{2,0}(G_1,C_1)$ will be monic. Thus in our analysis, we can assume that $d_2 : E_2^{0,1} \to E_2^{2,0}$ is an inclusion if we wish.

6.2. *Properties of d_r*

Using that $E_r^{*,*}$ is free over $E_r^{0,*}$, we will see that the differentials we are interested in are 'almost' determined by two standard properties.

The first is that d_r is a derivation.

The second is the transgression theorem: recall that $E_r^{0,*}$ is an unstable subalgebra of $H^*(C)$ and $E_r^{*,0}$ is an unstable quotient algebra of $H^*(Q)$. Given $x \in E_r^{0,r-1}$ and $a \in \mathcal{A}$, $ax \in E_{r+|a|}^{0,r+|a|-1}$, and $d_{r+|a|}(ax)$ is represented by $ad_r(x)$.

6.3. *The kernel of the DeRham and Koszul derivations*

Our differentials are modelled by two standard derivations.

Let $S^*(V)$ be the symmetric algebra on a graded $\mathbb{F}_p$-vector space V, with V concentrated in even degrees if p is odd. The DeRham derivation is the derivation

$$d_V : S^*(V) \to S^*(V) \otimes \Sigma V$$

determined by letting $d_V(v) = 1 \otimes \sigma v$ for $v \in V$.

We will need to know its kernel. Let $\Phi : S^*(V) \to S^*(V)$ denote the pth power map.

Lemma 6.2. *The kernel of d_V is $S^*(\Phi(V))$.*

This is a special case of a result due to Cartier [15]. For completeness, we sketch an elegant argument we learned from [18, proof of Proposition 3.3]. The kernel of d_V is H^0 of the DeRham complex $\Omega^*(V) = (S^*(V) \otimes \Lambda^*(\Sigma V), d_V)$. Since $\Omega^*(V \oplus W) \simeq \Omega^*(V) \otimes \Omega^*(W)$, the Kunneth theorem allows one to reduce to the case when V is one dimensional, where is it easily checked.

Let $\Lambda^*(V)$ be the exterior algebra on a graded $\mathbb{F}_p$-vector space V, with V concentrated in odd degrees. The Koszul derivation is the derivation

$$\delta_V : \Lambda^*(V) \to \Lambda^*(V) \otimes \Sigma V$$

determined by letting $\delta_V(v) = 1 \otimes \sigma v$ for $v \in V$. This is the bottom of the Koszul complex $(\Lambda^*(V) \otimes S^*(\Sigma V), \delta_V)$, which is acyclic, and we have

Lemma 6.3. *The kernel of δ_V is $\mathbb{F}_p$, i.e. δ_V is monic in positive degrees.*

6.4. *The structure of $E_r^{0,*}$ when $p = 2$*

If $p = 2$, $E_2^{0,*} = S^*(C^\#)$. As the squaring operation Φ in degree n corresponds to the Steenrod operation Sq^n, the image of $\Phi^{k-1} : E_2^{0,1} \to E_2^{0,2^k}$ lands in the subspace $E_{2^k+1}^{0,2^k}$. Thus we can define an increasing filtration of $C^\#$,

$$C_0^{\#} \subseteq C_1^{\#} \subseteq C_2^{\#} \subseteq \cdots$$

by letting $C_k^{\#}$ be the kernel of the composite

$$E_2^{0,1} \xrightarrow{\Phi^k} E_{2^k+1}^{0,2^k} \xrightarrow{d_{2^k+1}} E_{2^k+1}^{2^k+1,0}.$$

Theorem 6.4. *The only possible nonzero differentials*

$$d_r : E_r^{0,*} \to E_r^{r,*}$$

are d_{2^k+1} with $k = 0, 1, 2, \ldots$. For each such k, $E_{2^k+2}^{0,*} = \ker d_{2^k+1}$ equals

$$S^*\big(C_0^{\#} + \Phi\big(C_1^{\#}\big) + \cdots + \Phi^k\big(C_k^{\#}\big) + \Phi^{k+1}\big(C^{\#}\big)\big).$$

This polynomial algebra is noncanonically isomorphic to

$$S^*\big(V_0 \oplus \Phi(V_1) \oplus \cdots \oplus \Phi^k(V_k) \oplus \Phi^{k+1}\big(C^{\#}/C_k^{\#}\big)\big),$$

where $V_i = C_i^{\#}/C_{i-1}^{\#}$. In particular, one can choose a basis for $C^{\#}$ such that $E_\infty^{0,*}$ has the form described in Section 2.5.

Remark 6.5. In practical terms, the filtration of $C^{\#}$ is often determined by the extension homomorphism $\tau : H_2(Q) \to C$ together with the action of the Steenrod operations on $H^*(Q)$.

Note that $C_k^{\#}$ will also be the kernel of the composite

$$E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \xrightarrow{Sq^1} \cdots \xrightarrow{Sq^{2^{k-1}}} E_2^{2^k+1,0} \twoheadrightarrow E_{2^k+1}^{2^k+1,0},$$

where the last map is the evident quotient. Equivalently, $C_k^{\#}$ is the kernel of

$$C^{\#} \xrightarrow{\tau_G^{\#}} H^2(Q) \xrightarrow{Sq^1} \cdots \xrightarrow{Sq^{2^{k-1}}} H^{2^k+1}(Q) \twoheadrightarrow E_{2^k+1}^{2^k+1,0}.$$

Let $I_\tau(k) \subset H^*(Q)$ be the ideal generated by $\mathcal{A}(k-1) \cdot \mathrm{im}(\tau^{\#})$, where $\mathcal{A}(k) \subset \mathcal{A}$ is the subalgebra generated by $Sq^1, \ldots, Sq^{2^k}$. Then the quotient map $H^*(Q) \twoheadrightarrow E_{2^k+1}^{*,0}$ factors

$$H^*(Q) \twoheadrightarrow H^*(Q)/I_\tau(k-1) \twoheadrightarrow E_{2^k+1}^{*,0},$$

and the second map is often an isomorphism.

Proof of Theorem 6.4. By definition, $\Phi^k(C_k^{\#})$ is the kernel of the transgression $d_{2^k+1} : \Phi^k(C^{\#}) \to E_{2^k+1}^{2^k+1,0}$, and so consists of permanent cycles. It follows that, for each k, the subalgebra $S(k) = S^*(C_0^{\#} + \Phi(C_1^{\#}) + \cdots + \Phi^k(C_k^{\#}))$ is also contained in $E_\infty^{0,*}$.

By induction, we now show that

$$E_{2^k+1}^{0,*} = S^*\big(C_0^\# + \Phi\big(C_1^\#\big) + \cdots + \Phi^{k-1}\big(C_{k-1}^\#\big) + \Phi^k\big(C^\#\big)\big).$$

When $k = 0$, this just says that $E_2^{0,*} = S^*(C^\#)$, and so is certainly true. Now we assume the statement for k and prove it with k replaced by $k+1$.

Let $V = \Phi^k(C^\#)/\Phi^k(C_k^\#)$. Then $E_{2^k+1}^{0,*} \simeq S(k) \otimes S^*(V)$, ΣV identifies with the image of $d_{2^k+1} : \Phi^k(C^\#) \to E_{2^k+1}^{2^k+1,0}$, and we have a commutative diagram

$$\begin{tikzcd} E_{2^k+1}^{0,*} \arrow[r, "d_{2^k+1}"] \arrow[d, "\wr"] & E_{2^k+1}^{2^k+1,*} \\ S(k) \otimes S^*(V) \arrow[r, "1\otimes d_V"] & S(k) \otimes S^*(V) \otimes \Sigma V. \arrow[u] \end{tikzcd}$$

Here the right vertical map is induced from the inclusion $\Sigma V \subseteq E_{2^k+1}^{2^k+1,0}$ using the $E_{2^k+1}^{0,*}$-module structure on $E_{2^k+1}^{2^k+1,*}$.

We have reached the key point in our proof: as in our proof of Theorem 2.1, Lemma 5.2 shows that this module structure is free, and thus the right vertical map is monic. It follows that the kernel of the top map identifies with $\ker(1 \otimes d_V)$ which equals $S(k) \otimes S^*(\Phi(V))$, by Lemma 6.2. Otherwise said,

$$E_{2^k+2}^{0,*} = S^*\big(C_0^\# + \Phi\big(C_1^\#\big) + \cdots + \Phi^k\big(C_k^\#\big) + \Phi^{k+1}\big(C^\#\big)\big).$$

Finally, we note that once we know that $E_{2^k+2}^{0,*}$ has this form, $E_{2^{k+1}+1}^{0,*} = E_{2^k+2}^{0,*}$ follows by the transgression theorem. $\square$

6.5. The structure of $E_r^{0,*}$ when p is odd

When p is odd,

$$E_2^{0,*} = \Lambda^*\big(C^\#\big) \otimes S^*\big(\beta\big(C^\#\big)\big).$$

As the pth power operation Φ in degree $2n$ corresponds to the Steenrod operation $\mathcal{P}^n$, the image of $\Phi^k \circ \beta : E_2^{0,1} \to E_2^{0,2p^k}$ lands in the subspace $E_{2p^k+1}^{0,2p^k}$. Thus we can define an increasing filtration of $C^\#$,

$$C_0^\# \subseteq C_1^\# \subseteq C_2^\# \subseteq \cdots$$

by letting $C_0^\#$ be the kernel of $E_2^{0,1} \xrightarrow{d_2} E_2^{2,0}$, and, for $k \geqslant 0$, $C_{k+1}^\#$ be the kernel of the composite

$$E_2^{0,1} \xrightarrow{\Phi^k\circ\beta} E_{2p^k+1}^{0,2p^k} \xrightarrow{d_{2p^k+1}} E_{2p^k+1}^{2p^k+1,0}.$$

Theorem 6.6. *The only possible nonzero differentials*

$$d_r : E_r^{0,*} \to E_r^{r,*}$$

are d_2 and d_{2p^k+1} with $k = 0, 1, 2, \ldots$. Furthermore,

$$E_3^{0,*} = \Lambda^*\big(C_0^\#\big) \otimes S^*\big(\beta\big(C^\#\big)\big),$$

and, for each $k \geqslant 0$,

$$E_{2p^k+2}^{0,*} = \Lambda^*\big(C_0^\#\big) \otimes S^*\big(\beta\big(C_1^\#\big) + \Phi\beta\big(C_2^\#\big) + \cdots + \Phi^k\beta\big(C_{k+1}^\#\big) + \Phi^{k+1}\beta\big(C^\#\big)\big).$$

This free commutative algebra is noncanonically isomorphic to

$$\Lambda^*(V_0) \otimes S^*\big(\beta(V_0) \oplus \beta(V_1) \oplus \cdots \oplus \Phi^k\beta(V_{k+1}) \oplus \Phi^{k+1}\beta\big(C^\#/C_{k+1}^\#\big)\big),$$

where $V_i = C_i^\#/C_{i-1}^\#$. In particular, one can choose a basis for $C^\#$ such that $E_\infty^{0,*}$ has the form described in Section 2.5.

Remark 6.7. Similar to the case when $p = 2$, $C_{k+1}^\#$ will be the kernel of

$$C^\# \xrightarrow{\tau^\#} H^2(Q) \xrightarrow{\beta} H^3(Q) \xrightarrow{\mathcal{P}^1} \cdots \xrightarrow{\mathcal{P}^{p^{k-1}}} H^{2p^k+1}(Q) \twoheadrightarrow E_{2p^k+1}^{*,0},$$

and the quotient map $H^*(Q) \twoheadrightarrow E_{2p^k+1}^{*,0}$ factors as

$$H^*(Q) \twoheadrightarrow H^*(Q)/I_\tau(k) \twoheadrightarrow E_{2p^k+1}^{*,0}$$

where $I_\tau(k) \subset H^*(Q)$ is the ideal generated by $\mathcal{A}(k-1) \cdot \operatorname{im}(\tau^\#)$. Here $\mathcal{A}(k) \subset \mathcal{A}$ is the subalgebra generated by $\beta, \mathcal{P}^1, \mathcal{P}^p, \ldots, \mathcal{P}^{p^{k-1}}$.

Proof of Theorem 6.6. We just sketch the proof, as it follows along the lines of the proof of the $p = 2$ version of the theorem.

To compute $E_3^{0,*}$, let $V = C^\#/C_0^\#$. Then $\ker d_2$ identifies with the kernel of

$$\Lambda^*\big(C_0^\#\big) \otimes S^*\big(\beta\big(C^\#\big)\big) \otimes \Lambda^*(V) \xrightarrow{1\otimes 1\otimes \delta_V} \Lambda^*\big(C_0^\#\big) \otimes S^*\big(\beta\big(C^\#\big)\big) \otimes \Lambda^*(V) \otimes \Sigma V.$$

The formula for $E_3^{0,*}$ thus follows from Lemma 6.3.

To compute $E_{2p^k+2}^{0,*}$ for $k \geqslant 0$, let $V = \Phi^k\beta(C^\#)/\Phi^k\beta(C_{k+1}^\#)$. Then the subalgebra

$$S(k) = \Lambda^*\big(C_0^\#\big) \otimes S^*\big(\beta\big(C_1^\#\big) + \Phi\beta\big(C_2^\#\big) + \cdots + \Phi^k\beta\big(C_{k+1}^\#\big)\big)$$

is all permanent cycles. Using that $E_{2p^k+1}^{2p^k+1,*}$ is a free $E_{2p^k+1}^{0,*}$-module, $\ker d_{2p^k+1}$ identifies with the kernel of

$$S(k) \otimes S^*(V) \xrightarrow{1\otimes d_V} S(k) \otimes S^*(V) \otimes \Sigma V,$$

and the formula for $E^{0,*}_{2p^k+2}$ follows from Lemma 6.2. $\square$

7. p-central groups

In this section we prove our main theorems about p-central groups: Theorems 2.8 and 2.9.

We begin by recalling some notation from Section 2. If $C = C(G)$ is the maximal central p-elementary abelian subgroup of a finite group G, we have shown that $C^\# = H^1(C)$ admits an ordered basis $(x_1, \ldots, x_c)$ so that, if $y_j = \beta(x_j)$ for p odd,

$$\operatorname{Res}^G_C\big(H^*(G)\big) = \begin{cases} \mathbb{F}_2[x_1^{2^{j_1}}, \ldots, x_c^{2^{j_c}}] & \text{if } p = 2, \\ \mathbb{F}_p[y_1^{p^{j_1}}, \ldots, y_b^{p^{j_b}}, y_{b+1}, \ldots, y_c] \otimes \Lambda(x_{b+1}, \ldots, x_c) & \text{if } p \text{ is odd,} \end{cases}$$

with the j_i forming a sequence of nondecreasing nonnegative integers.

Then we say that G has type $[a_1, \ldots, a_c]$ where

$$(a_1, \ldots, a_c) = \begin{cases} (2^{j_1}, \ldots, 2^{j_c}) & \text{if } p = 2, \\ (2p^{j_1}, \ldots, 2p^{j_b}, 1, \ldots, 1) & \text{if } p \text{ is odd,} \end{cases}$$

and we let

$$e(G) = \sum_{i=1}^{c}(a_i - 1) \quad \text{and} \quad h(G) = \begin{cases} 2p^{k-1} & \text{if } a_1 = 2p^k \text{ with } k \geqslant 1, \\ 1 & \text{if } a_1 = 2, \\ 0 & \text{if } a_1 = 1. \end{cases}$$

We have the following lemma about products.

Lemma 7.1. *Suppose G_0 and G_1 have maximal central p-elementary abelian subgroups C_0 and C_1, and Duflot subalgebras A_0 and A_1. Then the following hold.*

(a) $C_0 \times C_1 = C(G_0 \times G_1)$, *and*

$$P_{C_0 \times C_1} H^*(G_0 \times G_1) = P_{C_0} H^*(G_0) \otimes P_{C_1} H^*(G_1).$$

(b) $A_0 \otimes A_1$ *will be a Duflot subalgebra for* $G_0 \times G_1$, *and*

$$Q_{A_0 \otimes A_1} H^*(G_0 \times G_1) = Q_{A_0} H^*(G_0) \otimes Q_{A_1} H^*(G_1).$$

(c) $e(G_0 \times G_1) = e(G_0) + e(G_1)$, *and* $h(G_0 \times G_1) = \max\{h(G_0), h(G_1)\}$.

Note that a subgroup H of a p-central group G is again p-central, and $C(H) = C(G) \cap H$. The next lemma is easily deduced.

Lemma 7.2. *Let G be a p-central group, and $j : H < G$ a subgroup. Then $e(H) \leqslant e(G)$ and $h(H) \leqslant h(G)$. If $e(H) = e(G)$ and A is a Duflot subalgebra of $H^*(G)$, then $j^*(A)$ will be a Duflot subalgebra of $H^*(H)$.*

Thanks to the first part of this lemma, Corollary 2.10 immediately follows from Theorem 2.9.

Remark 7.3. The example $H = \mathbb{Z}/4 < \mathbb{Z}/8 = G$ shows that the inequalities of the lemma can be equalities, even when H is a proper subgroup of a p-group G.

7.1. Benson–Carlson duality

If G is p-central, then $Q_A H^*(G)$ will be a finite dimensional $\mathbb{F}_p$-algebra if A is any Duflot subalgebra. Benson and Carlson tell us much more:

Theorem 7.4. *If G is p-central and A is a Duflot subalgebra of $H^*(G)$, then $Q_A H^*(G)$ is a Poincaré duality algebra with top class in degree $e(G)$.*

Under the assumption that A is a polynomial algebra (always true if $p = 2$), this is an immediate application of the main theorem in [6]. The general case reduces to this one: G and A will admit decompositions $G = C_0 \times G_1$ and $A = H^*(C_0) \otimes A_1$, with C_0 p-elementary, G_1 having no $\mathbb{Z}/p$ summands, and A_1 a (necessarily polynomial) Duflot subalgebra of $H^*(G_1)$. Then $Q_A H^*(G) = Q_{A_1} H^*(G_1)$, and $e(G) = e(G_1)$.

7.2. Proof of Theorem 2.8

Let G be p-central, $C = C(G)$, and $A \subset H^*(G)$ a Duflot subalgebra. We now prove the various parts of Theorem 2.8.

Firstly, Theorem 7.4 implies that $Q_A H^*(G)$ is zero in degrees greater than $e(G)$, and one dimensional in degree $e(G)$.

Now consider the Serre spectral sequence for $C \to G \to G/C$, as studied in Theorem 2.1. The bigraded algebra $Q_{E_\infty^{0,*}} E_\infty^{*,*}$ is the graded object associated to a decreasing filtration of the Poincaré duality algebra $Q_A H^*(G)$ with top degree $e(G)$. This forces the following to be true: there is a largest s, $s(G)$, such that $E_\infty^{s,*}$ is nonzero, $Q_{E_\infty^{0,*}} E_\infty^{s(G),*}$ will be one dimensional and concentrated in total degree $e(G)$, and nonzero classes in $E_\infty^{s(G),e(G)-s(G)} \subset H^{e(G)}(G)$ will be Poincaré duality classes.

These classes will also be $H^*(C)$-comodule primitives, as $E_\infty^{s(G),*}$ is a sub-$H^*(C)$-comodule of $H^*(G)$, and everything in the lowest degree must be primitive. As $P_C H^*(G)$ is contained in $Q_A H^*(G)$, we conclude that $P_C H^*(G)$ is also zero in degrees greater than $e(G)$, and one dimensional in degree $e(G)$.

By Corollary 2.4, $P_C H^{e(G)}(G)$ is also the top nonzero degree of $H^*(G)_{LF}$, and so consists of classes annihilated by all positive degree Steenrod operations.

It remains to show that, under the additional assumption that G is a p-group, $P_C H^{e(G)} H^*(G)$ is essential cohomology. This we prove in the next subsection.

7.3. p-central p-groups and essential cohomology

Let P be a p-central p-group. We have shown that $H^{e(P)}(P)_{LF} = P_{C(P)} H^{e(P)}(P)$ is a one dimensional subspace of $H^{e(P)}(P)$.

Proposition 7.5. *$H^{e(P)}(P)_{LF}$ is essential.*

Proof. As P is a p-group, maximal proper subgroups have the form $j: Q < P$, where Q is the kernel of a nonzero homomorphism $x: P \to \mathbb{Z}/p$. We need to show that $j^*(\zeta) = 0 \in H^*(Q)$ if $\zeta \in H^{e(P)}(P)_{LF}$ is nonzero.

The map $j^*: H^*(P) \to H^*(Q)$ will take $H^{e(P)}(P)_{LF}$ to $H^{e(P)}(Q)_{LF}$. If $e(Q) < e(P)$, we are done: $j^*(\zeta)$ will be an element of a zero group.

If $e(Q) = e(P)$, we reason as follows. Let A be a Duflot subalgebra of $H^*(P)$, so that $j^*(A)$ is a Duflot subalgebra of $H^*(Q)$. If $j^*(\zeta) \neq 0$, it will project to a nonzero element in $Q_{j^*(A)} H^*(Q)$. We show that this is impossible. Regard x as a nonzero element in $H^1(P)$. By construction, $j^*(x) = 0 \in H^1(Q)$. By the Poincaré duality, there exists $y \in H^*(P)$ such that $\zeta = xy \in Q_A H^*(P)$. But then $j^*(\zeta) = j^*(x) j^*(y) = 0 \in Q_{j^*(A)} H^*(Q)$. $\square$

Let $A(P, P)$ be the two sided Burnside ring over $\mathbb{F}_p$: the $\mathbb{F}_p$-algebra with basis given by equivalence classes of diagrams $P \geqslant Q \xrightarrow{\alpha} P$, and multiplication defined using the double coset formula.[14] If J is the ideal generated by all such diagrams with α *not* an isomorphism, then $A(P, P)/J \simeq \mathbb{F}_p[Out(P)]$, the group ring of the outer automorphism group.

Using transfers (a.k.a. induction), $A(P, P)$ acts on $H^*(P)$, with a basis element

$$\left[P \geqslant Q \xrightarrow{\alpha} P\right]$$

inducing

$$H^*(P) \xrightarrow{\alpha^*} H^*(Q) \xrightarrow{\mathrm{Tr}_Q^P} H^*(P).$$

As these are unstable $\mathcal{A}$-module maps, it follows that $H^{e(P)}(P)_{LF}$ is a one dimensional $A(P, P)$-submodule.

Corollary 7.6. *The ideal J acts trivially on $H^{e(P)}(P)_{LF}$.*

Proof. The previous proposition shows that if a homomorphism $\alpha: Q \to P$ is not onto, then $\alpha^*(H^{e(P)}(P)_{LF}) = 0$. $\square$

It follows that the $A(P, P)$-module $H^{e(P)}(P)_{LF}$ is the pullback of a one dimensional representation of $Out(P)$ over the prime field $\mathbb{F}_p$. We let $\omega(P)$ denote this representation. Clearly $\omega(P)$ will be trivial if $p = 2$, but this need not be the case when p is odd.

Example 7.7. Let $p = 3$. Then $\omega(\mathbb{Z}/9) = H^1(\mathbb{Z}/9)$ is nontrivial, as $-1: \mathbb{Z}/9 \to \mathbb{Z}/9$ induces multiplication by -1 on $H^1(\mathbb{Z}/9)$.

7.4. $d_0(G)$ when G has a p-central p-Sylow subgroup

We prove the parts of Theorem 2.9 involving d_0.

Firstly, if G is p-central, then Corollary 2.6 says that

$$\bar{R}_d H^*(G) \simeq H^*\big(C(G)\big) \otimes P_{C(G)} H^d(G).$$

[14] There are more elegant descriptions, but this is better for our purposes.

Since $d_0(G)$ is the largest d such that $\bar{R}_d H^*(G) \neq 0$, it follows that $d_0(G)$ will equal the top nonzero degree of $P_{C(G)} H^*(G)$, which we have computed to be $e(G)$.

Now suppose that G is not necessarily p-central, but has a p-central p-Sylow subgroup P. We show that then $d_0(G) = d_0(P)$.

We need to show that the largest d such that $\bar{R}_d H^*(G) \neq 0$ is $d = d_0(P) = e(P)$. Let $e_1 \in A(P, P)$ be an idempotent chosen so that $A(P, P)e_1$ is the projective cover of ϵ, the trivial $\mathbb{F}_p[Out(P)]$-module, pulled back to $A(P, P)$. Standard arguments show that there are inclusions

$$e_1 \bar{R}_d H^*(P) \subseteq \bar{R}_d H^*(G) \subseteq \bar{R}_d H^*(P).$$

Thus it suffices to show that $e_1 \bar{R}_{e(P)} H^*(P) \neq 0$. Otherwise said, it suffices to show that ϵ is a composition factor in the $A(P, P)$-modules $\bar{R}_{e(P)} H^*(P)$.

If $p = 2$, we are done: by Corollary 7.6, $\bar{R}_{e(P)} H^{e(P)}(P) = H^{e(P)}(P)_{LF} \simeq \epsilon$. As a bonus, we learn that $H^*(G)_{LF}$ is one dimensional in degree $e(P)$.

When p is odd, more care (and maybe luck) is needed. Recall that $\bar{R}_{e(P)} H^*(P) = H^*(C(P)) \otimes H^{e(P)}(P)_{LF}$. The fact that J acts as 0 on $H^{e(P)}(P)_{LF}$ implies that the same is true for $H^*(C(P)) \otimes H^{e(P)}(P)_{LF}$. Thus we just need to show that the trivial $Out(P)$-module occurs as a composition factor in $H^*(C(P)) \otimes H^{e(P)}(P)_{LF} = H^*(C(P)) \otimes \omega(P)$, or, equivalently, that $\omega(P)^{-1}$ occurs as an $Out(P)$-composition factor $H^*(C(P))$. We are done with the following lemma.[15]

Lemma 7.8. *If P is a p-central p-group with p an odd prime, then every irreducible $\mathbb{F}_p[Out(P)]$-module occurs as a composition factor of $H^*(C(P))$.*

The lemma, in a stronger form than stated, follows by combining [28, Proposition 5.7 and Corollary 6.8]. The key point is that, since $C(P) = \Omega_1(P)$, the kernel of $Aut(P) \to Aut(C(P))$ will be a p-group if p is odd [19, Theorem 5.3.10].

Example 7.9. Let $p = 3$ and G be the semidirect product $\mathbb{Z}/9 \rtimes \mathbb{Z}/2$. Then $d_0(G) = d_0(\mathbb{Z}/9) = 1$, but $\tilde{H}^*(G)_{LF} = \mathbf{0}$.

7.5. $d_1(G)$ *when G is p-central*

In this subsection, let G be p-central. We show that $d_1(G) = e(G) + h(G)$.

We get control of $d_1(G)$ by working directly with the desuspended composition factors $R_d H^*(G)$ of $H^*(G)$, rather than their $\mathcal{N}il_1$-localizations $\bar{R}_d H^*(G)$, as was done in our calculation of $d_0(G)$.

To simplify notation, write R_d for $R_d H^*(G)$, $\bar{R}_d$ for $\bar{R}_d H^*(G)$, and C for $C(G)$. We have that $\bar{R}_0 = H^*(C)$, and $R_0 = \mathrm{im}(i^*)$, where $i : C \hookrightarrow G$ is the inclusion.

In the nilpotent filtration of $H^*(G)$, the last nonzero submodule, $nil_{e(G)}\, H^*(G) = \Sigma^{e(G)} R_{e(G)}$, has been shown to be isomorphic to $\Sigma^{e(G)} R_0$ as an unstable module. Thus d_1 of this submodule of $H^*(G)$ equals $e(G) + d_1(R_0)$. By Proposition 3.13, the next lemma implies that if $d < e(G)$, $d_1(\Sigma^d R_d)$ is strictly smaller than this.

15 This lemma is false if $p = 2$, as the example $P = Q_8$ illustrates.

Lemma 7.10. *Each R_d with $d < e(G)$ admits a filtration by unstable modules with subquotients all of the form $\Sigma^k R_0$ with $d + k < e(G)$.*

Again appealing to Proposition 3.13, we then have the next corollary.

Corollary 7.11. $d_1(G) = d_1(\Sigma^{e(G)} R_{e(G)}) = e(G) + d_1(R_0)$.

Thus we will have proved that, when G is p-central, $d_1(G) = e(G) + h(G)$, once we have proved Lemma 7.10, and calculated that $d_1(R_0) = h(G)$. We begin the proof of Lemma 7.10 here, and then both finish it, and calculate $d_1(R_0)$, in the two subsections that follow, which correspond to the cases $p = 2$ and p odd.

Proof of Lemma 7.10. For all d, we have inclusions

$$R_0 \otimes P_C H^d(G) \subseteq R_d \subseteq \bar{R}_0 \otimes P_C H^d(G),$$

where $P_C H^d(G)$ is regarded as an unstable module concentrated in degree 0. These are inclusions of unstable modules, enriched with compatible R_0-module structures and $\bar{R}_0$-comodule structures. Call the category of such objects $R_0 - \bar{R}_0 - \mathcal{U}$.

Say that $M \in R_0 - \bar{R}_0 - \mathcal{U}$ admits a nice filtration if it admits a filtration in $R_0 - \bar{R}_0 - \mathcal{U}$ with subquotients all of the form $\Sigma^k R_0$. We will show that each R_d admits a nice filtration.

(That the composition factors will then also satisfy $d + k < e(G)$ follows immediately from the fact that $Q_{R_0}(\Sigma^d R_*)$ is a graded object associated to $Q_A H^*(G)$, which we know is one dimensional in degree $e(G)$ and zero above that.)

We claim that, if N admits a nice filtration, and $M \subseteq N$, then M also admits a nice filtration. To see this, suppose $F_0 N \subseteq F_1 N \subseteq \cdots$ is a filtration of N with $F_j N / F_{j-1} N = \Sigma^{k_j} R_0$. Let $F_j M = M \cap F_j N$. Then $F_j M / F_{j-1} M \subseteq F_j N / F_{j-1} N = \Sigma^{k_j} R_0$ will be an inclusion of objects in $R_0 - \bar{R}_0 - \mathcal{U}$ that will be split as R_0-modules, thanks to Corollary 5.4. We conclude that $F_j M / F_{j-1} M$ is either 0 or $\Sigma^{k_j} R_0$.

Thus to prove R_d has a nice filtration, it suffices to prove that $\bar{R}_0 \otimes P_C H^d(G)$ has a nice filtration, or just that $\bar{R}_0$ has a nice filtration. We show this in the next two subsections, which separately deal with the cases $p = 2$ and p is odd. □

7.6. *A calculation of $d_1(R_0)$, and a nice filtration of $\bar{R}_0$, when $p = 2$*

Suppose that $p = 2$, and that

$$R_0 = \mathbb{F}_2\left[x_1^{2^{j_1}}, \ldots, x_1^{2^{j_c}}\right] \subseteq \mathbb{F}_2[x_1, \ldots, x_c] = \bar{R}_0,$$

with $j_1 \geqslant \cdots \geqslant j_c$. We show the following.

Lemma 7.12. *$\bar{R}_0$ has a good filtration as an object in $R_0 - \bar{R}_0 - \mathcal{U}$.*

Lemma 7.13. *If $j_1 > 0$, $d_1(R_0) = 2^{j_1 - 1}$.*

Proof of Lemma 7.12. For all $1 \leqslant b \leqslant c$ and $1 \leqslant i_b \leqslant j_b$, the module

$$R(i_1,\dots,i_c)=\mathbb{F}_2\left[x_1^{2^{i_1}},\dots,x_c^{2^{i_c}}\right]$$

will be an object in $R_0-\bar{R}_0-\mathcal{U}$ in the evident way.

Clearly $R(j_1,\dots,j_c)=R_0$ admits a nice filtration. The short exact sequences

$$0\to R(i_1,\dots,i_c)\to R(i_1,\dots,i_b-1,\dots,i_c)\to \Sigma^{2^{i_b}}R(i_1,\dots,i_c)\to 0$$

then show that if $R(i_1,\dots,i_c)$ admits a nice filtration, so does $R(i_1,\dots,i_b-1,\dots,i_c)$. By downward induction, we conclude that $R(0,\dots,0)=\bar{R}_0$ admits a nice filtration. □

Proof of Lemma 7.13. By Proposition 3.12(c), it suffices to prove that, if $j>0$,

$$d_1\left(\mathbb{F}_2\left[x^{2^j}\right]\right)=2^{j-1}.$$

In the short exact sequence

$$0\to\mathbb{F}_2\left[x^{2^j}\right]\to\mathbb{F}_2\left[x^{2^{j-1}}\right]\to\Sigma^{2^{j-1}}\mathbb{F}_2\left[x^{2^j}\right]\to 0,$$

d_1 of the middle term is strictly less than $d_0(\Sigma^{2^{j-1}}\mathbb{F}_2[x^{2^j}])=2^{j-1}$: this is clear when $j=1$, and for larger j this follows by an inductive hypothesis. Thus Proposition 3.13(b) applies to say that $d_1(\mathbb{F}_2[x^{2^j}])=2^{j-1}$. □

7.7. A calculation of $d_1(R_0)$, and a nice filtration of $\bar{R}_0$, when p is odd

Suppose that p is odd. We can assume that

$$R_0=\mathbb{F}_p\left[y_1^{p^{j_1}},\dots,y_1^{p^{j_c}}\right]\subseteq\Lambda^*(x_1,\dots,x_c)\otimes\mathbb{F}_p[y_1,\dots,y_c]=\bar{R}_0,$$

with $j_1\geqslant\dots\geqslant j_c$, and we show the following.

Lemma 7.14. *$\bar{R}_0$ has a good filtration as an object in $R_0-\bar{R}_0-\mathcal{U}$.*

Lemma 7.15. *If $j_1=0$, $d_1(R_0)=1$. If $j_1>0$, $d_1(R_0)=2p^{j_1-1}$.*

Proof of Lemma 7.14. As a first step, we note that the filtration of $\bar{R}_0$ given by letting $F_k\bar{R}_0=\Lambda^{\leqslant k}(x_1,\dots,x_c)\otimes\mathbb{F}_p[y_1,\dots,y_c]$ is a filtration in the category $R_0-\bar{R}_0-\mathcal{U}$, and the associated subquotients are direct sums of suspensions of $\mathbb{F}_p[y_1,\dots,y_c]$. It follows that it suffices to prove the lemma with $\bar{R}_0$ replaced by $\mathbb{F}_p[y_1,\dots,y_c]$.

Our next reduction will allow us to reduce to the case when $c=1$.

If K is a sub-Hopf algebra of a Hopf algebra H, and both objects and all structure maps are in $\mathcal{U}$, one has a category $K-H-\mathcal{U}$, analogous to $R_0-\bar{R}_0-\mathcal{U}$. One can then say that $M\in K-H-\mathcal{U}$ has a good filtration if it has a filtration with subquotients that are all suspensions of K. It is easy to see that if $M_1\in K_1-H_1-\mathcal{U}$ and $M_2\in K_2-H_2-\mathcal{U}$ have a good filtration, then so does $M_1\otimes M_2$, viewed as an object in $K_1\otimes K_2-H_1\otimes H_2-\mathcal{U}$.

Applying this observation to the evident tensor decompositions of $K = R_0$ and $H = \mathbb{F}_p[y_1, \ldots, y_c]$, we are left just needing to show that $\mathbb{F}_p[y]$ has a nice filtration, when viewed as an object in $\mathbb{F}_p[y^{p^j}] - \mathbb{F}_p[y] - \mathcal{U}$.

By downwards induction on i, we show that, for $0 \leqslant i \leqslant j$, $\mathbb{F}_p[y^{p^i}]$ has a nice filtration, when viewed as an object in $\mathbb{F}_p[y^{p^j}] - \mathbb{F}_p[y] - \mathcal{U}$. The case $i = j$ is clear.

For the inductive step, we filter $\mathbb{F}_p[y^{p^i}]$. For $0 \leqslant r \leqslant p-1$, define $M(r)$ to be the span of $\{y^{p^i m} \mid m \equiv s \bmod p,\ \text{for some } 0 \leqslant s \leqslant r\}$. Using the formulae

$$\mathcal{P}^k y^n = \binom{n}{k} y^{n+k(p-1)} \quad \text{and} \quad \Delta\big(y^n\big) = \sum_k \binom{n}{k} y^k \otimes y^{n-k},$$

one easily checks that each $M(r)$ is an object in $\mathbb{F}_p[y^{p^j}] - \mathbb{F}_p[y] - \mathcal{U}$: if $\binom{n}{k} \not\equiv 0 \bmod p$, and n has the form $p^i(pa+s)$ with $0 \leqslant s \leqslant r \leqslant p-1$, then both $n + k(p-1)$ and k also have this form.

Thus we have a filtration in $\mathbb{F}_p[y^{p^j}] - \mathbb{F}_p[y] - \mathcal{U}$:

$$\mathbb{F}_p\big[y^{p^{i+1}}\big] = M(0) \subseteq M(1) \subseteq \cdots \subseteq M(p-1) = \mathbb{F}_p\big[y^{p^i}\big],$$

and we are assuming by induction that $\mathbb{F}_p[y^{p^{i+1}}]$ has a good filtration. Now one checks that $M(r)/M(r-1) \simeq \Sigma^{2p^i r} M(0)$ as objects in $\mathbb{F}_p[y^{p^j}] - \mathbb{F}_p[y] - \mathcal{U}$, so by upwards induction on r we conclude that each $M(r)$ has a good filtration. □

Proof of Lemma 7.15. By Proposition 3.12(c), it suffices to prove that $d_1(\mathbb{F}_p[y]) = 1$, and, if $j > 0$, $d_1(\mathbb{F}_p[y^{p^j}]) = 2p^{j-1}$.

Corollary 3.14 (or Proposition 3.13(b)), applied to the short exact sequence

$$0 \to \mathbb{F}_p[y] \to \Lambda^*(x) \otimes \mathbb{F}_p[y] \to \Sigma\mathbb{F}_p[y] \to 0,$$

shows that $d_1(\mathbb{F}_p[y]) = d_0(\Sigma\mathbb{F}_p[y]) = 1$.

If $j > 1$, we consider the short exact sequence:

$$0 \to \mathbb{F}_p\big[y^{p^j}\big] \to \mathbb{F}_p\big[y^{p^{j-1}}\big] \to \mathbb{F}_p\big[y^{p^{j-1}}\big]/\mathbb{F}_p\big[y^{p^j}\big] \to 0.$$

We claim that d_0 of the last term is $2p^{j-1}$, which, by induction, will be strictly more than d_1 of the middle term. Thus Proposition 3.13(b) applies to say that $d_1(\mathbb{F}_p[y^{p^j}]) = 2p^{j-1}$.

To verify the claim, one checks that the map $\mathbb{F}_p[y^{p^{j-1}}] \to \Sigma^{2p^{j-1}}\mathbb{F}_p[y^{p^{j-1}}]$ sending $y^{p^{j-1}n}$ to the $2p^{j-1}$th suspension of $ny^{p^{j-1}(n-1)}$ is a map of unstable $\mathcal{A}$-modules, and thus induces an embedding $\mathbb{F}_p[y^{p^{j-1}}]/\mathbb{F}_p[y^{p^j}] \hookrightarrow \Sigma^{2p^{j-1}}\mathbb{F}_p[y^{p^{j-1}}]$ in $\mathcal{U}$. Since the range of this embedding is the $2p^{j-1}$th suspension of a reduced module, the same is true of the domain, which thus has $d_0 = 2p^{j-1}$. □

7.8. $d_1(G)$ when G has a p-central p-Sylow subgroup

Now suppose that G is not necessarily p-central, but has a p-central p-Sylow subgroup P. Here we show that then $d_1(G) = d_1(P)$. As $d_1(G) \leqslant d_1(P)$ is always true, the point is to show that $d_1(G)$ is as big as it could be.

Let $e_\omega \in \mathbb{F}_p[Out(P)]$ be an idempotent chosen so that $\mathbb{F}_p[Out(P)]e_\omega$ is the projective cover of the one dimensional module $\omega(P)^{-1}$.

Lemma 7.16. *$d_1(G) = d_1(P)$ if and only if $d_1(R_{e(P)}H^*(G)) = h(P)$, and either of these equalities are implied by $d_1(e_\omega R_0 H^*(P)) = h(P)$.*

Proof. As we proved that $d_1(P) = e(P) + h(P)$, we showed that $d_1(nil_{e(P)}\, H^*(P)) = e(P) + h(P)$ and $d_1(H^*(P)/nil_{e(P)}\, H^*(P)) < e(P) + h(P)$. This second fact implies that $d_1(H^*(G)/nil_{e(P)}\, H^*(G)) < e(P) + h(P)$ also holds, as $H^*(G)$ is a direct summand of $H^*(P)$ in $\mathcal{U}$. We conclude that $d_1(G) = d_1(P)$ if and only if $d_1(nil_{e(P)}\, H^*(G)) = e(P) + h(P)$. As $d_1(nil_{e(P)}\, H^*(G)) = e(P) + d_1(R_{e(P)}H^*(G))$, we deduce that $d_1(G) = d_1(P)$ if and only if $d_1(R_{e(P)}H^*(G)) = h(P)$.

Now reasoning as in Section 7.4, this last equality would follow if one could show that $d_1(e_\omega R_0 H^*(P)) = h(P)$. □

We now sketch a proof that $d_1(e_\omega R_0 H^*(P)) = h(P)$. This involves redoing the calculation that $d_1(R_0 H^*(P)) = h(P)$ in a way that allows one to keep track of the $Out(P)$-action. Let $C = C(P)$.

The 0-line of the spectral sequence associated to $C \to P \to P/C$ is natural with respect to the action of $Out(P)$. Thus the filtration studied in Section 6,

$$H^*(C) = E_2^{0,*} \supset E_3^{0,*} \supset E_{2p+1}^{0,*} \supset \cdots \supset E_{2p^{k-1}+1}^{0,*} \supset E_{2p^k+1}^{0,*} = E_\infty^{0,*} = R_0 H^*(P),$$

is a filtration by unstable modules with an $Out(P)$-action.

Our work above shows that

$$d_1\big(E_r^{0,*}\big) = \begin{cases} 2p^{j-1} & \text{if } r = 2p^j + 1 \text{ with } j \geqslant 1, \\ 1 & \text{if } r = 3, \\ 0 & \text{if } r = 2. \end{cases}$$

We now suppose that $p > 2$ and $k \geqslant 1$: the cases when $p = 2$ or when $E_\infty^{0,*}$ equals $E_2^{0,*}$ or $E_3^{0,*}$ are similar and easier. Recall that then $h(P) = 2p^{k-1}$. Using Proposition 3.13 in the usual way, we conclude that $d_1(e_\omega R_0 H^*(P)) = h(P)$ if and only if $d_0(e_\omega B) = 2p^{k-1}$, where $B = E_{2p^{k-1}+1}^{0,*}/E_{2p^k+1}^{0,*}$.

From Section 6, we see that

$$B = \Lambda^*\big(C_0^\#\big) \otimes S^*\big(\beta\big(C_1^\#\big) + \Phi\beta\big(C_2^\#\big) + \cdots + \Phi^{k-1}\beta\big(C_k^\#\big)\big) \tag{7.1}$$

$$\otimes S^*\big(\Phi^{k-1}\beta\big(C^\#/C_k^\#\big)\big)/S^*\big(\Phi^k\beta\big(C^\#/C_k^\#\big)\big), \tag{7.2}$$

where $C_0^\# \subseteq C_1^\# \subseteq \cdots \subseteq C_k^\# \subseteq C^\#$ is a filtration of $C^\#$ as an $Out(P)$-module.

As an unstable module, B thus has the form

$$M \otimes \big(S^*\big(\Phi^{k-1}\beta(V)\big)/S^*\big(\Phi^k\beta(V)\big)\big),$$

where M is reduced. Now one observes that $S^*(\beta(V))/S^*(\Phi\beta(V)) = \Sigma^2 N$ where N is reduced. Thus

$$\begin{aligned} S^*\big(\Phi^{k-1}\beta(V)\big)/S^*\big(\Phi^k\beta(V)\big) &= \Phi^{k-1}\big(S^*\big(\beta(V)\big)/S^*\big(\Phi\beta(V)\big)\big) \\ &= \Phi^{k-1}\big(\Sigma^2 N\big) \\ &= \Sigma^{2p^{k-1}}\big(\Phi^{k-1}N\big), \end{aligned}$$

which is the $2p^{k-1}$st suspension of a reduced module.

We conclude that $d_0(e_\omega B) = 2p^{k-1}$ if and only if $e_\omega B$ is nonzero.

The image of $Out(P) \to GL(C)$ lands in the parabolic subgroup $GL(C, P)$ respecting the filtration of C, and the idempotent e_ω will project to a nonzero idempotent in $\mathbb{F}_p[GL(C, P)]$.

We claim that if $e \in \mathbb{F}_p[GL(C, P)]$ is any nonzero idempotent, then e acts nontrivially on B as described in (7.1). Equivalently, we claim that all irreducible $\mathbb{F}_p[GL(C, P)]$-modules occur as composition factors in B.

To prove the claim, we note that all irreducible $GL(C, P)$ modules will be pullbacks from the associated Levi factor (i.e. the product of 'block diagonal' $GL(V_j)$'s), as the projection from the one to the other has kernel which is a p-group. This reduces us quickly to verifying the following lemma.

Lemma 7.17. *Every irreducible $\mathbb{F}_p[GL(V)]$-module occurs as a composition factor in $S^*(\beta(V))/S^*(\Phi\beta(V))$.*

Proof. It is well known that every such irreducible S occurs in $S^*(\beta(V))$. Choosing an occurrence of the lowest polynomial degree, it is clear that it will remain nonzero in the quotient $S^*(\beta(V))/S^*(\Phi\beta(V))$. $\square$

8. Central essential cohomology

Recall that $Cess^*(G)$ is defined to be the kernel of the restriction map

$$H^*(G) \to \prod_{\substack{C(G)<U \\ C(G)\neq U}} H^*\big(C_G(U)\big).$$

The invariants $e'(G)$ and $e''(G)$ are then defined by letting

$$e'(G) = \max\big\{d \bigm| Q_A Cess^d(G) \neq 0\big\} \cup \{-1\},$$

where A is a Duflot subalgebra of $H^*(G)$, and

$$e''(G) = \max\big\{d \bigm| P_{C(G)} Cess^d(G) \neq 0\big\} \cup \{-1\}.$$

In this section we study $Cess^*(G)$, $e'(G)$, and $e''(G)$, and connect them to the invariant $d_0(G)$.

8.1. The structure of $Cess^*(G)$

We begin by proving Theorem 2.12. Most of this theorem is restated in the following. We let $C = C(G)$, as usual.

Proposition 8.1. *If A is a Duflot subalgebra of $H^*(G)$, then the following hold.*

(a) *$Cess^*(G)$ is a free A-module.*
(b) *The composite $P_C Cess^*(G) \hookrightarrow Cess^*(G) \twoheadrightarrow Q_A\, Cess^*(G)$ is monic.*
(c) *The sequence $0 \to Q_A\, Cess^*(G) \to Q_A H^*(G) \to \prod_{C(G)<U} Q_A H^*(C_G(U))$ is exact.*

Proof. It is convenient to let $M_1 = Cess^*(G)$, $M_2 = H^*(G)$, $M_3 = M_2/M_1$,

$$M_4 = \prod_{C(G)<U} H^*\bigl(C_G(U)\bigr),$$

and $M_5 = M_4/M_3$. We have short exact sequences of A-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

and

$$0 \to M_3 \to M_4 \to M_5 \to 0.$$

Statements (a) and (c) of the proposition follow if we verify that all the M_j are free A-modules. This we do by arguing as in [14, proof of Theorem 12.3.3].

The homomorphisms $C \times C_G(V) \to C_G(V)$ make each M_j into an $H^*(C)$ comodule and each $M_j \otimes H^*(C)$ into an A-module such that the comodule structure map $M_j \to M_j \otimes H^*(C)$ is a map of A-modules, and the composite $M_j \to M_j \otimes H^*(C) \to M_j$ is the identity. Thus M_j is a direct summand of $M_j \otimes H^*(C)$ as an A-module, and we conclude that M_j is free if $M_j \otimes H^*(C)$ is.

To show that $M_j \otimes H^*(C)$ is a free A-module, we give it a decreasing A-module filtration by letting $F^n = M_j^{\geqslant n} \otimes H^*(C)$. Then each $F^n/F^{n+1} = M_j^n \otimes H^*(C)$ is a direct sum of copies of $H^*(C)$, and so is a free A-module, and the freeness of M_j easily follows.

Finally, statement (b) follows from consideration of the commutative square

$$\begin{tikzcd} P_C\, Cess^*(G) \arrow[r] \arrow[d] & Q_A\, Cess^*(G) \arrow[d] \\ P_C H^*(G) \arrow[r] & Q_A H^*(G). \end{tikzcd}$$

The left map is clearly monic, and Theorem 2.1 says that the bottom map is also. Thus so is the top map. □

To finish the proof of Theorem 2.12, it remains to show that $Cess^*(G)$ is finitely generated as an A-module (with the corollary that $e'(G)$ and $e''(G)$ are well-defined finite numbers). As A

has Krull dimension equal to the rank of C, and $Cess^*(G)$ is a free A-module, it is equivalent to prove the next result.

Proposition 8.2. *The Krull dimension of $Cess^*(G)$ is at most the rank of C.*

Proof. The proposition follows from a result of Carlson. It is convenient to let $R = H^*(G)$, and $I = Cess^*(G)$. By definition, the Krull dimension of the R-module I is the Krull dimension of the algebra $R/Ann(I)$.

Let J be the image of

$$\sum_{\substack{C<U\\ C\neq U}} \mathrm{Ind}_{C_G(U)}^G : \bigoplus_{\substack{C<U\\ C\neq U}} H^*\big(C_G(U)\big) \to H^*(G).$$

By standard arguments,[16] $J \subset Ann(I)$. Thus $R/J \to R/Ann(I)$ is a surjection, and so the Krull dimension of $R/Ann(I)$ is at most the Krull dimension of R/J.

In the notation of [12], $J = J_{c+1}$, where c is the rank of C. Then [12, Corollary 2.2] says that the Krull dimension of R/J is at most c. □

The next proposition is easily verified.

Proposition 8.3. *$Cess^*(G \times H)$ is naturally isomorphic to $Cess^*(G) \otimes Cess^*(H)$. Thus*

$$e'(G \times H) = e'(G) + e'(H) \quad \textit{and} \quad e''(G \times H) = e''(G) + e''(H).$$

We end this subsection by proving Proposition 2.16, which we recall here.

Proposition 8.4. *Assuming that $V = C(C_G(V))$, $Cess^*(C_G(V)) = \mathbf{0}$ unless the rank of V is at least equal to the depth of $H^*(G)$.*

Proof. Let $r(U)$ denote the p-rank of an elementary abelian p-group U, and let d be the depth of $H^*(G)$. Suppose that $V = C(C_G(V))$ and $r(V) < d$. We wish to show that $Cess^*(C_G(V)) = \mathbf{0}$. Note that, if $V < U$, then $C_{C_G(V)}(U) = C_G(U)$. Thus $Cess^*(C_G(V))$ is the kernel of the restriction map

$$H^*\big(C_G(V)\big) \to \prod_{\substack{V<U<C_G(V)\\ V\neq U}} H^*\big(C_G(U)\big).$$

The kernel of this is contained in the kernel of

$$f : H^*\big(C_G(V)\big) \to \prod_{\substack{V<U<C_G(V)\\ r(U)=d}} H^*\big(C_G(U)\big).$$

Thus it suffices to show that f is monic.

[16] This follows immediately from the fact that $\mathrm{Ind}_{C_G(U)}^G : H^*(C_G(U)) \to H^*(G)$ is a map of $H^*(G)$-modules.

Meanwhile, Carlson's theorem [12] implies that the product of restriction maps

$$g : H^*(G) \to \prod_{\substack{U<G \\ r(U)=d}} H^*\bigl(C_G(U)\bigr)$$

is monic.

There is a commutative diagram

$$\begin{tikzcd} \coprod_{V<U<C_G(V),\, r(U)=d} V \times C_G(U) \arrow[r] \arrow[d] & V \times C_G(V) \arrow[d] \\ \coprod_{U<G,\, r(U)=d} C_G(U) \arrow[r] & G. \end{tikzcd}$$

This induces a commutative diagram in $\mathcal{U}$

$$\begin{tikzcd} H^*(V) \otimes H^*(C_G(V)) \arrow[r, "1\otimes f"] \arrow[d] & \prod_{V<U<C_G(V),\, r(U)=d} H^*(V) \otimes H^*(C_G(U)) \arrow[d] \\ H^*(G) \arrow[r, "g"] & \prod_{U<G,\, r(U)=d} H^*(C_G(U)). \end{tikzcd}$$

Adjointing, we get a commutative diagram

$$\begin{tikzcd} H^*(C_G(V)) \arrow[r, "f"] \arrow[d] & \prod_{V<U<C_G(V),\, r(U)=d} H^*(C_G(U)) \arrow[d] \\ T_V H^*(G) \arrow[r, "T_V g"] & \prod_{U<G,\, r(U)=d} T_V H^*(C_G(U)). \end{tikzcd}$$

The left vertical map here is an inclusion, and the exactness of T_V shows that $T_V g$ is monic. We conclude that f is also monic. □

8.2. *Proof of Theorem 2.14*

Theorem 2.14 says that

$$d_0(G) = \max\bigl\{e''\bigl(C_G(V)\bigr) \bigm| V < G\bigr\}.$$

In the right-hand side of this equation, one can restrict to subgroups V such that $V = C(C_G(V))$, because if $V < U$ and U is central in $C_G(V)$, then $C_G(V) = C_G(U)$, so that $e''(C_G(V)) = e''(C_G(U))$. Thus Theorem 2.14 will follow from the next theorem.

Theorem 8.5. *$\bar{R}_d H^*(G) \neq 0$ if and only if $P_V\, Cess^d(C_G(V)) \neq 0$ for some $V < G$ satisfying $V = C(C_G(V))$.*

As we begin the proof of this, it is convenient to let, for $V < G$,

$$Ess^*(V) = \ker\left\{ P_V H^*\big(C_G(V)\big) \to \prod_{V<U} P_V H^*\big(C_G(U)\big) \right\},$$

with the product over all U that are strictly bigger than V. One easily verifies that

$$Ess^*(V) = \begin{cases} P_V\, Cess^d(C_G(V)) & \text{if } V = C(C_G(V)), \\ 0 & \text{otherwise.} \end{cases}$$

Thus we wish to show that $\bar{R}_d H^*(G) = 0$ if and only if $Ess^d(V) = 0$ for all $V < G$.

We prove the 'only if' implication first. Let $c_V \in \mathbb{F}_p[GL(V)]$ be the top Dickson invariant, and let $W_G(V) = N_G(V)/V$. Then [28, Lemma 7.8] (proved by using the formula in this paper's Proposition 4.6) says that, for all V, there is an embedding

$$\big(c_V H^*(V) \otimes Ess^d(V)\big)^{W_G(V)} \subseteq \bar{R}_d H^*(G).$$

Thus $\bar{R}_d H^*(G) = 0$ implies that $(c_V H^*(V) \otimes Ess^d(V))^{W_G(V)} = 0$ for all V. But then $Ess^d(V) = 0$, by the next lemma.

Lemma 8.6. *If $W < GL(V)$ and M is an $\mathbb{F}_p[W]$-module, then $[c_V H^*(V) \otimes M]^W = 0$ implies that $M = 0$.*

Proof. It is well known (see, e.g. [4, p. 45]) that $\mathbb{F}_p[W]$ embeds in $H^*(V)$ as $\mathbb{F}_p[W]$-modules, and thus in $c_V H^*(V)$, as multiplication by c_V is a monic $GL(V)$-module map. Thus $(\mathbb{F}_p[W] \otimes M)^W$ embeds in $(c_V H^*(V) \otimes M)^W$. But $\mathbb{F}_p[W] \otimes M \simeq \mathbb{F}_p[W] \otimes M_{\mathrm{triv}}$ as $\mathbb{F}_p[W]$-modules, where M_{triv} denotes M with trivial W-action. Finally, $(\mathbb{F}_p[W] \otimes M_{\mathrm{triv}})^W \simeq M$ as $\mathbb{F}_p$-vector spaces. Putting this all together, we have shown that M embeds in $(c_V H^*(V) \otimes M)^W$, and the lemma follows. $\square$

Returning to the proof of the theorem, we now assume that $Ess^d(V) = 0$ for all V, and deduce that $\bar{R}_d H^*(G) = 0$. Using the formula in Proposition 4.6, given

$$x = (x_V) \in \bar{R}_d H^*(G) \subseteq \prod_V H^*(V) \otimes P_V H^d\big(C_G(V)\big),$$

we show that each component x_V of x is zero by downwards induction on V.

So assume that $x_U = 0$ for all $V < U$. Proposition 4.6 tells us that, under the restriction map

$$H^*(V) \otimes P_V H^d\big(C_G(V)\big) \to \prod_{V<U} H^*(V) \otimes P_V H^d\big(C_G(U)\big),$$

x_V will have the same image as (x_U) under the map

$$\prod_{V<U} H^*(U) \otimes P_U H^d\big(C_G(U)\big) \to \prod_{V<U} H^*(V) \otimes P_V H^d\big(C_G(U)\big).$$

Since the latter is zero by inductive assumption, we conclude that $x_V \in H^*(V) \otimes Ess^d(V)$, and is thus zero also.

Remark 8.7. Note that Theorem 8.5 includes a second proof that $e''(G)$ is a well-defined finite number.

8.3. The Depth Conjecture, the Regularity Conjecture, and a bound on $e'(G)$.

By Theorem 2.12, $e''(G) \leqslant e'(G)$. Thus we get the bound

$$d_0(G) \leqslant \max\bigl\{e'\bigl(C_G(V)\bigr) \bigm| V < G\bigr\}.$$

If G is p-central, we know that $e''(G) = e'(G) = e(G)$. Here we discuss work towards Conjecture 2.18 which said that, if G is not p-central, then $e'(G) < e(G)$. In particular, we show that this is true if the p-rank of G is no more than 2 more than the rank of $C(G)$, and link the general conjecture to Benson's Regularity Conjecture. En route, a similar argument will also prove Theorem 2.13, the special case of Carlson's Depth Conjecture in which the depth of $H^*(G)$ is as small as possible.

Thanks to Proposition 8.3, it suffices to prove either Theorem 2.13 or Conjecture 2.18 in the special case when G has no direct summands isomorphic to $\mathbb{Z}/p$. We will assume this. As a consequence, even in the odd prime case, a Duflot subalgebra A of $H^*(G)$ will have the form

$$A = \mathbb{F}_p[\xi_1, \dots, \xi_c],$$

where c is the rank of $C = C(G)$ and $e(G) = \sum_{j=1}^{c}(|\xi_j| - 1)$. The sequence $\xi_1, \dots, \xi_c$ is a *Duflot sequence*: the sequence restricts to a regular sequence in $H^*(C)$. We let r be the p-rank of G.

Lemma 8.8. *Suppose $c < r$. Given any Duflot sequence $\xi_1, \dots, \xi_c \in H^*(G)$, there exists $\xi \in H^*(G)$ such that, for all proper inclusions $C < V$, $\xi_1, \dots, \xi_c, \xi$ restricts to a regular sequence in $H^*(C_G(V))$.*

Proof. Let n be the rank of G/C (so $n > r - c$), and let ρ be the regular representation of G/C. For $1 \leqslant i \leqslant r - c$, let $\bar{\kappa}_i \in H^{2(p^n - p^{n-i})}(G/C)$ be the $(p^n - p^{n-i})$th Chern class of ρ, and then let $\kappa_i = \mathrm{Inf}_{G/C}^{G}(\bar{\kappa}_i) \in H^*(G)$. It is easy to check that $\xi_1, \dots, \xi_c, \kappa_1, \dots, \kappa_{r-c}$ is a polarized system of parameters in the sense of [20, Definition 2.2]. It follows that the element $\xi = \kappa_1$ satisfies the conclusion of the lemma. □

Proposition 8.9. *Suppose $c < r$ and $A = \mathbb{F}_p[\xi_1, \dots, \xi_c]$ is a Duflot algebra of $H^*(G)$. The following are equivalent for a fixed integer $e \geqslant 0$.*

(a) $e'(G) < e$.
(b) *With ξ as in the lemma, the kernel of multiplication by ξ,*

$$\ker\bigl\{\xi\cdot : H^d(G)/(\xi_1, \dots, \xi_c) \to H^{d+|\xi|}(G)/(\xi_1, \dots, \xi_c)\bigr\},$$

is zero for all $d \geqslant e$.

(c) $\bigcap_{\xi \in \tilde{H}^*(G)} \ker\{\xi\cdot : H^d(G)/(\xi_1, \dots, \xi_c) \to H^{d+|\xi|}(G)/(\xi_1, \dots, \xi_c)\}$

is zero for all $d \geqslant e$.

Proof. For each d and $\xi \in H^*(G)$, we have a commutative diagram

$$\begin{tikzcd} H^d(G)/(\xi_1,\dots,\xi_c) \arrow[r, "f(d)"] \arrow[d, "\xi\cdot"] & \bigoplus_{C<V} H^d(C_G(V))/(\xi_1,\dots,\xi_c) \arrow[d, "\xi\cdot"] \\ H^{d+|\xi|}(G)/(\xi_1,\dots,\xi_c) \arrow[r, "f(d+|\xi|)"] & \bigoplus_{C<V} H^{d+|\xi|}(C_G(V))/(\xi_1,\dots,\xi_c), \end{tikzcd} \tag{8.1}$$

where $f(d)$ is induced by the evident restriction maps.

By Theorem 2.12, $f(d)$ is monic for all large d, and $e'(G)$ is the largest d such that $f(d)$ is not monic. Thus statement (a) is equivalent to the statement that $\ker f(d)$ is zero for all $d \geqslant e$.

We show that statement (a) implies statement (b). Thus suppose that $\ker f(d)$ is zero for all $d \geqslant e$, and let $\xi \in H^*(G)$ be as in the lemma. Then, in diagram (8.1), the right map is monic for all d, and the top map is monic for all $d \geqslant e$. Thus the left map is monic in the same range.

Statement (b) obviously implies statement (c).

Finally, we show that statement (c) implies statement (a). Assuming statement (c), we prove, by downwards induction on d, that $f(d)$ is monic. Thus assume $f(d')$ is monic for all $d' > d \geqslant e$. Given $0 \neq \kappa \in H^d(G)/(\xi_1,\dots,\xi_c)$, we need to show that $f(d)(\kappa) \neq 0$. By (c), there exists $\xi \in \tilde{H}^*(G)$ such that $\xi\cdot\kappa \neq 0$. As $f(d+|\xi|)$ is monic by inductive assumption, $f(d+|\xi|)(\xi\cdot\kappa) \neq 0$. But this equals $\xi \cdot f(d)(\kappa)$, and so $f(d)(\kappa) \neq 0$. □

Proof of Theorem 2.13. With notation as in the proposition just proved, we wish to prove that $Cess^*(G) = 0$ if and only if the depth of $H^*(G)$ is greater than c. Thanks to Theorem 2.12, $Cess^*(G) = 0$ if and only if statement (a) of the last proposition holds when $e = 0$. But then statement (b) is true with $e = 0$, and thus the depth of $H^*(G)$ is at least $c + 1$.

Conversely, if the depth of $H^*(G)$ is at least $c + 1$, there exists a ξ such that $\xi_1,\dots,\xi_c,\xi$ is a regular sequence on $H^*(G)$, and so statement (c) certainly holds with $e = 0$. Thus statement (a) does as well. □

Now we study statements (b) and (c) of the last proposition, using work by Carlson and Benson.

Proposition 8.10. *If $r - c = 1$, then $Q_A\,Cess^*(G)$ satisfies the Poincaré duality with duality degree equal to $e(G)$. In other words, the Poincaré polynomial $p_{Q_A\,Cess^*(G)}(t)$ satisfies*

$$p_{Q_A\,Cess^*(G)}(t) = t^{e(G)}\,p_{Q_A\,Cess^*(G)}(1/t).$$

Proof. The conclusion of the proposition is obvious if $Cess^*(G) = 0$, so we can assume that the depth of $H^*(G)$ is precisely c. Let $\xi_1,\dots,\xi_c$ be as in Proposition 8.9, and choose ξ as in the lemma. Replacing ξ by a large power of itself, if necessary, we can assume that, in diagram (8.1), $f(d + |\xi|)$ is monic for all d. Thus $Q_A\,Cess^*(G)$, the kernel of the top map in (8.1), identifies with the kernel of multiplication by ξ, the left map in (8.1). But a careful reading of [7, Lemma 3.2 and its proof] reveals that the Poincaré series of this kernel is precisely the polynomial called '$p_r(t)$' there, and then [7, Theorem 3.9] says that the functional equation of the proposition holds. □

Corollary 8.11. *If $r - c = 1$, then*

$$e'(G) = e(G) - \min\{d \mid Cess^d(G) \neq 0\} < e(G).$$

To state what we know about the situation when $r - c > 1$, we need to introduce local cohomology. If I is a homogeneous ideal in a graded ring R, and M is a graded R-module, $H_I^{0,*}(M)$ is defined to be the I-torsion in M, i.e. the set of $x \in M$ such that $I^k x = 0$ for some k. This is a left exact functor of M, and $H_I^{d,*}(M)$ is defined to be the associated dth right derived functor.

In [5], Benson conjectured

Conjecture 8.12 *(Strong Regularity Conjecture).*

$$H^{i,j}_{\tilde{H}^*(G)}\big(H^*(G)\big) = 0 \quad \text{for } \begin{cases} j \geqslant -i & \text{if } c \leqslant i < r, \\ j > -i & \text{if } i = r. \end{cases}$$

Proof of Proposition 2.19. This proposition asserted that, for a fixed finite group G, Conjecture 2.18 is implied by the Strong Regularity Conjecture.

In the terminology of [5], if the Strong Regularity Conjecture holds, then, by [5, Theorem 4.5], every filter regular sequence is of type beginning with the sequence $(-1, -2, \ldots, -(c+1))$. In particular, with ξ as in statement (b) of Proposition 8.9, the sequence $\xi_1, \ldots, \xi_c, \xi$ is the beginning of such a sequence. From the definition of filter regular, we see that statement (b) of Proposition 8.9 thus holds with $e = e(G)$. $\square$

As mentioned in the introduction, in [5], Benson shows that his conjecture is true if $r - c \leqslant 2$. I have my own 'heuristic' proof of statement (c) with $e = e(G)$ under the same condition, and the failure of the method to go beyond $r - c \leqslant 2$ makes one wonder if a counterexample to both of our conjectures is lurking among the groups of order 128 or 256.

Remark 8.13. Slightly milder than the Strong Regularity Conjecture is Benson's Regularity Conjecture, which asserts that the Castelnuovo–Mumford regularity of $H^*(G)$ is precisely 0. In terms of local cohomology, this is the statement that $H^{i,j}_{\tilde{H}^*(G)}(H^*(G)) = 0$ for $j > -i$. For our purposes, this is enough to deduce that $e'(G) \leqslant e(G)$.

9. Examples

Example 9.1. Let $W(2)$ be the universal 2-central group whose quotient by its center C is $V_2 = (Z/2)^2$. Thus there is a central extension

$$H_2(V_2; \mathbb{F}_2) \xrightarrow{i} W(2) \xrightarrow{q} V_2,$$

where $C = H_2(V_2; \mathbb{F}_2) \simeq (\mathbb{Z}/2)^3$. In terms of Hall–Senior numbering, and thus also the numbering in [14], $W(2)$ is 32#18.

In the associated spectral sequence, one has that

$$E_2^{*,*} = \mathbb{F}_2[x, y, a, b, c],$$

with $a, b, c \in E_2^{0,1}$ and $x, y \in E_2^{1,0}$, and $d_2(a) = x^2$, $d_2(b) = xy$, and $d_2(c) = y^2$.

$$
\begin{array}{c|cccc}
q & & & & \\
 & & & & \\
2 & & & & \\
1 & & u, v & xv & \\
0 & 1 & x, y & & \\
\hline
 & 0 & 1 & 2 & p
\end{array}
$$

Fig. 1. $E_3^{p,q} = E_\infty^{p,q}$ modulo (a^2, b^2, c^2).

As $E_3^{*,0} = \mathbb{F}_2[x, y]/(x^2, xy, y^2)$, it follows that a^2, b^2, and c^2 must be permanent cycles. We conclude that $W(2)$ will have type $[2, 2, 2]$ so that $d_0(W(2)) = e(W(2)) = 3$ and $d_1(W(2)) = 4$.

With a bit more work, one can show that $E_3^{*,*}/(a^2, b^2, c^2)$ is six dimensional with generators as indicated in Fig. 1, where u and v are respectively represented by $bx + ay$ and $cx + by$. This is a Poincaré duality algebra with relations $x^2 = xy = y^2 = u^2 = v^2 = uv = xu = yv = xv + yu = 0$.

It follows that $E_3^{*,*} = E_\infty^{*,*}$, and then that

$$H^*\big(W(2)\big) \simeq \mathbb{F}_2[\alpha, \beta, \gamma, x, y, u, v]/\big(x^2, xy, y^2, u^2, uv, v^2, xu, yv, xv + yu\big),$$

with $x, y \in H^1$ and $\alpha, \beta, \gamma, u, v \in H^2$. Here α, β, and γ are represented by a^2, b^2, and c^2 in the spectral sequence.

The polynomial subalgebra $A = \mathbb{F}_2[\alpha, \beta, \gamma]$ is a Duflot subalgebra. With respect to the $H^*(C) = \mathbb{F}_2[a, b, c]$ comodule structure, the elements $1, x, y$ are in the image of the inflation map q^* and so are primitive. The top class xv is not in the image of inflation, but is primitive, by our general theory. The elements u and v are *not* primitive, as $m^*(u) = 1 \otimes u + b \otimes x + a \otimes y$ and $m^*(v) = 1 \otimes v + c \otimes x + b \otimes y$ in $H^*(C) \otimes E_\infty^{*,*}$. Thus each of the inclusions

$$\operatorname{im}(q^*) \hookrightarrow P_C H^*\big(W(2)\big) \hookrightarrow Q_A H^*\big(W(2)\big)$$

is *proper*.

The nilpotent filtration works as follows.

$R_0 = H^*(W(2))/(x, y, u, v) \simeq \mathbb{F}_2[\alpha, \beta, \gamma]$ and $\bar{R}_0 = \mathbb{F}_2[a, b, c]$. The embedding $R_0 \subset \bar{R}_0$ sends α to a^2, β to b^2, and γ to c^2.

R_1 is the free $\mathbb{F}_2[\alpha, \beta, \gamma]$-module on generators $\bar{x}, \bar{y}, \bar{u}, \bar{v}$ of respective degrees 0, 0, 1, 1, and $\bar{R}_1$ is the free $\mathbb{F}_2[a, b, c]$-module on $\bar{x}, \bar{y}$. The embedding $R_1 \subset \bar{R}_1$ sends $\bar{u}$ to $b\bar{x} + a\bar{y}$ and $\bar{v}$ to $c\bar{x} + b\bar{y}$. Thus $Sq^1(\bar{u}) = \beta\bar{x} + \alpha\bar{y}$ and $Sq^1(\bar{v}) = \gamma\bar{x} + \beta\bar{y}$.

R_2 and $\bar{R}_2$ are both 0, as $P_C H^2(W(2)) = 0$.

R_3 is the free $\mathbb{F}_2[\alpha, \beta, \gamma]$-module on a single generator $\bar{x}\bar{v}$ of degree 0, and $\bar{R}_3$ is the free $\mathbb{F}_2[a, b, c]$-module on this same element.

Finally, $H^*(W(2))_{LF} \subset H^*(W(2))$ is the algebra spanned by $1, x, y, xv$. All nontrivial products and Steenrod operations are zero.

Example 9.2. Let G be the group of order 64 with Hall–Senior number #108. Using information from [14], we analyzed $H^*(G)$ in detail for other purposes in [28]. Here we summarize relevant bits to illustrate how one can calculate $H^*(G)_{LF}$ and $\bar{R}_d H^*(G)$ by using Propositions 4.7 and 4.6.

The commutator subgroup $Z = [G, G]$ has order 2. The center C is elementary abelian of rank 2, and $C = \Phi(G)$, so $Z < C$ and G/C is elementary abelian of rank 4. There is a unique maximal elementary abelian group V of rank 3, and its centralizer K has order 32, so that

$N_G(V)/C_G(V) = G/K \simeq \mathbb{Z}/2$. More precisely, K is isomorphic to $(\mathbb{Z}/2)^2 \times Q_8$, with Q_8 embedded so that $V \cap Q_8 = Z$.

We have the following picture of $\mathcal{A}^C(G)$:

$$\begin{tikzcd} C \arrow[r] & V \arrow[loop right, "\mathbb{Z}/2"] \end{tikzcd}$$

and from this it is already clear that $\bar{R}_0 H^*(G) = H^*(V)^{\mathbb{Z}/2}$.

We have maps of unstable algebras equipped with $Aut(G)$ action:

$$P_V H^*(K) \hookrightarrow P_C H^*(K) \xleftarrow{j^*} P_C H^*(G),$$

where $j : K \to G$ is the inclusion. It is easily checked that j^* is onto in degree 1.

The maps of pairs $(Q_8, Z) \to ((\mathbb{Z}/2)^2 \times Q_8, (\mathbb{Z}/2)^2 \times Z) = (K, V)$ induce an isomorphism of algebras:

$$P_V H^*(K) \simeq P_Z H^*(Q_8).$$

The algebra $P_Z H^*(Q_8)$ is familiar: the calculation of $H^*(Q_8)$ using the Serre spectral sequence associated to $Z \to Q_8 \to Q_8/Z$ reveals that $P_Z H^*(Q_8) = \mathrm{Im}\{H^*(Q_8/Z) \to H^*(Q_8)\} = B^*$, where B^* is the Poincaré duality algebra $\mathbb{F}_2[x, w]/(x^2 + xw + w^2, x^2 w + xw^2)$, where x and w both have degree 1. B^* has dimension $1, 2, 2, 1$ in degrees $0, 1, 2, 3$.

From this we learn that $P_C H^*(K) \simeq B^*[y]$ where y is also in degree 1, and thus is generated by elements in degree 1. It follows that $j^* : P_C H^*(G) \to P_C H^*(K)$ is onto, and then that $Inn(G)$ acts trivially on both $P_V H^*(K)$ and $P_C H^*(K)$.

Proposition 4.7 then tells us that there is a pullback diagram of unstable algebras:

$$\begin{tikzcd} H^*(G)_{LF} \arrow[r] \arrow[d] & P_C H^*(G) \arrow[d, "j^*"] \\ P_V H^*(K) \arrow[r] & P_C H^*(K). \end{tikzcd}$$

Similarly, Proposition 4.6 tells us that, for all d, there is a pullback diagram of unstable modules:

$$\begin{tikzcd} \bar{R}_d H^*(G) \arrow[r] \arrow[d] & H^*(C) \otimes P_C H^d(G) \arrow[d, "1 \otimes j^*"] \\ H^*(V)^{\mathbb{Z}/2} \otimes P_V H^d(K) \arrow[r] & H^*(C) \otimes P_C H^d(K). \end{tikzcd}$$

Note that the kernel of $j^* : H^*(G) \to H^*(K)$ is precisely $Cess^*(G)$, which is described in [14]. In our terminology, we learn that a Duflot subalgebra A is polynomial on classes of degree 2 and 8 (so G has type $[8, 2]$), and $Q_A\, Cess^*(G)$ is a graded vector space of dimension $1, 3, 5, 6, 5, 3, 1$ in degrees $1, 2, 3, 4, 5, 6, 7$. Note that this evident Poincaré duality is predicted by Proposition 8.10.

$Q_A\, Cess^*(G)$ has a basis in which every element is a product of one dimensional classes, and thus $P_C\, Cess^*(G) = Q_A\, Cess^*(G)$. In [28, Proposition 10.2], we further showed that

$$P_C\, Cess^*(G) \simeq \Sigma B^*[y]/\big(y^4\big)$$

as unstable modules.

It follows that there are short exact sequences in $\mathcal{U}$:

$$0 \to \Sigma B^*[y]/\big(y^4\big) \to H^*(G)_{LF} \to B^* \to 0,$$

and

$$0 \to H^*(C) \otimes \big[\Sigma B^*[y]/\big(y^4\big)\big]^d \to \bar{R}_d H^*(G) \to H^*(V)^{\mathbb{Z}/2} \otimes B^d \to 0.$$

Furthermore, $d_0(G) = e''(G) = e'(G) = 7$, and $e(G) = 8$.

Example 9.3. One can often determine $e(G)$ using minimal information about the extension class $\tau^*: C^* \to H^2(G/C)$ (where $C = C(G)$), and in situations where $H^*(G)$ has yet to be calculated.

For example, suppose that $p = 2$ and G has no $\mathbb{Z}/2$ direct summands (so that τ^* is monic). If the image of τ^* has a basis consisting of products of one dimensional classes, then G has type $[2,\dots,2]$ and so $e(G)$ equals the rank of C. To see this, we note that, if $d_2(a) = xy$, then

$$d_3\big(a^2\big) = d_3\big(Sq^1 a\big) = Sq^1\big(d_2(a)\big) = Sq^1(xy) = x^2y + xy^2 \equiv 0 \mod (xy).$$

This criterion holds for the important family of groups studied in [3]. There the authors associate a 2-central Galois group $\mathcal{G}_{\mathbb{F}}$ to every field $\mathbb{F}$ of characteristic different from 2 that is not formally real. They call this group a W-group due to its connections to the Witt ring $W\mathbb{F}$ [33]. Thus $d_0(\mathcal{G}_{\mathbb{F}}) = e(\mathcal{G}_{\mathbb{F}}) = r$ and $d_1(\mathcal{G}_{\mathbb{F}}) = r + 1$, where $\mathcal{G}_{\mathbb{F}}$ has rank r. Included among these groups are the universal W-groups $W(n)$, the 2-central group with extension sequence

$$H_2\big((\mathbb{Z}/2)^n; \mathbb{F}_2\big) \to W(n) \to (\mathbb{Z}/2)^n.$$

Thus $d_0(W(n)) = \binom{n+1}{2}$ and $d_1(W(n)) = \binom{n+1}{2} + 1$.

At odd primes p, analogous criteria exist, ensuring that G is p-central of type $[2,\dots,2]$. Interesting families of such groups were studied by Browder–Pakianathan [11] and Adem–Pakianathan [2]. Included among these are the universal groups $W(n,p)$, with extension sequence

$$H_2\big((\mathbb{Z}/p)^n; \mathbb{F}_p\big) \to W(n,p) \to (\mathbb{Z}/p)^n.$$

Thus $d_0(W(n,p)) = \binom{n+1}{2}$ and $d_1(W(n,p)) = \binom{n+1}{2} + 1$.

Example 9.4. Compared to the families in the last example, at the other extreme among 2-central 2-groups is the 2-Sylow subgroup P of the simple group $SU(3,4)$. This group has order 64 and Hall–Senior number #187. Its center C is elementary abelian of rank 2. In [21], Green analyzed

q									
8				2					
7									
6				6	8	8	8	4	1
5									
4			8	12	8	7	4		
3									
2			4	7	8	8	8		
1									
0	1	4	8	10	8	6			
	0	1	2	3	4	5	6	7	8 p

Fig. 2. The dimension of $E_\infty^{p,q}$ modulo (a^8, b^8).

the associated spectral sequence.[17] In particular, P has type $[8, 8]$, so that $d_0(P) = e(P) = 14$ and $d_1(P) = 18$, and the analogue of Fig. 1 is the impressively complex Fig. 2 (reproduced from [21]).

In spite of this complexity, it is interesting to note that one can get the bound $e(P) \leqslant 14$ quite easily, by using representation theory and characteristic classes.

We thank David Green for the following description of some complex representations of P. Let $H^*(C) = \mathbb{F}_2[a, b]$, and then let ρ_a and ρ_b be the one dimensional complex representations of C with respective total Stiefel–Whitney classes $w(\rho_a) = 1 + a^2$, $w(\rho_b) = 1 + b^2$. These representations extend to one dimensional representations $\tilde{\rho}_a$ and $\tilde{\rho}_b$ of subgroups Q_a and Q_b of index 4 in P. Let ω_a and ω_b be the four dimensional representations one gets by inducing $\tilde{\rho}_a$ and $\tilde{\rho}_b$ up to P: these turn out to be irreducible.

By construction $\operatorname{Res}^P_C(\omega_a) = 4\rho_a$ and $\operatorname{Res}^P_C(\omega_b) = 4\rho_b$. It follows that the total Stiefel–Whitney classes of ω_a and ω_b restrict to $(1 + a^2)^4 = 1 + a^8$ and $(1 + b^2)^4 = 1 + b^8$ in $H^*(C)$. Thus $\operatorname{im}\operatorname{Res}^P_C$ contains $\mathbb{F}_2[a^8, b^8]$ and so $e(P) \leqslant 14$ must hold.

Alternatively, one can just use the single eight dimensional representation $\omega_a \oplus \omega_b$. This is faithful, as it is faithful when restricted to C, the subgroup of all elements of order 2. It has characteristic classes that restrict to $a^8 + b^8$ and $a^8 b^8$ in $H^*(C)$. From this, one can formally deduce that the special Hopf algebra $\operatorname{im}\operatorname{Res}^P_C$ must contain $\mathbb{F}_2[a^8, b^8]$, so that $d_0(P) \leqslant 14$ and $d_1(P) \leqslant 18$. By contrast, the estimate of Henn, Lannes, and Schwartz in [25] just lets one conclude that $d_0(P) \leqslant 64$ and $d_1(P) \leqslant 120$ if one knows that P has a faithful eight dimensional complex representation. This suggests that there might be some general bounds for $d_s(G)$ for an arbitrary group G, determined by the dimensions of its faithful representations, that are much better than those in [25].

Acknowledgments

Early versions of some of these results were presented at a talk in Oberwolfach in September, 2005: see [27]. Some of this research was done during a visit to the Cambridge University Pure Mathematics Department in the first half of 2006, and the author thanks DPMMS for its hospitality. Conversations with Dave Benson about the local cohomology spectral sequence have been enlightening, as have conversations with David Green.

[17] A key simplification comes by computing with $\mathbb{F}_4$ coefficients rather than $\mathbb{F}_2$ coefficients.

Appendix A. Tables of group invariants

Here are various tables of some of our invariants for 2-groups of order dividing 64. The tables were compiled by hand using the calculations in [14] and the website version [13]. The type of a group G, and thus $e(G)$ and $h(G)$, can be deduced by inspecting the description of restriction to maximal elementary abelian subgroups; this is particularly easy when G is 2-central. If G is not 2-central, one can immediately determine if $Cess^*(G) \neq 0$, since both the rank of $Z(G)$ and the depth of $H^*(G)$ are given, and then read off the number $e'(G)$ from the description of depth essential cohomology. The website source allows one to identify centralizers of elementary abelian subgroups as needed.

We say a group is indecomposable if it cannot be written as a nontrivial direct product of two subgroups. The numbering of groups is as in [14] which follows the Hall–Senior numbering [23].

In Tables 1 and 2, recall that, since G is 2-central, $d_0(G) = e(G) = e'(G) = e''(G)$, and $d_1(G) = e(G) + h(G)$.

Table 1
Indecomposable, 2-central, 2-groups of order $\leqslant 32$

Order	#	Type	$d_0(G)$	$d_1(G)$	Notes
2	1	[1]	0	0	$\mathbb{Z}/2$
4	2	[2]	1	2	$\mathbb{Z}/4$
8	3	[2]	1	2	$\mathbb{Z}/8$
	5	[4]	3	5	Q_8
16	5	[2]	1	2	$\mathbb{Z}/16$
	14	[4]	3	5	Q_{16}
32	18	[2, 2, 2]	3	4	
	19	[2, 2]	2	3	
	21	[2, 2]	2	3	
	28	[4, 2]	4	6	
	29	[2, 2]	2	3	
	30	[2, 2]	2	3	
	35	[4, 2]	4	6	
	40	[4, 4]	6	8	
	51	[4]	3	5	Q_{32}

Table 2
Indecomposable, 2-central, groups of order 64

#	Type	$d_0(G)$	$d_1(G)$	Notes
11	[2]	1	2	$\mathbb{Z}/64$
30	[2, 2, 2]	3	4	
37	[2, 2, 2]	3	4	
38	[2, 2]	2	3	
39	[2, 2]	2	3	
41	[2, 2]	2	3	
59	[2, 2, 2]	3	4	
63	[4, 2]	4	6	
64	[2, 2]	2	3	
65	[2, 2]	2	3	
82	[2, 2, 2]	3	4	
87	[4, 2, 2]	5	7	
88	[2, 2, 2]	3	4	
90	[2, 2, 2]	3	4	
92	[4, 2, 2]	5	7	

Table 2 (*continued*)

#	Type	$d_0(G)$	$d_1(G)$	Notes
93	[2, 2, 2]	3	4	
101	[4, 4]	6	8	
119	[4, 2]	4	6	
139	[4, 2]	4	6	
140	[2, 2]	2	3	
141	[2, 2]	2	3	
145	[4, 2, 2]	5	7	
149	[4, 2, 2]	5	7	
152	[4, 2, 2]	5	7	
153	[4, 4, 4]	9	11	2-Sylow of $Sz(8)$
162	[4, 4]	6	8	
187	[8, 8]	14	18	2-Sylow of $U_3(\mathbb{F}_4)$
190	[4, 2]	4	6	
191	[4, 4]	6	8	
192	[4, 2]	4	6	
194	[4, 4]	6	8	
199	[4, 4]	6	8	
210	[4, 4]	6	8	
211	[4, 2]	4	6	
212	[4, 4]	6	8	
222	[4, 4]	6	8	
227	[4, 4]	6	8	
233	[4, 4]	6	8	
235	[4, 2]	4	6	
236	[4, 2]	4	6	
240	[4, 4]	6	8	
267	[4]	3	5	Q_{64}

In Table 3, ‘**2**’ means $\mathbb{Z}/2$, etc. To compute $d_0(G)$, we needed to observe that, in all cases covered by this table, $e''(G) = e'(G)$. Except when G is 32#41, this can be checked by noticing that elements in the top degree in $Q_A\,Cess^*(G)$ are represented by classes in the image of $\mathrm{Inf}^G_{G/C}$, and so are primitive. When G is 32#41, elements in the $Q_A\,Cess^5(G)$ are represented by essential classes of the lowest degree, and so are primitive.

Table 3
Indecomposable, non-2-central, 2-groups of order ⩽ 32

Order	#	Type	Depth	Rank	$e(G)$	$e'(G)$	$d_0(G)$	$C_G(V)$'s	Notes
8	4	[2]	2	2	1	−1	0	$\mathbf{2}^2$	D_8
16	8	[4]	1	2	3	−1	1	$\mathbf{4}\times\mathbf{2}$	AES_{16}
	9	[2, 2]	2	3	2	1	1	$\mathbf{2}^2$	
	11	[4]	1	2	3	2	2	$\mathbf{4}\times\mathbf{2}$	
	12	[2]	1	2	1	−1	0	$\mathbf{2}^2$	D_{16}
	13	[4]	1	2	3	2	2	$\mathbf{2}^2$	SD_{16}
32	16	[4, 2]	2	3	4	3	3	$\mathbf{4}\times\mathbf{2}^2$	
	17	[4]	2	2	3	−1	1	$\mathbf{8}\times\mathbf{2}$	
	20	[2, 2]	2	3	2	1	1	$\mathbf{4}\times\mathbf{2}^2$	
	22	[4]	1	2	3	2	2	$\mathbf{8}\times\mathbf{2}$	
	26	[4]	2	2	3	−1	1	$\mathbf{8}\times\mathbf{2}$, $\mathbf{4}\times\mathbf{2}$	
	27	[2, 2]	2	3	2	1	1	$\mathbf{2}^3$	
	31	[4]	2	2	3	−1	2	$\mathbf{4}\times\mathbf{4}$, $\mathbf{4}\times\mathbf{2}$	

(*continued on next page*)

Table 3 (*continued*)

Order	#	Type	Depth	Rank	$e(G)$	$e'(G)$	$d_0(G)$	$C_G(V)$'s	Notes
	32	[4]	1	2	3	2	2	$\mathbf{8}\times\mathbf{2}$	
	33	[2, 2]	3	4	2	−1	0	$\mathbf{2}^4, \mathbf{2}^3$	
	34	[2, 2]	3	3	2	−1	0	$\mathbf{2}^3$	
	36	[2, 2]	3	3	2	−1	1	$\mathbf{4}\times\mathbf{2}^2, \mathbf{2}^3$	
	37	[4, 2]	2	3	4	3	3	$\mathbf{4}\times\mathbf{2}^2$	
	38	[4, 2]	2	3	4	2	2	$\mathbf{4}\times\mathbf{2}^2, \mathbf{2}^3$	
	39	[4, 2]	2	3	4	3	3	$\mathbf{2}^3$	
	41	[4, 4]	2	3	6	5	5	$\mathbf{2}^3$	
	42	[4]	3	3	3	−1	0	$\mathbf{2}^3$	$D_8 * D_8$
	43	[8]	2	2	7	−1	3	$Q_8\times\mathbf{2}$	$D_8 * Q_8$
	44	[4]	2	3	3	−1	1	$\mathbf{4}\times\mathbf{2}, \mathbf{2}^3$	
	45	[8]	1	2	7	4	4	$Q_8\times\mathbf{2}, \mathbf{4}\times\mathbf{2}$	
	46	[4]	2	3	3	−1	3	$Q_8\times\mathbf{2}, \mathbf{2}^3$	
	47	[4]	1	3	3	1	1	$D_8\times\mathbf{2}, \mathbf{2}^3$	
	48	[8]	1	2	7	6	6	$Q_8\times\mathbf{2}$	
	49	[2]	2	2	1	−1	0	$\mathbf{2}^2$	D_{32}
	50	[4]	1	2	3	2	2	$\mathbf{2}^2$	SD_{32}

Table 4
Indecomposable, non-2-central, order 64, with $Cess^*(G) \neq \mathbf{0}$

#	Type	Rank	$e'(G)$	#	Type	Rank	$e'(G)$	#	Type	Rank	$e'(G)$
42	[4]	2	2	173	[4, 4]	4	3	193	[4, 2]	3	3
67	[4]	2	2	175	[4, 4]	4	3	196	[4, 2]	3	2
143	[4]	2	2	183	[4, 4]	4	5	197	[4, 2]	3	3
182	[4]	2	2	202	[4, 2]	4	2	198	[4, 2]	3	3
245	[8]	2	4					200	[4, 4]	3	5
246	[4]	2	2	32	[4, 2]	3	3	204	[4, 2]	3	3
249	[8]	2	6	33	[4, 2]	3	3	206	[4, 2]	3	3
255	[8]	2	6	40	[2, 2]	3	1	207	[4, 2]	3	3
258	[8]	2	4	54	[4, 2]	3	3	208	[4, 2]	3	3
266	[4]	2	2	60	[4, 2]	3	3	209	[4, 2]	3	3
				61	[4, 2]	3	3	213	[4, 2]	3	2
121	[8]	3	3	62	[2, 2]	3	1	214	[4, 2]	3	2
130	[8]	3	3	79	[4, 4]	3	5	215	[4, 4]	3	5
133	[8]	3	4	80	[4, 4]	3	4	216	[4, 4]	3	5
180	[8]	3	3	95	[4, 2]	3	3	218	[4, 2]	3	3
181	[8]	3	3	97	[4, 2]	3	3	219	[4, 2]	3	2
247	[4]	3	1	98	[4, 2]	3	3	220	[4, 2]	3	3
251	[8]	3	4	99	[4, 2]	3	2	221	[4, 4]	3	5
253	[8]	3	3	100	[4, 2]	3	3	223	[4, 4]	3	5
254	[8]	3	4	102	[4, 4]	3	5	224	[4, 4]	3	5
257	[8]	3	3	108	[8, 2]	3	7	225	[4, 2]	3	2
262	[8]	3	3	115	[8, 2]	3	7	226	[4, 2]	3	3
				116	[4, 2]	3	3	228	[4, 2]	3	2
81	[2, 2, 2]	5	1	118	[4, 2]	3	3	229	[4, 2]	3	3
				129	[4, 2]	3	3	230	[4, 2]	3	3
83	[2, 2, 2]	4	2	132	[4, 2]	3	3	231	[4, 4]	3	5
85	[2, 2, 2]	4	2	138	[2, 2]	3	1	232	[4, 4]	3	5
86	[2, 2, 2]	4	2	161	[4, 4]	3	4	234	[2, 2]	3	1
89	[4, 2, 2]	4	4	165	[4, 4]	3	4	238	[4, 4]	3	5
91	[2, 2, 2]	4	2	166	[4, 4]	3	4	239	[4, 2]	3	3

Table 4 (*continued*)

#	Type	Rank	$e'(G)$	#	Type	Rank	$e'(G)$	#	Type	Rank	$e'(G)$
146	[2, 2, 2]	4	2	167	[4, 4]	3	4				
147	[4, 2, 2]	4	4	168	[4, 4]	3	5				
148	[2, 2, 2]	4	2	172	[8, 2]	3	5				
150	[2, 2, 2]	4	2	174	[4, 4]	3	5				
151	[2, 2, 2]	4	2	177	[4, 4]	3	3				
				178	[4, 4]	3	4				
94	[4, 2]	4	2	179	[4, 4]	3	5				
113	[4, 2]	4	2	185	[4, 4]	3	3				
131	[4, 2]	4	2	186	[4, 4]	3	4				
163	[4, 4]	4	3	189	[4, 2]	3	3				

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