Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of Computational and Applied Mathematics 171 (2004) 199–234
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

The Sobolev orthogonality and spectral analysis of the Laguerre polynomials $\{L_n^{-k}\}$ for positive integers k ☆

W.N. Everitt[a], L.L. Littlejohn[b,*], R. Wellman[c]

[a] Department of Mathematics, The University of Birmingham, Birmingham, B15 2TT, UK
[b] Department of Mathematics and Statistics, Utah State University, Logan, UT 84322-3900, USA
[c] Department of Mathematics, Westminster College, Foster Hall, Salt Lake City, UT 84105, USA

Received 11 June 2003

Abstract

For $k \in \mathbb{N}$, we consider the analysis of the classical Laguerre differential expression

$$\ell_{-k}[y](x) = \frac{1}{x^{-k}\mathrm{e}^{-x}}(-(x^{-k+1}\mathrm{e}^{-x}y'(x))' + rx^{-k}\mathrm{e}^{-x}y(x)) \quad (x \in (0,\infty)),$$

where $r \geqslant 0$ is fixed, in several nonisomorphic Hilbert and Hilbert–Sobolev spaces.

In one of these spaces, specifically the Hilbert space $L^2((0,\infty); x^{-k}\mathrm{e}^{-x})$, it is well known that the Glazman–Krein–Naimark theory produces a self-adjoint operator A_{-k}, generated by $\ell_{-k}[\cdot]$, that is bounded below by rI, where I is the identity operator on $L^2((0,\infty); x^{-k}\mathrm{e}^{-x})$. Consequently, as a result of a general theory developed by Littlejohn and Wellman, there is a continuum of left-definite Hilbert spaces $\{H_{s,-k} = (V_{s,-k}, (\cdot,\cdot)_{s,-k})\}_{s>0}$ and left-definite self-adjoint operators $\{B_{s,-k}\}_{s>0}$ associated with the pair $(L^2((0,\infty); x^{-k}\mathrm{e}^{-x}), A_{-k})$. For A_{-k} and each of the operators $B_{s,-k}$, it is the case that the *tail-end sequence* $\{L_n^{-k}\}_{n=k}^{\infty}$ of Laguerre polynomials form a complete set of eigenfunctions in the corresponding Hilbert spaces.

In 1995, Kwon and Littlejohn introduced a Hilbert–Sobolev space $W_k[0,\infty)$ in which the *entire* sequence of Laguerre polynomials is orthonormal. In this paper, we construct a self-adjoint operator in this space, generated by the second-order Laguerre differential expression $\ell_{-k}[\cdot]$, having $\{L_n^{-k}\}_{n=0}^{\infty}$ as a complete set of eigenfunctions. The key to this construction is in identifying a certain closed subspace of $W_k[0,\infty)$ with the kth left-definite vector space $V_{k,-k}$.

MSC: primary 33C45; 34B24; secondary 42C05; 34B40

Keywords: Laguerre polynomials; Orthogonal polynomials; Left-definite theory; Self-adjoint operators; Sobolev spaces

☆ The authors L.L.L. and R.W. are very pleased to dedicate this paper to their co-author W.N. Everitt as a small means of large thanks for his many years of teaching, mentoring and guiding us as well as for his constant encouragement.

* Corresponding author.
E-mail addresses: w.n.everitt@bham.ac.uk (W.N. Everitt), lance@math.usu.edu (L.L. Littlejohn), rwellman@westminstercollege.edu (R. Wellman).

0377-0427/$ - see front matter © 2004 Published by Elsevier B.V.
doi:10.1016/j.cam.2004.01.017

1. Introduction

For $\alpha > -1$, the analysis of the classical Laguerre differential expression

$$\ell_\alpha[y](x) := -xy''(x) + (x-1-\alpha)y'(x) + ry(x)$$
$$= \frac{1}{x^\alpha e^{-x}}(-(x^{\alpha+1}e^{-x}y'(x))' + rx^\alpha e^{-x}y(x)) \quad (x > 0;\ r \geqslant 0 \text{ fixed})$$

is well understood and documented from the viewpoints of differential equations, special functions, and spectral theory. Indeed, the nth Laguerre polynomial $y = L_n^\alpha(x)$ is a solution of

$$\ell_\alpha[y](x) = (n+r)y(x) \quad (n \in \mathbb{N}_0)$$

and classical properties of these polynomials are numerous and well known (see [2,15,16]). The right-definite operator-theoretic properties and spectral analysis of this Lagrangian symmetrizable expression $\ell_\alpha[\cdot]$, when $\alpha > -1$, are also detailed in the literature (see [1,5,13,14]). More specifically, as an application of the classical Glazman–Krein–Naimark (GKN) theory, there is a self-adjoint operator A_α, generated from this Laguerre differential expression, in the weighted Hilbert space $L^2((0,\infty); x^\alpha e^{-x})$ having the Laguerre polynomials $\{L_n^\alpha\}_{n=0}^\infty$ as a complete set of eigenfunctions.

However, the functional analytic theory of this expression, specifically when $-\alpha := k \in \mathbb{N}$, is less clear and is the principle focus of this paper. One of the main differences in this case, compared to the classical situation of $\alpha > -1$, stems from the fact that the Laguerre polynomials of degree $< k$ do not belong to the Hilbert space $L^2((0,\infty); x^{-k}e^{-x})$. However, the tail-end sequence of Laguerre polynomials $\{L_n^{-k}\}_{n=k}^\infty$ still form a complete orthogonal set in $L^2((0,\infty); x^{-k}e^{-x})$ (see Section 2). In this space, we construct, again with the aid of the GKN theory, a self-adjoint operator A_{-k}, bounded below by rI in $L^2((0,\infty); x^{-k}e^{-x})$, having these Laguerre polynomials $\{L_n^\alpha\}_{n=k}^\infty$ as eigenfunctions. Consequently, a general left-definite theory, recently developed by Littlejohn and Wellman [11], may be applied to this operator to assert the existence of a continuum of (left-definite) Hilbert–Sobolev spaces $\{H_{s,-k} = (V_{s,-k}, (\cdot,\cdot)_{s,-k})\}_{s>0}$ and (left-definite) self-adjoint operators $\{B_{s,-k}\}_{s>0}$, generated from A_{-k}. We explicitly construct these function spaces $V_{s,-k}$, their associated inner products $(\cdot,\cdot)_{s,-k}$, and operators $B_{s,-k}$ for all $s \in \mathbb{N}$. Furthermore, we show that the Laguerre polynomials $\{L_n^{-k}\}_{n=k}^\infty$ form a complete set of eigenfunctions for each of these operators.

The main portion of this paper, however, is to study the *entire* sequence of Laguerre polynomials $\{L_n^{-k}\}_{n=0}^\infty$ in a Sobolev space $W_k[0,\infty)$ that was recently discovered by Kwon and Littlejohn [9] (see also [10]). We show that these Laguerre polynomials form a complete orthonormal set in $W_k[0,\infty)$. Furthermore, we construct a self-adjoint operator T_k in $W_k[0,\infty)$ having these polynomials as eigenfunctions. Interestingly, the key to the construction of T_k is in identifying a certain subspace of $W_k[0,\infty)$ with the kth left-definite function space $V_{k,-k}$ and the kth left-definite operator $B_{k,-k}$ associated with the pair $(L^2((0,\infty); x^{-k}e^{-x}), A_{-k})$.

This paper is an extension of results given in [5] where the authors develop the spectral properties of the Laguerre differential expression $\ell_{-k}[\cdot]$ in various Hilbert and Hilbert–Sobolev spaces but only for the cases $k=1$ and 2. At the time of publication of [5], the general left-definite theory, developed in [11], which is instrumental in the results of this paper, was not fully developed. Consequently, the analytic methods used in [5] were not readily applicable to the general case of k being an arbitrary positive integer.

The contents of this paper are as follows. In Section 2, we review some important properties of the Laguerre polynomials, including a remarkable identity when the parameter α is a negative integer. Section 3 summarizes various properties of the Laguerre differential expression in the right-definite setting $L^2((0,\infty);x^\alpha e^{-x})$ for $\alpha > -1$ and $-\alpha = k \in \mathbb{N}$. A review of the general left-definite theory is given in Section 4 and this theory is applied to the Laguerre expression in Section 5. We remark that in [11], the authors give a detailed left-definite analysis of the Laguerre expression when $\alpha > -1$; some care must be exercised in extending these results to the case when $-\alpha = k \in \mathbb{N}$ but the results in this case are very similar and details will be omitted in this paper. In Section 5, we also establish some important properties of functions in the kth left-definite space $H_{k,-k} = (V_{k,-k}, (\cdot,\cdot)_{k,-k})$; a key to developing these properties is an important integral inequality established by Chisholm and Everitt [3]. Section 6 reviews the Kwon–Littlejohn discovery of a Sobolev space $W_k[0,\infty)$ where the entire sequence of Laguerre polynomials $\{L_n^{-k}\}_{n=0}^{\infty}$ is orthonormal. The completeness of $\{L_n^{-k}\}_{n=0}^{\infty}$ in $W_k[0,\infty)$ is established in Section 7. A fundamental decomposition of $W_k[0,\infty)$ is developed in Section 8; this decomposition is both important and necessary in the construction of *three* self-adjoint operators generated by $\ell_{-k}[\cdot]$. Sections 9 and 10 are concerned with explicitly constructing two of these self-adjoint operators $T_{k,1}$ and $T_{k,2}$ in certain closed subspaces $W_{k,1}[0,\infty) = (S_{k,1}[0,\infty), (\cdot,\cdot)_k)$ and its orthogonal complement $W_{k,2}[0,\infty)$, respectively, of $W_k[0,\infty)$. The fundamental decomposition, obtained in Section 8, as well as the important equality $S_{k,1}[0,\infty) = V_{k,-k}$, which we establish in Section 5, plays a key role in the construction of $T_{k,1}$. Lastly, in Section 11, the self-adjoint operator $T_k = T_{k,1} \oplus T_{k,2}$, generated by the Laguerre differential expression $\ell_{-k}[\cdot]$, is constructed and various properties of this operator are developed, including the fact that the Laguerre polynomials $\{L_n^{-k}\}_{n=0}^{\infty}$ are a (complete) set of eigenfunctions of T_k.

Notation. In this paper, we fix $k \in \mathbb{N}$. For $\alpha \in \mathbb{R}$, let $L^2_\alpha(0,\infty) := L^2((0,\infty);x^\alpha e^{-x})$ (we use both notations in this paper) denote the Hilbert function space defined by

$$L^2_\alpha(0,\infty) := \left\{ f : (0,\infty) \to \mathbb{C} \mid f \text{ is Lebesgue measurable} \right.$$
$$\left. \text{and } \int_0^\infty |f(x)|^2 x^\alpha e^{-x}\, dx < \infty \right\} \tag{1.1}$$

with inner product and norm, respectively, given by

$$(f,g)_{L^2(\alpha)} := \int_0^\infty f(x)\bar{g}(x) x^\alpha e^{-x}\, dx \text{ and } \|f\|_{L^2(\alpha)} = (f,f)^{1/2}_{L^2(\alpha)} \quad (f,g \in L^2_\alpha(0,\infty)).$$

Occasionally, we shall refer to the Hilbert space $L^2(I)$ in this paper; this is the usual Lebesgue square integrable space consisting of all complex-valued (Lebesgue) measurable functions that are (Lebesgue) square integrable on the real interval I. Let $\mathscr{P}[0,\infty)$ denote the vector space of all complex-valued polynomials $p : [0,\infty) \to \mathbb{C}$ of the real variable x. The set $\mathbb{N}$ will denote the positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, while $\mathbb{R}$ and $\mathbb{C}$ will denote, respectively, the real and complex number fields. The term AC will denote absolute continuity; for an open interval $I \subset \mathbb{R}$, the notation $AC_{\mathrm{loc}}(I)$ will denote those functions $f : I \to \mathbb{C}$ that are absolutely continuous on all compact subintervals of I. If A is a linear operator, $\mathscr{D}(A)$ will denote its domain. The identity operator will be denoted by I and will be used in several Hilbert spaces in this paper. Lastly, a word is in order regarding displayed, bracketed information. For example,

$$f(t) \text{ has property } P \quad (t \in I)$$

and

$$g_m \text{ has property } Q \quad (m \in \mathbb{N}_0)$$

mean, respectively, that $f(t)$ has property P *for all* $t \in I$ and g_m has property Q *for all* $m \in \mathbb{N}_0$. Further notations are introduced as needed throughout the paper.

2. Preliminaries: properties of the Laguerre polynomials

For any $\alpha \in \mathbb{R}$, the Laguerre polynomials $\{L_n^\alpha\}_{n=0}^\infty$ are defined by

$$L_n^\alpha(x) = \sum_{j=0}^{n} (-1)^j \binom{n+\alpha}{n-j} \frac{x^j}{j!} \quad (n \in \mathbb{N}_0),$$

observe that $L_n^\alpha(x)$ is a polynomial of degree exactly n for any choice of $\alpha \in \mathbb{R}$. Moreover, in this case, $y = L_n^\alpha(x)$ is a solution of the differential equation

$$\ell_\alpha[y](x) = (n+r)y(x) \quad (n \in \mathbb{N}_0),$$

where

$$\begin{aligned} \ell_\alpha[y](x) &:= -xy'' + (x-1-\alpha)y'(x) + ry(x) \\ &= \frac{1}{x^\alpha e^{-x}}(-(x^{\alpha+1}e^{-x}y'(x))' + rx^\alpha e^{-x}y(x)). \end{aligned} \tag{2.1}$$

The parameter r in (2.1), which can be viewed as a spectral *shift* parameter, is a fixed nonnegative constant and is usually presented in the literature as zero. However we can assume, without loss of generality, that $r > 0$; as we will see this assumption is critical for many of the results in this paper.

When $\alpha > -1$, the Laguerre polynomials $\{L_n^\alpha\}_{n=0}^\infty$ form a complete orthogonal set in the Hilbert space $L_\alpha^2(0,\infty)$ (see (1.1)), with inner product

$$(f,g)_{L^2(\alpha)} := \int_0^\infty f(x)\bar{g}(x)x^\alpha e^{-x}\,dx \quad (f,g \in L_\alpha^2(0,\infty)), \tag{2.2}$$

and norm

$$\|f\|_{L^2(\alpha)} := (f,f)_{L^2(\alpha)}^{1/2} \quad (f \in L_\alpha^2(0,\infty)), \tag{2.3}$$

see [16, Chapter V] for an in-depth discussion of these orthogonal polynomials. In fact, in this case, we have

$$(L_n^\alpha, L_m^\alpha)_{L^2(\alpha)} = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{n,m} \quad (n,m \in \mathbb{N}_0). \tag{2.4}$$

When $\alpha < -1$, $-\alpha \notin \mathbb{N}$, the Laguerre polynomials $\{L_n^\alpha\}_{n=0}^\infty$ are also orthogonal on the real line but, in this case, with respect to a signed measure; this is a consequence of Favard's theorem [2, Theorem 6.4, p. 75].

In the case where $-\alpha := k \in \mathbb{N}$, the Laguerre polynomials $\{L_n^{-k}\}_{n=0}^{\infty}$ cannot be orthogonal with respect to any Lebesgue–Stieltjes bilinear form of the type

$$\int_{\mathbb{R}} f(x)\bar{g}(x)\,\mathrm{d}\mu, \tag{2.5}$$

where μ is a (signed) Borel measure. Indeed, the three-term recurrence relation for the Laguerre polynomials $\{L_n^{\alpha}\}_{n=0}^{\infty}$ is

$$L_{-1}^{\alpha}(x) = 0, \quad L_0^{\alpha}(x) = 1,$$

$$(n+1)L_{n+1}^{\alpha}(x) + (x - \alpha - 2n - 1)L_n^{\alpha}(x) + (n+\alpha)L_{n-1}^{\alpha}(x) = 0 \quad (n \in \mathbb{N}_0),$$

observe that the coefficient of $L_{n-1}^{\alpha}(x)$ in this recurrence relation vanishes when $\alpha = -n$. Consequently, Favard's theorem says that the full sequence of Laguerre polynomials $\{L_n^{\alpha}\}_{n=0}^{\infty}$, when α is a negative integer, cannot be orthogonal on the real line with respect to a bilinear form of the type (2.5). However, as we will demonstrate shortly, the *tail-end sequence* $\{L_n^{-k}\}_{n=k}^{\infty}$ is orthogonal in the Hilbert space $L_{-k}^2(0,\infty)$ with inner product

$$(f,g)_{L^2(-k)} := \int_0^{\infty} f(x)\bar{g}(x)x^{-k}\mathrm{e}^{-x}\,\mathrm{d}x \quad (f,g \in L_{-k}^2(0,\infty)) \tag{2.6}$$

and norm

$$\|f\|_{L^2(-k)} = (f,f)_{L^2(-k)}^{1/2} \quad (f \in L_{-k}^2(0,\infty)), \tag{2.7}$$

moreover $\{L_n^{-k}\}_{n=0}^{k-1} \notin L_{-k}^2(0,\infty)$.

One of the more remarkable properties of the Laguerre polynomials $\{L_n^{-k}\}_{n=k}^{\infty}$, when $k \in \mathbb{N}$, is the formula (see [16, p. 102])

$$L_n^{-k}(x) = (-1)^k \frac{(n-k)!}{n!} x^k L_{n-k}^k(x) \quad (k \in \mathbb{N};\ n \geqslant k). \tag{2.8}$$

Formula (2.8) plays a key role throughout this paper in our analysis of the second-order Laguerre differential expression

$$\begin{aligned}\ell_{-k}[y](x) &:= \frac{1}{x^{-k}\mathrm{e}^{-x}}(-(x^{-k+1}\mathrm{e}^{-x}y'(x))' + rx^{-k}\mathrm{e}^{-x}y(x)) \\ &= -xy''(x) + (x-1+k)y'(x) + ry(x) \quad (x>0).\end{aligned} \tag{2.9}$$

We now establish the following result.

Theorem 2.1. *The Laguerre polynomials* $\{L_n^{-k}\}_{n=k}^{\infty}$ *form a complete orthogonal set in the space* $L_{-k}^2(0,\infty)$. *Equivalently, the set* $\mathscr{P}_k[0,\infty)$ *of all polynomials* p *of degree at least* k *satisfying*

$$p(0) = p'(0) = \cdots = p^{(k-1)}(0) = 0$$

is dense in $L_{-k}^2(0,\infty)$.

Proof. Observe that

$$\int_0^{\infty} |f(x)|^2 x^{-k}\mathrm{e}^{-x}\,\mathrm{d}x = \int_0^{\infty} |x^{-k}f(x)|^2 x^k \mathrm{e}^{-x}\,\mathrm{d}x,$$

so that

$$\|f\|_{L^2(-k)} = \|x^{-k} f\|_{L^2(k)}, \tag{2.10}$$

where $\|\cdot\|_{L^2(k)}$ and $\|\cdot\|_{L^2(-k)}$ are the norms defined in (2.3) and (2.7), respectively. Hence, $f \in L^2_{-k}(0,\infty)$ if and only if $x^{-k} f \in L^2_k(0,\infty)$.

Let $f \in L^2_{-k}(0,\infty)$ and let $\varepsilon > 0$. Hence $x^{-k} f \in L^2_k(0,\infty)$. Since the space $\mathscr{P}[0,\infty)$ of polynomials is dense in $L^2_k(0,\infty)$ (see [16, Theorem 5.7.2]), there exists $q \in \mathscr{P}[0,\infty)$ such that

$$\|x^{-k} f - q\|_{L^2(k)} < \varepsilon. \tag{2.11}$$

Let $p(x) = x^k q(x)$ so $p \in P_k(0,\infty)$; by (2.10), we see that

$$\begin{aligned} \|f - p\|_{L^2(-k)} &= \|x^{-k}(f - p)\|_{L^2(k)} \\ &= \|x^{-k} f - q\|_{L^2(k)} \\ &< \varepsilon \text{ by (2.11)}. \end{aligned}$$

This completes the proof of the theorem. □

3. Right-definite analysis of the Laguerre differential expression

When $\alpha > -1$, the Laguerre polynomials $\{L_n^\alpha\}_{n=0}^\infty$ form a complete set of eigenfunctions of the self-adjoint operator

$$A_\alpha : \mathscr{D}(A_\alpha) \subset L^2_\alpha(0,\infty) \to L^2_\alpha(0,\infty)$$

defined by

$$\begin{cases} A_\alpha f := \ell_\alpha[f], \\ f \in \mathscr{D}(A_\alpha), \end{cases} \tag{3.1}$$

where $\ell_\alpha[\cdot]$ is the Laguerre differential expression, defined in (2.1). Here, the domain $\mathscr{D}(A_\alpha)$ of A_α is given by

$$\mathscr{D}(A_\alpha) := \begin{cases} \Delta_\alpha & \text{if } \alpha \geqslant 1, \\ \left\{ f \in \Delta_\alpha \mid \lim_{x\to 0^+} x^{\alpha+1} f'(x) = 0 \right\} & \text{if } -1 < \alpha < 1, \end{cases} \tag{3.2}$$

where Δ_α is the maximal domain in $L^2_\alpha(0,\infty)$, defined by

$$\Delta_\alpha := \{ f : (0,\infty) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(0,\infty);\, f, \ell_\alpha[f] \in L^2_\alpha(0,\infty)\}.$$

Moreover, A_α is bounded below by rI in $L^2_\alpha(0,\infty)$; that is,

$$(A_\alpha f, f)_{L^2(\alpha)} \geqslant r(f,f)_{L^2(\alpha)} \quad (f \in \mathscr{D}(A_\alpha)),$$

where $(\cdot,\cdot)_{L^2(\alpha)}$ is the inner product defined in (2.2). Furthermore, the spectrum of A_α is discrete and given by $\sigma(A_\alpha) = \{m + r \mid m \in \mathbb{N}_0\}$. We recommend the sources [11,14,19, Section 12] for explicit and further details concerning both analytic and algebraic properties of the operator A_α.

In the case $\alpha < -1$ and $-\alpha \notin \mathbb{N}$, we recommend the contribution [7], where a spectral analysis of the Laguerre expression (2.1) is carried out in a Krein space setting.

Turning to the case $\alpha = -k$, where $k \in \mathbb{N}$, the authors in [5] show that the differential operator $A_{-k} : \mathscr{D}(A_{-k}) \subset L^2_{-k}(0,\infty) \to L^2_{-k}(0,\infty)$ defined by

$$\begin{cases} A_{-k} := \ell_{-k}[f], \\ f \in \mathscr{D}(A_{-k}) := \{f : (0,\infty) \to \mathbb{C} \mid f, f' \in AC_{\mathrm{loc}}(0,\infty); f, \ell_{-k}[f] \in L^2_{-k}(0,\infty)\}, \end{cases} \tag{3.3}$$

where $\ell_{-k}[\cdot]$ is the Laguerre differential expression defined in (2.9), is self-adjoint and bounded below by rI in $L^2_{-k}(0,\infty)$; that is,

$$(A_{-k}f, f)_{L^2(-k)} \geqslant r(f,f)_{L^2(-k)} \quad (f \in \mathscr{D}(A_{-k})), \tag{3.4}$$

where $(\cdot,\cdot)_{L^2(-k)}$ is the inner product defined in (2.6). Note that $\mathscr{D}(A_{-k})$ is, in fact, the maximal domain Δ_{-k} of $\ell_{-k}[\cdot]$ in $L^2_{-k}(0,\infty)$. This is a consequence of the expression $\ell_{-k}[\cdot]$ being strong limit-point and Dirichlet at both $x = 0$ and ∞; see [5, Theorem 2.2]. We note the spectrum of A_{-k} is discrete and given by $\sigma(A_{-k}) = \{m + r \mid m \geqslant k\}$. Moreover, the Laguerre polynomials $\{L_n^{-k}\}_{n=k}^{\infty}$ form a complete orthogonal set of eigenfunctions of A_{-k} in $L^2_{-k}(0,\infty)$; further details can be found in [5, Theorem 2.2].

4. General left-definite theory

Let V denote a vector space (over the complex field $\mathbb{C}$) and suppose that $(\cdot,\cdot)$ is an inner product with norm $\|\cdot\|$, generated from $(\cdot,\cdot)$, such that $H = (V, (\cdot,\cdot))$ is a Hilbert space. Suppose V_r (the subscripts will be made clear shortly) is a linear manifold (subspace) of the vector space V and let $(\cdot,\cdot)_r$ and $\|\cdot\|_r$ denote an inner product and associated norm, respectively, over V_r (quite possibly different from $(\cdot,\cdot)$ and $\|\cdot\|$). We denote the resulting inner product space by $W_r = (V_r, (\cdot,\cdot)_r)$.

Throughout this section, we assume that $A : \mathscr{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by rI, for some $r > 0$; that is,

$$(Ax, x) \geqslant r(x,x) \quad (x \in \mathscr{D}(A)).$$

It follows that A^s, for each $s > 0$, is a self-adjoint operator that is bounded below in H by $r^s I$.

We now define an sth left-definite space associated with (H, A).

Definition 4.1. Let $s > 0$ and suppose V_s is a linear manifold of the Hilbert space $H = (V, (\cdot,\cdot))$ and $(\cdot,\cdot)_s$ is an inner product on $V_s \times V_s$. Let $W_s = (V_s, (\cdot,\cdot)_s)$. We say that W_s is an sth *left-definite space* associated with the pair (H, A) if each of the following conditions hold:

(1) W_s is a Hilbert space,
(2) $\mathscr{D}(A^s)$ is a linear manifold of V_s,
(3) $\mathscr{D}(A^s)$ is dense in W_s,
(4) $(x,x)_s \geqslant r^s(x,x)$ $(x \in V_s)$, and
(5) $(x,y)_s = (A^s x, y)$ $(x \in \mathscr{D}(A^s),\ y \in V_s)$.

It is not clear, from the definition, if such a self-adjoint operator A generates a left-definite space for a given $s>0$. However, in [11], the authors prove the following theorem; the Hilbert space spectral theorem plays a prominent role in establishing this result.

Theorem 4.1 (See Littlejohn and Wellman [11, Theorem 3.1]). *Suppose $A:\mathscr{D}(A)\subset H\to H$ is a self-adjoint operator that is bounded below by rI, for some $r>0$. Let $s>0$. Define $W_s=(V_s,(\cdot,\cdot)_s)$ by*

$$V_s=\mathscr{D}(A^{s/2}) \tag{4.1}$$

and

$$(x,y)_s=(A^{s/2}x,A^{s/2}y)\quad (x,y\in V_s). \tag{4.2}$$

Then W_s is a left-definite space associated with the pair (H,A). Moreover, suppose $W_s':=(V_s',(\cdot,\cdot)_s')$ is another sth left-definite space associated with the pair (H,A). Then $V_s=V_s'$ and $(x,y)_s=(x,y)_s'$ for all $x,y\in V_s=V_s'$; i.e., *$W_s=W_s'$. That is to say, $W_s=(V_s,(\cdot,\cdot)_s)$ is the* unique *left-definite space associated with (H,A).*

Remark 4.1. Although all five conditions in Definition 4.1 are necessary in the proof of Theorem 4.1, the most important property, in a sense, is the one given in (5). Indeed, this property asserts that the sth left-definite inner product is generated from the sth power of A. In particular, if A is generated from a Lagrangian symmetric differential expression $\ell[\cdot]$, the sth left-definite inner product $(\cdot,\cdot)_s$ is determined by the sth power of $\ell[\cdot]$. Consequently, even though these left-definite spaces and left-definite inner products exist for all $s>0$, we can only *explicitly* obtain these spaces and inner products when s is a positive integer. We refer the reader to [11] where an example is discussed in which the *entire* continuum of left-definite spaces and inner products are explicitly obtained.

Definition 4.2. For $s>0$, let $W_s=(V_s,(\cdot,\cdot)_s)$ denote the sth left-definite space associated with (H,A). If there exists a self-adjoint operator $B_s:\mathscr{D}B_s)\subset W_s\to W_s$ satisfying

$$B_sf=Af\quad (f\in\mathscr{D}(B_s)\subset\mathscr{D}(A)),$$

we call such an operator an sth *left-definite operator associated with* (H,A).

Again, it is not immediately clear that such an B_s exists for a given $s>0$; in fact, however, as the next theorem shows, B_s exists and is unique.

Theorem 4.2 (See Little John and Wellman [11, Theorem 3.2]). *Suppose A is a self-adjoint operator in a Hilbert space H that is bounded below by rI, for some $r>0$. For any $s>0$, let $W_s=(V_s,(\cdot,\cdot)_s)$ be the sth left-definite space associated with (H,A). Then there exists a unique left-definite operator B_s in W_s associated with (H,A). Moreover,*

$$\mathscr{D}(B_s)=V_{s+2}\subset\mathscr{D}(A).$$

The next theorem gives further explicit information regarding the left-definite spaces and left-definite operators associated with (H,A).

Theorem 4.3 (See Littlejohn and Wellman [11, Theorem 3.4]). *Suppose A is a self-adjoint operator in a Hilbert space H that is bounded below by rI, for some $r > 0$. Let $\{H_s = (V_s, (\cdot,\cdot)_s)\}_{s>0}$ and $\{B_s\}_{s>0}$ be the left-definite spaces and left-definite operators, respectively, associated with (H, A). Then the following results are true*:

(1) *Suppose A is bounded. Then, for each $s > 0$,*
  (i) $V = V_s$;
  (ii) *the inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_s$ are equivalent*;
  (iii) $A = B_s$.

(2) *Suppose A is unbounded. Then*
  (i) *V_s is a proper subspace of V*;
  (ii) *V_s is a proper subspace of V_t whenever $0 < t < s$*;
  (iii) *the inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_s$ are not equivalent for any $s > 0$*;
  (iv) *the inner products $(\cdot,\cdot)_t$ and $(\cdot,\cdot)_s$ are not equivalent for any $s, t > 0$, $s \neq t$*;
  (v) *$\mathscr{D}(B_s)$ is a proper subspace of $\mathscr{D}(A)$ for each $s > 0$*;
  (vi) *$\mathscr{D}(B_t)$ is a proper subspace of $\mathscr{D}(B_s)$ whenever $0 < s < t$.*

Remark 4.2. A statement is in order regarding the apparent ambiguity between part (v) of Definition 4.1 and the definition of $(\cdot,\cdot)_s$ given in (4.2) of Theorem 4.1. From part (2)(ii) of Theorem 4.3, we see that $\mathscr{D}(A^s) = V_{2s} \subset V_s$. Consequently, if $x \in \mathscr{D}(A^s)$ and $y \in V_s$, we see from the self-adjointness of $A^{s/2}$ that

$$(x, y)_s = (A^s x, y) = (A^{s/2}(A^{s/2}x), y) = (A^{s/2}x, A^{s/2}y).$$

The fact that $A^{s/2}x \in \mathscr{D}(A^{s/2})$ follows from [11, Theorem 4.3, Eq. (4.3), Lemma 5.3, Eqs. (5.8) and (5.9)].

The last theorem that we state in this section shows that the point spectrum, continuous spectrum, and resolvent set of a self-adjoint, bounded below operator A and each of its associated left-definite operators B_s $(s > 0)$ are identical; see [8, Section 7.2] for the definitions concerning the various components of the spectrum listed below and the resolvent set of a general linear operator.

Theorem 4.4 (See Littlejohn and Wellman [11, Theorem 3.6]). *For each $s > 0$, let B_s denote the sth left-definite operator associated with the self-adjoint operator A that is bounded below by rI in H, for some $r > 0$. Then*

(a) *the point spectra of A and B_s coincide*; *that is*, $\sigma_p(B_s) = \sigma_p(A)$;
(b) *the continuous spectra of A and B_s coincide*; *that is*, $\sigma_c(B_s) = \sigma_c(A)$;
(c) *the resolvent sets of A and B_s are equal*; *that is*, $\rho(B_s) = \rho(A)$.

We refer the reader to [11] for other theorems, and examples, associated with the general left-definite theory of self-adjoint operators A that are bounded below.

5. Left-definite analysis of the Laguerre differential expression

Since, for $\alpha > -1$, the Laguerre differential operator, defined in (3.1) and (3.2), is self-adjoint and bounded below by rI, there exists a continuum of left-definite spaces and left-definite operators associated with $(L^2_\alpha(0,\infty), A_\alpha)$. Indeed, this is an immediate consequence of the results in the previous section. We remind the reader of the definition of the space $L^2_\alpha(0,\infty)$ in (1.1).

In [11], the authors show that the nth left-definite space associated with $(L^2_\alpha(0,\infty), A_\alpha)$, when $\alpha > -1$, is given by

$$H_{n,\alpha} := (V_{n,\alpha}, (\cdot,\cdot)_{n,\alpha}) \quad (n \in \mathbb{N}),$$

where

$$V_{n,\alpha} := \{f : (0,\infty) \to \mathbb{C} \mid f \in AC^{(n-1)}_{\mathrm{loc}}(0,\infty);\ f^{(j)} \in L^2_{\alpha+j}(0,\infty)\ (j = 0,1,\ldots,n)\} \tag{5.1}$$

and where the inner product $(\cdot,\cdot)_{n,\alpha}$ is given by

$$(f,g)_{n,\alpha} := \sum_{j=0}^{n} b_j(n,r) \int_0^\infty f^{(j)}(t)\bar{g}^{(j)}(t) t^{\alpha+j} \mathrm{e}^{-t}\, \mathrm{d}t \quad (f,g \in V_{n,\alpha}). \tag{5.2}$$

Here, the numbers $b_j(n,r)$ $(j = 0,1,\ldots,n)$ are defined to be

$$b_0(n,r) := \begin{cases} 0 & \text{if } r = 0, \\ r^n & \text{if } r > 0 \end{cases} \tag{5.3}$$

and, for $j \in \{1,2,\ldots,n\}$,

$$b_j(n,r) := \begin{cases} S_n^{(j)} & \text{if } r = 0, \\ \displaystyle\sum_{m=0}^{n-1} \binom{n}{m} S_{n-m}^{(j)} r^m & \text{if } r > 0, \end{cases} \tag{5.4}$$

where $\{S_n^{(j)}\}$ are the classical Stirling numbers of the second kind, defined by

$$S_n^{(j)} := \sum_{i=0}^{j} \frac{(-1)^{i+j}}{j!} \binom{j}{i} i^n \quad (n,j \in \mathbb{N}_0). \tag{5.5}$$

We note that each $b_j(n,r)$ is positive for $r > 0$ and $b_n(n,r) = 1$ for $n \in \mathbb{N}$. The Laguerre polynomials $\{L_n^\alpha\}_{n=0}^\infty$ form a complete orthogonal set in each $H_{n,\alpha}$; in fact,

$$(L_j^\alpha, L_m^\alpha)_{n,\alpha} = \frac{(j+r)^n \Gamma(j+\alpha+1)}{j!} \delta_{j,m} \quad (j,m \geqslant 0).$$

Moreover, for each $n \in \mathbb{N}$, the nth left-definite operator

$$B_{n,\alpha} : \mathscr{D}(B_{n,\alpha}) \subset H_{n,\alpha} \to H_{n,\alpha}$$

associated with $(A_\alpha, L^2_\alpha(0,\infty))$ is given explicitly by

$$B_{n,\alpha} f := \ell_\alpha[f]$$

$$f \in \mathscr{D}(B_{n,\alpha}) := V_{n+2,\alpha},$$

where $\ell_\alpha[\cdot]$ is the Laguerre differential expression defined in (2.1); the Laguerre polynomials $\{L_n^\alpha\}_{n=0}^\infty$ are a complete set of eigenfunctions of each operator $B_{n,\alpha}$ and the spectrum of each $B_{n,\alpha}$ is discrete and given by $\sigma(B_{n,\alpha}) = \sigma(A_\alpha) = \{m + r \mid m \in \mathbb{N}_0\}$.

When $\alpha = -k$, for some $k \in \mathbb{N}$, the self-adjoint operator A_{-k}, defined in (3.3), is bounded below in $L^2_{-k}(0,\infty)$ by rI (see (3.4)). In this case, the above results extend *mutatis mutandis* so we will not prove these results here. The following theorem summarizes various properties of the left-definite spaces and operators associated with $(L^2_{-k}(0,\infty), A_{-k})$.

Theorem 5.1. *Let $k \in \mathbb{N}$ and let A_{-k} denote the self-adjoint operator, defined in (3.3), that is bounded below by rI in $L^2_{-k}(0,\infty)$. Then the sequence of left-definite spaces associated with $(L^2_{-k}(0,\infty), A_{-k})$ is given by*

$$\{H_{n,-k} := (V_{n,-k}, (\cdot,\cdot)_{n,-k})\}_{n=1}^\infty, \tag{5.6}$$

where

$$\begin{aligned} V_{n,-k} := \{f : (0,\infty) \to \mathbb{C} \mid f^{(j)} \in AC_{\mathrm{loc}}(0,\infty)\ (j = 0,1,\ldots,n-1);\\ f^{(j)} \in L^2_{j-k}(0,\infty)\ (j = 0,1,\ldots,n)\} \end{aligned} \tag{5.7}$$

and

$$(f,g)_{n,-k} := \sum_{j=0}^{n} b_j(n,r) \int_0^\infty f^{(j)}(x)\bar{g}^{(j)}(x) x^{j-k} \mathrm{e}^{-x}\, \mathrm{d}x \quad (f,g \in V_{n,-k}). \tag{5.8}$$

In particular, the kth left-definite space $H_{k,-k} = (V_{k,-k}, (\cdot,\cdot)_{k,-k})$ associated with $(L^2_{-k}(0,\infty), A_{-k})$ is given by

$$\begin{aligned} V_{k,-k} := \{f : (0,\infty) \to \mathbb{C} \mid f^{(j)} \in AC_{\mathrm{loc}}(0,\infty)\ (j = 0,1,\ldots,k-1);\\ f^{(j)} \in L^2_{j-k}(0,\infty)\ (j = 0,1,\ldots,k)\}, \end{aligned} \tag{5.9}$$

and

$$(f,g)_{k,-k} := \sum_{j=0}^{k} b_j(k,r) \int_0^\infty f^{(j)}(x)\bar{g}^{(j)}(x) x^{j-k} \mathrm{e}^{-x}\, \mathrm{d}x \quad (f,g \in V_{k,-k}). \tag{5.10}$$

The Laguerre polynomials $\{L_n^{-k}\}_{n=k}^\infty$ form a complete orthogonal set in each left-definite space $H_{n,-k}$; in fact,

$$(L_j^{-k}, L_m^{-k})_{n,-k} = \frac{(j+r)^n (j-k)!}{j!} \delta_{j,m} \quad (j,m \geqslant k). \tag{5.11}$$

Furthermore, the sequence $\{B_{n,-k}\}_{n=1}^\infty$ of left-definite (self-adjoint) operators associated with the pair $(L^2_{-k}(0,\infty), A_{-k})$ is given explicitly by

$$B_{n,-k} : \mathscr{D}(B_{n,-k}) \subset H_{n,-k} \to H_{n,-k},$$

where

$$B_{n,-k} f := \ell_{-k}[f],$$

$$\mathscr{D}(B_{n,-k}) := V_{n+2,-k}$$
$$= \{f \in V_{n,-k} \mid f^{(j)} \in AC_{\mathrm{loc}}(0,\infty);\ f^{(j+1)} \in L^2_{j+1-k}(0,\infty)\ (j = n, n+1)\}$$

and where $\ell_{-k}[\cdot]$ *is the Laguerre differential expression defined in* (2.9). *The Laguerre polynomials* $\{L_n^{-k}\}_{n=k}^{\infty}$ *form a complete set of eigenfunctions of each* $B_{n,-k}$; *furthermore*, *the spectrum of* $B_{n,-k}$ *is discrete and given by*

$$\sigma(B_{n,-k}) = \sigma(A_{-k}) = \{m + r \mid m \geqslant k\}.$$

In particular, *we note that the kth left-definite operator* $B_{k,-k} : \mathscr{D}(B_{k,-k}) \subset H_{k,-k} \to H_{k,-k}$ *is given by*

$$\begin{cases} B_{k,-k} f := \ell_{-k}[f], \\ \mathscr{D}(B_{k,-k}) := V_{k+2,-k}, \end{cases} \tag{5.12}$$

where

$$V_{k+2,-k} := \{f : (0,\infty) \to \mathbb{C} \mid f^{(j)} \in AC_{\mathrm{loc}}(0,\infty)\ (j = 0,1,\ldots,k+1);$$
$$f^{(j)} \in L^2_{j-k}(0,\infty)\ (j = 0,1,\ldots,k+2)\}, \tag{5.13}$$

or, *equivalently*,

$$V_{k+2,-k} := \{f \in V_{k,-k} | f^{(k)}, f^{(k+1)} \in AC_{\mathrm{loc}}(0,\infty);\ f^{(k+1)} \in L^2((0,\infty); x\mathrm{e}^{-x}),$$
$$f^{(k+2)} \in L^2((0,\infty); x^2\mathrm{e}^{-x})\}. \tag{5.14}$$

As we will see in Section 9, both the operator $B_{k,-k}$ and the kth left-definite vector space $V_{k,-k}$ play an important role in obtaining a certain self-adjoint operator $T_{k,1}$.

We seek to obtain a new characterization of the kth left-definite vector space $V_{k,-k}$ associated with $(A_{-k}, L^2_{-k}(0,\infty))$; this characterization will be important in the developments in the rest of this paper. Before obtaining this characterization, we state an important operator inequality result that will be used on several occasions in this paper.

Theorem 5.2. *Let* $(a,b) \subset \mathbb{R}$ *with* $-\infty \leqslant a < b \leqslant \infty$ *and suppose* $\varphi, \psi : (a,b) \to \mathbb{C}$ *satisfy*

$$\varphi \in L^2(a,c), \psi \in L^2(c,b) \quad (c \in (a,b)).$$

Define the linear operators $S, T : L^2(a,b) \to L^2_{\mathrm{loc}}(a,b)$ *by*

$$Sf(x) = \varphi(x) \int_x^b \psi(x) f(x)\,\mathrm{d}x \quad (x \in (a,b)),$$

$$Tf(x) = \psi(x) \int_a^x \varphi(x) f(x)\,\mathrm{d}x \quad (x \in (a,b)).$$

Then S and T are bounded operators into $L^2(a,b)$ *if and only if there exists a positive constant K such that*

$$\int_a^x |\varphi(x)|^2\,\mathrm{d}x \int_x^b |\psi(x)|^2\,\mathrm{d}x \leqslant K \quad (x \in (a,b)). \tag{5.15}$$

Moreover, for fixed $f \in L^2(a,b)$,

$$\varphi(x)\int_x^b \psi(x)f(x)\,\mathrm{d}x \in L^2(a,b)$$

and

$$\psi(x)\int_a^x \varphi(x)f(x)\,\mathrm{d}x \in L^2(a,b)$$

if and only if (5.15) *holds for some positive constant* K.

Remark 5.1. This theorem was established by Chisholm and Everitt [3] in 1971. Results were extended to the general case of conjugate indices p and q $(p,q>1)$ in [4] in 1999. It recently came to our attention that this general result is contained in a result due to Muckenhoupt [12] in 1972. Moreover, the contributions by Talenti [17] and Tomaselli [18], both in 1969, also contain results equivalent to Theorem 5.2.

Theorem 5.3. *Let*

$$S_{k,1}[0,\infty) := \{f:[0,\infty)\to\mathbb{C}\mid f^{(j)}\in AC_{\mathrm{loc}}[0,\infty), f^{(j)}(0)=0\ (j=0,1,\dots,k-1);\\ f^{(k)}\in L_0^2(0,\infty)\}. \tag{5.16}$$

Then

$$V_{k,-k} = S_{k,1}[0,\infty), \tag{5.17}$$

where $V_{k,-k}$ *is defined in* (5.9)

Remark 5.2. The subscript 1 in $S_{k,1}[0,\infty)$ will be made clearer in Section 8. We note that functions in $V_{k,-k}$ are defined on the interval $(0,\infty)$ whereas functions in $S_{k,1}[0,\infty)$ have domain $[0,\infty)$. In the course of the proof of Theorem 5.3, we will see that the limits

$$\lim_{x\to 0^+} f^{(j)}(x)\quad (f\in V_{k,-k}; j=0,1,\dots,k-1)$$

exist and are finite so, in this case, we define $f^{(j)}(0) := \lim_{x\to 0^+} f^{(j)}(x)$. Using standard arguments, we then show that $f^{(j)}(0)=0$ for $j=0,1,\dots,k-1$.

Proof. $\underline{S_{k,1}[0,\infty)\subset V_{k,-k}}$: Let $f\in S_{k,1}[0,\infty)$. We need to show that

$$f^{(j)}\in L^2_{j-k}(0,\infty)\quad (j=0,1,\dots,k). \tag{5.18}$$

By definition of $S_{k,1}[0,\infty)$, the claim in (5.18) is true for $j=k$. Suppose, using mathematical induction, it is the case that

$$f^{(r)}\in L^2_{r-k}(0,\infty)\quad (r=k,k-1,\dots,j+1), \tag{5.19}$$

where $j\in\{0,1,\dots,k-1\}$; we need to prove

$$f^{(j)}\in L^2_{j-k}(0,\infty). \tag{5.20}$$

Before proving (5.20), we first show

$$f^{(r)} \in L_0^2(0,\infty) \quad (r = 0, 1, \ldots, k). \tag{5.21}$$

Again, from the definition of $S_{k,1}[0,\infty)$, we see that

$$f^{(k)} \in L_0^2(0,\infty),$$

so (5.21) is true for $r = k$. Suppose

$$f^{(r)} \in L_0^2(0,\infty) \quad (r = k, k-1, \ldots, j+1) \tag{5.22}$$

for some $j \in \{0, 1, \ldots, k-1\}$. Since $f^{(j)}(0) = 0$ and $f^{(j)} \in AC_{\text{loc}}[0,\infty)$, we see that

$$f^{(j)}(x)\mathrm{e}^{-x/2} = \mathrm{e}^{-x/2} \int_0^x \mathrm{e}^{t/2}(\mathrm{e}^{-t/2} f^{(j+1)}(t))\,\mathrm{d}t. \tag{5.23}$$

By assumption, $f^{(j+1)} \in L_0^2(0,\infty)$ or, equivalently, $\mathrm{e}^{-t/2} f^{(j+1)} \in L^2(0,\infty)$. We apply Theorem 5.2 with $a = 0$, $b = \infty$, $\varphi(x) = \mathrm{e}^{x/2}$, and $\psi(x) = \mathrm{e}^{-x/2}$ to see that $\mathrm{e}^{-x/2} f^{(j)}(x) \in L^2(0,\infty)$ or, equivalently, $f^{(j)} \in L_0^2(0,\infty)$. This completes the induction and establishes (5.21). To prove (5.20), notice from our induction hypothesis in (5.19) that $f^{(j+1)} \in L_{j+1-k}^2(0,\infty)$ or, equivalently,

$$x^{(j+1-k)/2}\mathrm{e}^{-x/2} f^{(j+1)} \in L^2(0,\infty).$$

In particular,

$$\frac{f^{(j+1)}(x)}{x^{(k-j-1)/2}} \in L^2(0,1).$$

Since $f^{(j)}(0) = 0$, we see that, for $0 < x < 1$,

$$\frac{f^{(j)}(x)}{x^{(k-j)/2}} = \frac{1}{x^{(k-j)/2}} \int_0^x t^{(k-j-1)/2} \frac{f^{(j+1)}(t)}{t^{(k-j-1)/2}}\,\mathrm{d}t. \tag{5.24}$$

Again, we apply Theorem 5.2 with $a = 0$, $b = 1$, $\varphi(x) = x^{(k-j-1)/2}$, and $\psi(x) = 1/x^{(k-j)/2}$; since

$$\int_0^x t^{k-j-1}\,\mathrm{d}t \int_x^1 \frac{1}{t^{k-j}}\,\mathrm{d}t$$

$$= \begin{cases} -x\ln x & \text{if } j = k-1 \\ \dfrac{1}{(k-j)(j-k+1)}(x^{k-j} - x) & \text{if } j < k-1 \end{cases} \quad (0 < x < 1) \;\leqslant K$$

for some $0 < K < \infty$, we see that $f^{(j)}(x)/x^{(k-j)/2} \in L^2(0,1)$ or, equivalently,

$$f^{(j)} \in L^2\left((0,1); \frac{\mathrm{e}^{-x}}{x^{k-j}}\right). \tag{5.25}$$

For $x \geqslant 1$, $x^{k-j} \geqslant 1$ so that

$$\int_1^\infty |f^{(j)}(x)|^2 \frac{\mathrm{e}^{-x}}{x^{k-j}}\,\mathrm{d}x \leqslant \int_1^\infty |f^{(j)}(x)|^2 \mathrm{e}^{-x}\,\mathrm{d}x < \infty$$

from (5.21). Consequently, we see that

$$f^{(j)} \in L^2\left((1,\infty); \frac{\mathrm{e}^{-x}}{x^{k-j}}\right). \tag{5.26}$$

Combining (5.25) and (5.26), we obtain

$$f^{(j)} \in L^2\left((0,\infty); \frac{\mathrm{e}^{-x}}{x^{k-j}}\right)$$
$$= L^2_{j-k}(0,\infty)$$

and this completes the induction on (5.18). Hence $S_{k,1}[0,\infty) \subset V_{k,-k}$, as required.

$V_{k,-k} \subset S_{k,1}[0,\infty)$: Let $f \in V_{k,-k}$. In particular,

$$f^{(k)} \in L^2_0(0,\infty); \text{ that is, } \int_0^\infty |f^{(k)}(x)|^2 \mathrm{e}^{-x}\,\mathrm{d}x < \infty. \tag{5.27}$$

Hence, $f^{(k)} \in L^2(0,1)$ and thus, using a standard measure theory argument, $f^{(j)} \in AC[0,1]$ for $j = 0,1,\ldots,k-1$. By assumption, $f^{(j)} \in AC_{\mathrm{loc}}(0,\infty)$ $(j = 0,1,\ldots,k-1)$, so it follows that

$$f^{(j)} \in AC_{\mathrm{loc}}[0,\infty) \quad (j = 0,1,\ldots,k-1). \tag{5.28}$$

By definition of $V_{k,-k}$, we see that

$$\int_0^\infty |f^{(j)}(x)|^2 \frac{\mathrm{e}^{-x}}{x^{k-j}}\,\mathrm{d}x < \infty \quad (j = 0,1,\ldots,k),$$

in particular,

$$\int_0^1 \frac{|f^{(j)}(x)|^2}{x^{k-j}}\,\mathrm{d}x < \infty \quad (j = 0,1,\ldots,k).$$

If, for some $j \in \{0,1,\ldots,k-1\}$, $f^{(j)}(0) \neq 0$, then there exists $\varepsilon \in (0,1)$ and $c > 0$ such that

$$|f^{(j)}(x)| \geqslant c \quad (x \in [0,\varepsilon]).$$

But then

$$\infty > \int_0^1 \frac{|f^{(j)}(x)|^2}{x^{k-j}}\,\mathrm{d}x \geqslant \int_0^\varepsilon \frac{c^2}{x^{k-j}}\,\mathrm{d}x = \infty,$$

a contradiction. Hence

$$f^{(j)}(0) = 0 \quad (j = 0,1,\ldots,k-1). \tag{5.29}$$

Combining (5.27)–(5.29), we see that $V_{k,-k} \subset S_{k,1}[0,\infty)$, completing the proof of the theorem. □

We note the following result, that will be used later in this paper, whose proof follows along the same lines as given above.

Corollary 5.1. *For each $n \in \mathbb{N}$ and $f \in V_{n,-k}$, where $V_{n,-k}$ is defined in (5.7), we have*

$$f^{(j)}(0) = 0 \quad (j = 0,1,\ldots,n-1).$$

A key result in establishing the self-adjointness of the operator $T_{k,1}$ in Section 9 is the following theorem.

Theorem 5.4. *Let $f, g \in \mathscr{D}(B_{k,-k}) = V_{k+2,-k}$, where $V_{k+2,-k}$ is defined in* (5.13). *Then*

(a) $x^{1/2} f^{(k+1)} \in L_0^2(0,\infty)$; *that is*, $\int_0^\infty |f^{(k+1)}(x)|^2 x\mathrm{e}^{-x}\,\mathrm{d}x < \infty$,
(b) $\lim_{x\to\infty} x\mathrm{e}^{-x} f^{(k+1)}(x)\bar{g}^{(k)}(x) = 0$.

Proof. We first prove part (a). Let $f \in V_{k+2,-k} \subset V_{k,-k}$ and, without loss of generality, assume f is real-valued. Since $B_{k,-k}f \in V_{k,-k}$, we see that

$$f^{(k)}, (\ell_{-k}[f])^{(k)} \in L_0^2(0,\infty).$$

Hence, $(\ell_{-k}[f])^{(k)} f^{(k)} \in L_0^1(0,\infty)$ satisfies

$$\lim_{x\to\infty} \int_0^x (\ell_{-k}[f](t))^{(k)} f^{(k)}(t)\mathrm{e}^{-t}\,\mathrm{d}t = \int_0^\infty (\ell_{-k}[f](t))^{(k)} f^{(k)}(t)\mathrm{e}^{-t}\,\mathrm{d}t < \infty.$$

Since

$$\begin{aligned}(\ell_{-k}[f](t))^{(k)} &= -tf^{(k+2)}(t) + (t-1)f^{(k+1)}(t) + (k+r)f^{(k)}(t)\\ &= \frac{1}{\mathrm{e}^{-t}}[-(t\mathrm{e}^{-t}f^{(k+1)}(t))' + (k+r)f^{(k)}(t)\mathrm{e}^{-t}],\end{aligned} \tag{5.30}$$

we see that, for $x > 0$,

$$\begin{aligned}&\int_0^x (\ell_{-k}[f](t))^{(k)} f^{(k)}(t)\mathrm{e}^{-t}\,\mathrm{d}t\\ &\quad= \int_0^x [-(t\mathrm{e}^{-t}f^{(k+1)}(t))' f^{(k)}(t) + (k+r)(f^{(k)}(t))^2\mathrm{e}^{-t}]\,\mathrm{d}t.\end{aligned}$$

Consequently, since $\int_0^\infty (f^{(k)}(t))^2\mathrm{e}^{-t}\,\mathrm{d}t < \infty$, we see that

$$\int_0^\infty (t\mathrm{e}^{-t}f^{(k+1)}(t))' f^{(k)}(t)\,\mathrm{d}t < \infty. \tag{5.31}$$

By integration by parts, we see that

$$\begin{aligned}&-\int_0^x (t\mathrm{e}^{-t}f^{(k+1)}(t))' f^{(k)}(t)\,\mathrm{d}t + x\mathrm{e}^{-x}f^{(k+1)}(x)f^{(k)}(x)\\ &\quad= \int_0^x (f^{(k+1)}(t))^2 t\mathrm{e}^{-t}\,\mathrm{d}t \quad (0 < x < \infty).\end{aligned}$$

Hence, from (5.31), we see that if

$$\int_0^\infty (f^{(k+1)}(t))^2 t\mathrm{e}^{-t}\,\mathrm{d}t = \infty,$$

then

$$\lim_{x\to\infty} x\mathrm{e}^{-x}f^{(k+1)}(x)f^{(k)}(x) = \infty. \tag{5.32}$$

Hence, there exists $x_0 > 0$ such that

$$f^{(k+1)}(x)f^{(k)}(x) \geqslant \frac{\mathrm{e}^x}{x} \quad (x \geqslant x_0).$$

Integrate this inequality over $[x_0,x]$ to obtain

$$\frac{(f^{(k)}(x))^2}{2}-\frac{(f^{(k)}(x_0))^2}{2}=\int_{x_0}^{x} f^{(k+1)}(t)f^{(k)}(t)\,\mathrm{d}t$$
$$\geqslant \int_{x_0}^{x}\frac{\mathrm{e}^t}{t}\,\mathrm{d}t. \tag{5.33}$$

Moreover, integration by parts yields

$$\int_{x_0}^{x}\frac{\mathrm{e}^t}{t}\,\mathrm{d}t=\frac{\mathrm{e}^x}{x}-\frac{\mathrm{e}^{x_0}}{x_0}+\int_{x_0}^{x}\frac{\mathrm{e}^t}{t^2}\,\mathrm{d}t$$
$$\geqslant \frac{\mathrm{e}^x}{x}-\frac{\mathrm{e}^{x_0}}{x_0}.$$

Hence (5.33) implies that

$$\frac{(f^{(k)}(x))^2}{2}\geqslant \frac{\mathrm{e}^x}{x}-\frac{\mathrm{e}^{x_0}}{x_0}+\frac{(f^{(k)}(x_0))^2}{2}=\frac{\mathrm{e}^x}{x}+c\quad (x\geqslant x_0).$$

Therefore

$$\infty>\int_{x_0}^{\infty}\frac{(f^{(k)}(t))^2}{2}\mathrm{e}^{-t}\,\mathrm{d}t\geqslant\int_{x_0}^{\infty}\left(\frac{\mathrm{e}^t}{t}+c\right)\mathrm{e}^{-t}\,\mathrm{d}t$$
$$=\int_{x_0}^{\infty}\frac{1}{t}\,\mathrm{d}t+c\int_{x_0}^{\infty}\mathrm{e}^{-t}\,\mathrm{d}t$$
$$=\infty,$$

a contradiction. Hence, we must have

$$\int_0^{\infty}(f^{(k+1)}(t))^2 t\mathrm{e}^{-t}\,\mathrm{d}t<\infty,$$

which proves part (a) of the theorem.

To prove (b), we again assume that $f,g\in V_{k+2,-k}$ are both real-valued. Since

$$\int_0^x f^{(k+1)}(t)g^{(k+1)}(t)t\mathrm{e}^{-t}\,\mathrm{d}t=x\mathrm{e}^{-x}f^{(k+1)}(x)g^{(k)}(x)-\int_0^x(t\mathrm{e}^{-t}f^{(k+1)}(t))'g^{(k)}(t)\,\mathrm{d}t\quad (x>0),$$

we see, from part (a) and the definition of $V_{k+2,-k}$, that

$$\lim_{x\to\infty} x\mathrm{e}^{-x}f^{(k+1)}(x)g^{(k)}(x):=2c$$

exists and is finite. If this limit is not zero, we may assume that $c>0$. Hence, there exists $x_0^*>0$ such that

$$x\mathrm{e}^{-x}f^{(k+1)}(x)g^{(k)}(x)\geqslant c$$

with

$$f^{(k+1)}(x)>0,\quad g^{(k)}(x)>0\quad (x\geqslant x_0^*), \tag{5.34}$$

so that

$$xe^{-x}f^{(k+1)}(x)|g^{(k+1)}(x)| \geqslant c\,\frac{|g^{(k+1)}(x)|}{g^{(k)}(x)} \quad (x \geqslant x_0^*).$$

Consequently,

$$\begin{aligned}\int_{x_0^*}^{x} te^{-t}f^{(k+1)}(t)|g^{(k+1)}(t)|\,dt &\geqslant c\int_{x_0^*}^{x}\frac{|g^{(k+1)}(t)|}{g^{(k)}(t)}\,dt\\ &\geqslant c\left|\int_{x_0^*}^{x}\frac{g^{(k+1)}(t)}{g^{(k)}(t)}\,dt\right|\\ &= c|\ln(g^{(k)}(t))|_{x_0^*}^{x}\\ &\geqslant c|\ln(g^{(k)}(x))| - c_1 \quad (x \geqslant x_0^*).\end{aligned}$$

From part (a), we see that

$$\lim_{x\to\infty}\int_{x_0^*}^{x} te^{-t}f^{(k+1)}(t)|g^{(k+1)}(t)|\,dt < \infty,$$

so we must have

$$\limsup_{x\to\infty}|\ln(g^{(k)}(x))| < \infty. \tag{5.36}$$

It follows that there exists constants $M_1, M_2 > 0$ such that

$$M_1 < g^{(k)}(x) < M_2 \quad (x \geqslant x_0^*). \tag{5.37}$$

For if $g^{(k)}$ is unbounded on $[x_0^*, \infty)$, there exists a sequence $\{x_n\} \subset [x_0^*, \infty)$ such that $x_n \to \infty$ and $g^{(k)}(x_n) \to \infty$, contradicting (5.36). Furthermore, if $g^{(k)}$ is not bounded away from zero, then there exists a sequence $\{y_n\} \subset [x_0^*, \infty)$ such that $g^{(k)}(y_n) \to 0$ and, thus, $\ln(g^{(k)}(y_n)) \to -\infty$; however, this also contradicts (5.36). From (5.34) and (5.37), we see that

$$xe^{-x}f^{(k+1)}(x) \geqslant \frac{c}{M_2} := \tilde{c},$$

so that

$$(f^{(k+1)}(x))^2 xe^{-x} \geqslant (\tilde{c})^2\,\frac{e^x}{x} \quad (x \geqslant x_0^*).$$

Integrating on $[x_0^*, \infty)$ yields

$$\int_{x_0^*}^{\infty}(f^{(k+1)}(t))^2 te^{-t}\,dt \geqslant (\tilde{c})^2\int_{x_0^*}^{\infty}\frac{e^t}{t}\,dt = \infty,$$

contradicting part (a). Hence, we must have

$$\lim_{x\to\infty} xe^{-x}f^{(k+1)}(x)g^{(k)}(x) = 0$$

and this completes the proof of the theorem. □

Remark 5.3. Property (a) of Theorem 5.4 says that $\ell_{-k}[\,\cdot\,]$ is *Dirichlet* at $x = \infty$ on $V_{k+2,-k}$ in the kth left-definite space $H_{k,-k}$, while property (b) shows that $\ell_{-k}[\,\cdot\,]$ is *strong limit-point* at $x = \infty$ on $V_{k+2,-k}$.

6. Sobolev orthogonality of the Laguerre polynomials

In [9], the authors show that, for each $k \in \mathbb{N}$, the *entire* sequence of Laguerre polynomials $\{L_n^{-k}\}_{n=0}^{\infty}$ is, remarkably, *orthonormal* with respect to the positive-definite inner product $(\cdot,\cdot)_k$, defined on $\mathscr{P}[0,\infty) \times \mathscr{P}[0,\infty)$ by

$$(p,q)_k := \sum_{m=0}^{k-1} \sum_{j=0}^{m} B_{m,j}(k)[p^{(m)}(0)\bar{q}^{(j)}(0) + p^{(j)}(0)\bar{q}^{(m)}(0)] + \int_0^{\infty} p^{(k)}(x)\bar{q}^{(k)}(x)\mathrm{e}^{-x}\,\mathrm{d}x$$

$$= \sum_{r=0}^{k-1} \langle w_{k,r}, p^{(r)}\bar{q}^{(r)} \rangle + \int_0^{\infty} p^{(k)}(x)\bar{q}^{(k)}(x)\mathrm{e}^{-x}\,\mathrm{d}x \quad (p,q \in \mathscr{P}[0,\infty)), \tag{6.1}$$

where the numbers $B_{m,j}(k)$ are given by

$$B_{m,j}(k) = \begin{cases} \sum_{p=0}^{j} (-1)^{m+j} \binom{k-1-p}{m-p} \binom{k-1-p}{j-p} & \text{if } 0 \leqslant j < m \leqslant k-1, \\ \frac{1}{2} \sum_{p=0}^{m} \binom{k-1-p}{m-p}^2 & \text{if } 0 \leqslant j = m \leqslant k-1 \end{cases} \tag{6.2}$$

and where $w_{k,r}$ is the linear functional defined by

$$w_{k,r} = \binom{k}{r} \sum_{j=0}^{k-r-1} \binom{k-r-1}{j} \delta^{(j)}.$$

Here $\delta^{(j)}$ is the classic Dirac delta distribution defined, in this case, on the polynomial space $\mathscr{P}[0,\infty)$ through the standard formula

$$\langle \delta^{(j)}, p \rangle = (-1)^j p^{(j)}(0) \quad (p \in \mathscr{P}[0,\infty)).$$

That is to say,

$$(L_n^{-k}, L_m^{-k})_k = \delta_{n,m} \quad (n,m \in \mathbb{N}_0). \tag{6.3}$$

We note that it is precisely the identity in (2.8) that led the authors in [9] to constructing this inner product $(\cdot,\cdot)_k$.

For example, the Laguerre polynomials $\{L_n^{-1}\}_{n=0}^{\infty}$ are orthonormal with respect to

$$(p,q)_1 = p(0)\bar{q}(0) + \int_0^{\infty} p'(x)\bar{q}'(x)\mathrm{e}^{-x}\,\mathrm{d}x \quad (p,q \in \mathscr{P}[0,\infty)), \tag{6.4}$$

while $\{L_n^{-3}\}_{n=0}^{\infty}$ are orthonormal with respect to

$$\begin{aligned}(p,q)_3 = {} & p(0)\bar{q}(0) - 2[p'(0)\bar{q}(0) + p(0)\bar{q}'(0)] + 5p'(0)\bar{q}'(0) \\ & +[p''(0)\bar{q}(0) + p(0)\bar{q}''(0)] - 3[p''(0)\bar{q}'(0) + p'(0)\bar{q}''(0)] \\ & +3p''(0)\bar{q}''(0) + \int_0^{\infty} p'''(x)\bar{q}'''(x)\mathrm{e}^{-x}\,\mathrm{d}x \quad (p,q \in \mathscr{P}[0,\infty)).\end{aligned} \tag{6.5}$$

As discussed in Section 2, we remark that these Laguerre polynomials $\{L_n^{-3}\}$ of degree $\geqslant 3$ are orthogonal as well in the Hilbert space $L^2_{-3}(0,\infty)$; indeed, it is the case that

$$(L_n^{-3}, L_m^{-3})_{L^2(-3)} = \frac{1}{n(n-1)(n-2)}\delta_{n,m} \quad (n,m \geqslant 3),$$

where $(\cdot,\cdot)_{L^2(-3)}$ is the inner product defined in (2.6).

The discovery of the orthonormality of $\{L_n^{-1}\}_{n=0}^{\infty}$ with respect to the inner product $(\cdot,\cdot)_1$ in (6.4) was first reported in the paper [10] by Kwon and Littlejohn. Subsequently, in [9], the authors extended this result and determined explicitly the inner product $(\cdot,\cdot)_k$, given in (6.1), for each $k \in \mathbb{N}$.

For $k \in \mathbb{N}$, we define the function space $S_k[0,\infty)$ to be

$$S_k[0,\infty) := \{f : [0,\infty) \to \mathbb{C} \mid f, f', \ldots, f^{(k-1)} \in AC_{\mathrm{loc}}[0,\infty);\ f^{(k)} \in L^2((0,\infty); \mathrm{e}^{-x})\}. \tag{6.6}$$

Observe that $\mathscr{P}[0,\infty) \subset S_k[0,\infty)$. Furthermore, notice that, for $f,g \in S_k[0,\infty)$, $(f,g)_k$ is well-defined, where $(\cdot,\cdot)_k$ is given in (6.1). However, even though it is clear that

$$(\cdot,\cdot)_k : S_k[0,\infty) \times S_k[0,\infty) \to \mathbb{C}$$

is a *bilinear form*, it is not immediately obvious, for large values of $k \in \mathbb{N}$, that it is an *inner product* on $S_k[0,\infty) \times S_k[0,\infty)$. Indeed, the authors in [9] only showed that $(\cdot,\cdot)_k$ is an inner product on the proper subspace $\mathscr{P}[0,\infty) \times \mathscr{P}[0,\infty)$ of $S_k[0,\infty) \times S_k[0,\infty)$.

In fact, it is not difficult to see that $(\cdot,\cdot)_1$, defined in (6.4), is an inner product on $S_1[0,\infty) \times S_1[0,\infty)$. As for $(\cdot,\cdot)_3$, given in (6.5), a calculation shows that

$$(f,f)_3 = |f(0) - 2f'(0) + f''(0)|^2 + |f'(0) - f''(0)|^2 + |f''(0)|^2 + \int_0^\infty |f'''(x)|^2 \mathrm{e}^{-x}\,\mathrm{d}x \quad (f \in S_3[0,\infty)),$$

from which it follows that $(\cdot,\cdot)_3$ is an inner product on $S_3[0,\infty) \times S_3[0,\infty)$. In general, we have the following result which readily shows that $(\cdot,\cdot)_k$ is an inner product on $S_k[0,\infty) \times S_k[0,\infty)$.

Lemma 6.1. *Let $k \in \mathbb{N}$. Then, for $f \in S_k[0,\infty)$,*

$$(f,f)_k = \sum_{r=0}^{k-1} \left| \sum_{j=r}^{k-1} \binom{k-r-1}{j-r} (-1)^{j-r} f^{(j)}(0) \right|^2 + \int_0^\infty |f^{(k)}(x)|^2 \mathrm{e}^{-x}\,\mathrm{d}x. \tag{6.7}$$

Proof. Expanding

$$\sum_{r=0}^{k-1} \left| \sum_{j=r}^{k-1} \binom{k-r-1}{j-r} (-1)^{j-r} f^{(j)}(0) \right|^2,$$

we obtain

$$\left(f(0)-\binom{k-1}{1}f'(0)+\binom{k-1}{2}f''(0)-\cdots+(-1)^{k-1}f^{(k-1)}(0)\right)$$
$$\times\left(\bar f(0)-\binom{k-1}{1}\bar f'(0)+\binom{k-1}{2}\bar f''(0)-\cdots+(-1)^{k-1}\bar f^{(k-1)}(0)\right)$$
$$+\left(f'(0)-\binom{k-2}{1}f''(0)+\binom{k-2}{2}f'''(0)-\cdots+(-1)^{k-2}f^{(k-1)}(0)\right)$$
$$\times\left(\bar f'(0)-\binom{k-2}{1}\bar f''(0)+\binom{k-2}{2}\bar f'''(0)-\cdots+(-1)^{k-2}\bar f^{(k-1)}(0)\right)$$
$$+\ldots$$
$$+\left(f^{(j)}(0)-\binom{k-j-1}{1}f^{(j+1)}(0)+\binom{k-j-1}{2}f^{(j+2)}(0)-\cdots\right.$$
$$\left.+(-1)^{k-1-j}f^{(k-1)}(0)\right)$$
$$\times\left(\bar f^{(j)}(0)-\binom{k-j-1}{1}\bar f^{(j+1)}(0)+\binom{k-j-1}{2}\bar f^{(j+2)}(0)-\cdots\right.$$
$$\left.+(-1)^{k-1-j}\bar f^{(j-1)}(0)\right)$$
$$+\ldots$$
$$+f^{(k-1)}(0)\bar f^{(k-1)}(0).$$

It is straightforward to check that the coefficient of $[f^{(m)}(0)\bar f^{(j)}(0)+f^{(j)}(0)\bar f^{(m)}(0)]$ in this above expression is given by

$$c_{m,j}(k):=\begin{cases}\displaystyle\sum_{p=0}^{m}(-1)^{m+j}\binom{k-1-p}{m-p}\binom{k-1-p}{j-p} & \text{if } 0\leqslant j<m\leqslant k-1,\\[2ex] \displaystyle\frac{1}{2}\sum_{p=0}^{m}\binom{k-1-p}{m-p}^{2} & \text{if } 0\leqslant j=m\leqslant k-1.\end{cases}$$

But this coefficient is exactly $B_{m,j}(k)$, defined in (6.2). By comparing (6.1) and (6.7), we see that the proof of this lemma is now complete. □

Let

$$W_k[0,\infty):=(S_k[0,\infty),(\cdot,\cdot)_k) \tag{6.8}$$

be this inner product space; for each $k \in \mathbb{N}$, we write

$$\|f\|_k := (f,f)_k^{1/2} \quad (f \in S_k[0,\infty))$$

for the norm $\|\cdot\|_k$ obtained from $(\cdot,\cdot)_k$.

We are now in position to prove the following theorem.

Theorem 6.1. *For each $k \in \mathbb{N}$, $W_k[0,\infty)$ is a Hilbert space.*

Proof. Suppose $\{f_n\}_{n=1}^{\infty} \subset W_k[0,\infty)$ is a Cauchy sequence. From (6.7), we see that

$$\|f_n - f_m\|_k^2 = \sum_{r=0}^{k-2} \left| \sum_{j=r}^{k-1} \binom{k-r-1}{j-r} (-1)^{j-r} (f_n^{(j)}(0) - f_m^{(j)}(0)) \right|^2 + |f_n^{(k-1)}(0) - f_m^{(k-1)}(0)|^2 + \int_0^\infty |f_n^{(k)}(x) - f_m^{(k)}(x)|^2 \mathrm{e}^{-x}\,\mathrm{d}x, \tag{6.9}$$

from which we see that

$$\|f_n - f_m\|_k^2 \geqslant |f_n^{(k-1)}(0) - f_m^{(k-1)}(0)|^2 \tag{6.10}$$

and

$$\|f_n - f_m\|_k^2 \geqslant \int_0^\infty |f_n^{(k)}(x) - f_m^{(k)}(x)|^2 \mathrm{e}^{-x}\,\mathrm{d}x. \tag{6.11}$$

From the completeness of $\mathbb{C}$ and $L^2((0,\infty);\mathrm{e}^{-x})$, we see that there exists $A_{k-1} \in \mathbb{C}$ and $g \in L^2((0,\infty);\mathrm{e}^{-x})$ such that

$$f_n^{(k-1)}(0) \to A_{k-1} \text{ in } \mathbb{C}$$

and

$$f_n^{(k)} \to g \text{ in } L^2((0,\infty);\mathrm{e}^{-x}).$$

It follows then that

$$g \in L^1_{\mathrm{loc}}(0,\infty).$$

Returning to (6.9), we see that

$$\|f_n - f_m\|_k^2 = \sum_{r=0}^{k-3} \left| \sum_{j=r}^{k-1} \binom{k-r-1}{j-r} (-1)^{j-r} (f_n^{(j)}(0) - f_m^{(j)}(0)) \right|^2 + |(f_n^{(k-2)}(0) - f_m^{(k-2)}(0)) - (f_n^{(k-1)}(0) - f_m^{(k-1)}(0))|^2 + |f_n^{(k-1)}(0) - f_m^{(k-1)}(0)|^2 + \int_0^\infty |f_n^{(k)}(x) - f_m^{(k)}(x)|^2 \mathrm{e}^{-x}\,\mathrm{d}x.$$

From this it follows that $\{f_n^{(k-2)}(0)\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{C}$ and hence there exists $A_{k-2} \in \mathbb{C}$ such that

$$f_n^{(k-2)}(0) \to A_{k-2} \text{ in } \mathbb{C}.$$

Continuing, by induction, we see that, for $j = 0, 1, \ldots, k-1$, there exists $A_j \in \mathbb{C}$ such that

$$f_n^{(j)}(0) \to A_j \text{ in } \mathbb{C} \quad (j = 0, 1, \ldots, k-1). \tag{6.12}$$

Define $f : [0, \infty) \to \mathbb{C}$ by

$$f(x) = \sum_{j=0}^{k-1} \frac{A_j x^j}{j!} + \int_0^x \int_0^{t_1} \cdots \int_0^{t_{k-1}} g(u)\, \mathrm{d}u\, \mathrm{d}t_{k-1} \ldots \mathrm{d}t_1. \tag{6.13}$$

Then f satisfies the following properties:

(i) $f, f', \ldots, f^{(k-1)} \in AC_{\mathrm{loc}}[0, \infty)$;
(ii) $f^{(j)}(0) = A_j$ $(j = 0, 1, 2, \ldots, k-1)$;
(iii) $f^{(k)}(x) = g(x)$ for a.e. $x > 0$ and $f^{(k)} = g \in L^2((0, \infty); \mathrm{e}^{-x})$.

Hence $f \in S_k[0, \infty)$ and

$$\begin{aligned} \|f_n - f\|_k^2 = & \sum_{r=0}^{k-1} \left| \sum_{j=r}^{k-1} \binom{k-r-1}{j-r} (-1)^{j-r} (f_n^{(j)}(0) - f^{(j)}(0)) \right|^2 \\ & + \int_0^\infty |f_n^{(k)}(x) - f^{(k)}(x)|^2 \mathrm{e}^{-x}\, \mathrm{d}x \\ & \to 0 \text{ as } n \to \infty. \end{aligned}$$

This completes the proof of the theorem. □

7. The completeness of the Laguerre polynomials $\{L_n^{-k}\}_{n=0}^\infty$ in $W_k[0, \infty)$

We now prove

Theorem 7.1. *$\mathscr{P}[0, \infty)$ is dense in the Hilbert space $W_k[0, \infty)$ for each $k \in \mathbb{N}$. Equivalently, the Laguerre polynomials $\{L_n^{-k}\}_{n=0}^\infty$ form a complete orthonormal set in $W_k[0, \infty)$.*

Proof. We remind the reader that the orthonormality of $\{L_n^{-k}\}_{n=0}^\infty$ (see (6.3) is established in [9]. Let $f \in W_k[0, \infty)$ and let $\varepsilon > 0$. Since $\mathscr{P}[0, \infty)$ is dense in $L^2((0, \infty); \mathrm{e}^{-x})$ and $f^{(k)} \in L^2((0, \infty); \mathrm{e}^{-x})$, there exists $q \in \mathscr{P}[0, \infty)$ such that

$$\|f^{(k)} - q\|_{L^2(0)}^2 = \int_0^\infty |f^{(k)}(x) - q(x)|^2 \mathrm{e}^{-x}\, \mathrm{d}x < \varepsilon^2, \tag{7.1}$$

where $\|\cdot\|_{L^2(0)}$ is the norm defined in (2.3). Define $p \in \mathscr{P}[0, \infty)$ by

$$p(x) = \sum_{j=0}^{k-1} \frac{f^{(j)}(0) x^j}{j!} + \int_0^x \int_0^{t_1} \cdots \int_0^{t_{k-1}} q(u)\, \mathrm{d}u\, \mathrm{d}t_{k-1} \ldots \mathrm{d}t_1. \tag{7.2}$$

Note that

$$p(0)=f(0),\ \ p'(0)=f'(0),\dots,p^{(k-1)}(0)=f^{(k-1)}(0) \tag{7.3}$$

and

$$p^{(k)}(x)=q(x)\quad (x\in[0,\infty)). \tag{7.4}$$

From identity (6.7), together with the properties in (7.1), (7.3), and (7.4), we see that

$$\begin{aligned}\|f-p\|_k^2 &= \sum_{r=0}^{k-1}\left|\sum_{j=r}^{k-1}\binom{k-r-1}{j-r}(-1)^{j-r}(f^{(j)}(0)-p^{(j)}(0))\right|^2 \\ &\quad + \int_0^\infty |f^{(k)}(x)-p^{(k)}(x)|^2 \mathrm{e}^{-x}\,\mathrm{d}x \\ &= \int_0^\infty |f^{(k)}(x)-p^{(k)}(x)|^2 \mathrm{e}^{-x}\,\mathrm{d}x \\ &= \int_0^\infty |f^{(k)}(x)-q(x)|^2 \mathrm{e}^{-x}\,\mathrm{d}x < \varepsilon^2,\end{aligned}$$

i.e., $\|f-p\|_k<\varepsilon$. This completes the proof of this theorem. □

8. A fundamental decomposition and identification of two inner product spaces

We begin with the following definition.

Definition 8.1. Let $S_{k,1}[0,\infty)$ be the function space defined in (5.16) so that

$$S_{k,1}[0,\infty)=\{f\in S_k[0,\infty)\mid f^{(j)}(0)=0\ \ (j=0,1,\dots,k-1)\} \tag{8.1}$$

and let $S_{k,2}[0,\infty)$ be the function space defined by

$$S_{k,2}[0,\infty):=\{f\in S_k[0,\infty)\mid f^{(k)}(x)=0\ \ (x\in[0,\infty))\}. \tag{8.2}$$

Define the inner product spaces $W_{k,1}[0,\infty)$ and $W_{k,2}[0,\infty)$ by

$$W_{k,1}[0,\infty):=(S_{k,1}[0,\infty),(\cdot,\cdot)_k) \tag{8.3}$$

and

$$W_{k,2}[0,\infty):=(S_{k,2}[0,\infty),(\cdot,\cdot)_k). \tag{8.4}$$

We remind the reader that, by Theorem 5.3 $S_{k,1}[0,\infty)=V_{k,-k}$, where $V_{k,-k}$ is the kth left-definite vector space, defined in (5.9), associated with the pair $(L^2_{-k}(0,\infty),A_{-k})$.

Remark 8.1. It follows from (2.8) that $\{L_n^{-k}\}_{n=k}^{\infty}\subset S_{k,1}[0,\infty)$. It is precisely this remarkable property (2.8) of the Laguerre polynomials that prompts our definition of $S_{k,1}[0,\infty)$.

Theorem 8.1. *$W_{k,1}[0,\infty)$ and $W_{k,2}[0,\infty)$ are closed, orthogonal subspaces of $W_k[0,\infty)$, where $W_k[0,\infty)$ is defined in (6.8), with*

$$W_k[0,\infty) = W_{k,1}[0,\infty) \oplus W_{k,2}[0,\infty). \tag{8.5}$$

Furthermore, the Laguerre polynomials $\{L_n^{-k}\}_{n=k}^{\infty}$ and $\{L_n^{-k}\}_{n=0}^{k-1}$ are complete orthonormal sequences in $W_{k,1}[0,\infty)$ and $W_{k,2}[0,\infty)$, respectively.

Proof. Since $W_{k,2}[0,\infty)$ is k-dimensional, it is a closed subspace of $W_k[0,\infty)$ and it is straightforward to check that

$$W_{k,1}[0,\infty) \subset W_{k,2}^{\perp}[0,\infty),$$

where $W_{k,2}^{\perp}[0,\infty)$ is the orthogonal complement of $W_{k,2}[0,\infty)$. Let $f \in W_k[0,\infty)$. Define

$$f_1(x) := f(x) - \sum_{j=0}^{k-1} f^{(j)}(0)\frac{x^j}{j!},$$

$$f_2(x) := \sum_{j=0}^{k-1} f^{(j)}(0)\frac{x^j}{j!}.$$

A calculation shows that $f_1^{(j)}(0) = 0$ for $j = 0, 1, \ldots, k-1$ so that $f_1 \in W_{k,1}[0,\infty)$; similarly, it is clear that $f_2 \in W_{k,2}[0,\infty)$. Furthermore,

$$\begin{aligned}(f_1, f_2)_k &= \sum_{m=0}^{k-1}\sum_{j=0}^{m} B_{m,j}[f_1^{(m)}(0)\bar{f}_2^{(j)}(0) + f_1^{(j)}(0)\bar{f}_2^{(m)}(0)] + \int_0^{\infty} f_1^{(k)}(x)\bar{f}_2^{(k)}(x)\mathrm{e}^{-x}\,\mathrm{d}x \\ &= 0,\end{aligned}$$

since $f_1^{(j)}(0) = 0$ for $j = 0, 1, \ldots, k-1$ and $f_2^{(k)}(x) \equiv 0$. This establishes (8.5). A proof that the Laguerre polynomials $\{L_n^{-k}\}_{n=k}^{\infty}$ form a complete orthonormal sequence in $W_{k,1}[0,\infty)$ is identical to that given in Theorem 7.1; the proof that $\{L_n^{-k}\}_{n=0}^{k-1}$ is complete in $W_{k,2}[0,\infty)$ is obvious since $W_{k,2}[0,\infty)$ is k-dimensional. □

Observe that, for $f, g \in S_{k,1}[0,\infty)$, we have

$$(f, g)_k = \int_0^{\infty} f^{(k)}(x)\bar{g}^{(k)}(x)\mathrm{e}^{-x}\,\mathrm{d}x, \tag{8.6}$$

indeed, this follows since $f^{(j)}(0) = 0$ for $j = 0, 1, \ldots, k-1$.

Interestingly, the two inner products $(\cdot,\cdot)_k$ and the kth left-definite inner product $(\cdot,\cdot)_{k,-k}$ are equivalent on $V_{k,-k} = S_{k,1}[0,\infty)$, as we show.

Theorem 8.2. *The two inner products $(\cdot,\cdot)_{k,-k}$ and $(\cdot,\cdot)_k$, defined in (5.10) and (6.1), respectively, are equivalent on $V_{k,-k} = S_{k,1}[0,\infty)$.*

Proof. Let $f \in V_{k,-k} = S_{k,1}[0,\infty)$. Since $b_j(k,r) \geqslant 0$ $(j = 0,1,\ldots,k)$ and $b_k(k,r) = 1$ (recall the definition of these numbers in (5.4)), we see from (8.6) that

$$\begin{aligned}(f,f)_k &= \int_0^\infty \left|f^{(k)}(x)\right|^2 \mathrm{e}^{-x}\,\mathrm{d}x \\ &\leqslant \sum_{j=0}^{k} b_j(k,r)\int_0^\infty |f^{(j)}(x)|^2 x^{j-k}\mathrm{e}^{-x}\,\mathrm{d}x \\ &= (f,f)_{k,-k}. \end{aligned} \tag{8.7}$$

Recall (see Theorem 5.1) that the Laguerre polynomials $\{L_n^{-k}\}_{n=k}^{\infty}$ are a complete orthogonal set in $H_{k,-k}$ with

$$(L_j^{-k}, L_m^{-k})_{k,-k} = \frac{(j+r)^k (j-k)!}{j!}\,\delta_{j,m} \quad (j,m \geqslant k).$$

Consequently,

$$\left\{\frac{(j!)^{1/2}}{(j+r)^{k/2}((j-k)!)^{1/2}} L_j^{-k}\right\}_{j=k}^{\infty} \tag{8.8}$$

is a complete orthonormal set in $H_{k,-k}$. Furthermore, since $(j+r)^k(j-k)!/j! \geqslant 1$ for $j \geqslant k$ and

$$\lim_{j\to\infty} \frac{(j+r)^k (j-k)!}{j!} = 1,$$

there exists $L = L(k,r)$ satisfying

$$0 < L < 1 \quad \text{and} \quad \frac{(j+r)^k (j-k)!}{j!} < \frac{1}{L} \quad (j \geqslant k). \tag{8.9}$$

Let $\{\xi_j\}_{j=k}^{\infty}$ be the Fourier coefficients of f in $H_{k,-k}$ relative to the orthonormal basis given in (8.8); that is,

$$\xi_j = \frac{(j!)^{1/2}}{(j+r)^{k/2}((j-k)!)^{1/2}} (f, L_j^{-k})_{k,-k} \quad (j \geqslant k). \tag{8.10}$$

Then, from the classical theory, as $n \to \infty$, we have

$$f_n := \sum_{j=k}^{n} \xi_j \frac{(j!)^{1/2}}{(j+r)^{k/2}((j-k)!)^{1/2}} L_j^{-k} \to f \quad \text{in } H_{k,-k}.$$

Observe, from (8.7), it is also the case that

$$f_n \to f \quad \text{in } W_{k,1}[0,\infty),$$

indeed,

$$\begin{aligned}\|f_n - f\|_k^2 &= (f_n - f, f_n - f)_k \\ &\leqslant (f_n - f, f_n - f)_{k,-k} \\ &= \|f_n - f\|_{k,-k}^2 \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Furthermore, from the orthonormality of $\{L_j^{-k}\}_{j=k}^{\infty}$ in $W_{k,1}[0,\infty)$, we see from (8.9) that

$$\begin{aligned}(f_n, f_n)_k &= \sum_{j=k}^{n} |\xi_j|^2 \frac{j!}{(j+r)^k (j-k)!} \\ &> L \sum_{j=k}^{n} |\xi_j|^2 \\ &= L(f_n, f_n)_{k,-k} \quad (n \geqslant k). \end{aligned} \tag{8.11}$$

Letting $n \to \infty$ in (8.11) yields

$$(f, f)_k \geqslant L(f, f)_{k,-k}. \tag{8.12}$$

Together, (8.7) and (8.12) complete the proof of the theorem. □

Remark 8.2. We note that the equivalence of the inner products $(\cdot,\cdot)_k$ and $(\cdot,\cdot)_{k,-k}$ on $V_{k,-k} = S_{k,1}[0,\infty)$ also follows from the inequality in (8.7), the completeness of the two inner product spaces $W_{k,1}[0,\infty) = (S_{k,1}[0,\infty), (\cdot,\cdot)_k)$ and $H_{k,-k} = (V_{k,-k}, (\cdot,\cdot)_{k,-k})$, and an application of the Open Mapping Theorem. Indeed, if $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are Banach spaces and there exists a positive constant c such that

$$\|x\|_1 \leqslant c\|x\|_2 \quad (x \in X),$$

then, from the Open Mapping Theorem, the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on X; see [8, Chapter 4, Problem 8, p. 291].

9. The self-adjoint Laguerre operator $T_{k,1}$

Definition 9.1. The operator $T_{k,1} : \mathscr{D}(T_{k,1}) \subset W_{k,1}[0,\infty) \to W_{k,1}[0,\infty)$ is given by

$$\begin{cases} T_{k,1} f := \ell_{-k}[f], \\ f \in \mathscr{D}(T_{k,1}) := V_{k+2,-k}, \end{cases} \tag{9.1}$$

where $\ell_{-k}[\cdot]$ is the Laguerre expression defined in (2.9) and where $V_{k+2,-k}$ is given in (5.13).

Since the kth left-definite operator $B_{k,-k}$, defined in (5.12), has the same form and domain as does $T_{k,1}$ and since $B_{k,-k} f \in V_{k,-k} = S_{k,1}[0,\infty)$ for $f \in V_{k+2,-k} \subset V_{k,-k}$, it is clear that $T_{k,1}$ does indeed map $V_{k+2,-k} \subset W_{k,1}[0,\infty)$ into $W_{k,1}[0,\infty)$.

We remind the reader that, from Theorem 5.3, $V_{k+2,-k}$, defined in (5.14), consists of precisely those functions $f : [0,\infty) \to \mathbb{C}$ satisfying

$$\begin{cases} \text{(i)}\ f^{(j)} \in AC_{\text{loc}}[0,\infty) \text{ for } j = 0,1,\ldots,k-1, \\ \text{(ii)}\ f^{(k+j)} \in AC_{\text{loc}}(0,\infty) \text{ for } j = 0,1, \\ \text{(iii)}\ f^{(j)}(0) = 0 \text{ for } j = 0,1,\ldots,k-1, \\ \text{(iv)}\ f^{(k+j)} \in L^2((0,\infty); x^j e^{-x}) \text{ for } j = 0,1,2. \end{cases} \tag{9.2}$$

Lemma 9.1. *The space* $V_{k+2,-k}$ *is dense in* $W_{k,1}[0,\infty)$; *that is to say*, $T_{k,1}$ *is a densely defined operator*.

Proof. Let $f \in W_{k,1}[0,\infty) = (S_{k,1}[0,\infty), (\cdot,\cdot)_k)$, where $S_{k,1}[0,\infty) = V_{k,-k}$ by Theorem 5.3, and let $\varepsilon > 0$. Since the kth left-definite operator $B_{k,-k}$, with domain $\mathscr{D}(B_{k,-k}) = V_{k+2,-k}$ is densely defined in $H_{k,-k} = (V_{k,-k}, (\cdot,\cdot)_{k,-k})$, there exists $g \in V_{k+2,-k}$ such that

$$\|f-g\|_{k,-k} < \varepsilon,$$

where

$$\|f-g\|_{k,-k} = \left(\sum_{j=0}^{k} b_j(k,r)\int_0^\infty |f^{(j)}(x)-g^{(j)}(x)|^2 x^{j-k}\mathrm{e}^{-x}\,\mathrm{d}x\right)^{1/2}.$$

However, since each $b_j(k,r) \geqslant 0$ and $b_k(k,r) = 1$, we see that

$$\begin{aligned}\varepsilon &> \left(\sum_{j=0}^{k} b_j(k,r)\int_0^\infty |f^{(j)}(x)-g^{(j)}(x)|^2 x^{j-k}\mathrm{e}^{-x}\,\mathrm{d}x\right)^{1/2}\\ &\geqslant b_k^{1/2}(k,r)\left(\int_0^\infty |f^{(k)}(x)-g^{(k)}(x)|^2\mathrm{e}^{-x}\,\mathrm{d}x\right)^{1/2}\\ &= \left(\int_0^\infty |f^{(k)}(x)-g^{(k)}(x)|^2\mathrm{e}^{-x}\,\mathrm{d}x\right)^{1/2}. \end{aligned} \tag{9.3}$$

On the other hand, since $f \in S_k[0,\infty) = V_{k,-k}$ and $g \in V_{k+2,-k} \subset V_{k,-k}$, we see that

$$f^{(j)}(0) = g^{(j)}(0) = 0 \quad (j = 0,1,\ldots,k-1).$$

Consequently,

$$\begin{aligned}\|f-g\|_k^2 &= \sum_{r=0}^{k-1}\left|\sum_{j=r}^{k-1}\binom{k-r-1}{j-r}(-1)^{j-r}(f^{(j)}(0)-g^{(j)}(0))\right|^2\\ &\quad + \int_0^\infty |f^{(k)}(x)-g^{(k)}(x)|^2\mathrm{e}^{-x}\,\mathrm{d}x\\ &= \int_0^\infty |f^{(k)}(x)-g^{(k)}(x)|^2\mathrm{e}^{-x}\,\mathrm{d}x < \varepsilon^2 \quad \text{by (9.3).}\end{aligned}$$

This shows that $T_{k,1}$ is densely defined in $W_{k,1}[0,\infty)$ and completes the proof of the lemma. □

We can now show, among other results, that $T_{k,1}$ is symmetric in $W_{k,1}[0,\infty)$.

Theorem 9.1. *The operator* $T_{k,1}$, *defined in* (9.1), *is symmetric and bounded below by* $(r+k)I$ *in* $W_{k,1}[0,\infty)$; *that is*,

$$(T_{k,1}f, f)_k \geqslant (k+r)(f,f)_k \quad (f \in \mathscr{D}(T_{k,1})). \tag{9.4}$$

Furthermore, the Laguerre polynomials $\{L_n^{-k}\}_{n=k}^{\infty}$ *form a complete orthonormal set of eigenfunctions of* $T_{k,1}$ *with* $y = L_n^{-k}(x)$ *corresponding to the simple eigenvalue* $\lambda_n = n + r$ *for each integer* $n \geqslant k$.

Proof. From Lemma 9.1, to show symmetry of $T_{k,1}$, it suffices to show that $T_{k,1}$ is Hermitian; that is

$$(T_{k,1}f, g)_k = (f, T_{k,1}g)_k \quad (f, g \in \mathscr{D}(T_{k,1})).$$

Let $f, g \in \mathscr{D}(T_{k,1}) = V_{k+2,-k}$. Then, from Corollary 5.1,

$$f^{(j)}(0) = g^{(j)}(0) = 0 \quad (j = 0, 1, \ldots, k+1); \tag{9.5}$$

moreover, since $T_{k,1}[f] = \ell_{-k}[f] = B_{k,-k}f \in V_{k,-k} = S_{k,1}[0,\infty)$, we see from the definition of $V_{k,-k}$ that

$$(\ell_{-k}[f])^{(j)}(0) = 0 \quad (j = 0, 1, \ldots, k-1). \tag{9.6}$$

Consequently,

$$\begin{aligned}
(T_{k,1}f, g)_k
&= \sum_{m=0}^{k-1}\sum_{j=0}^{m} B_{m,j}[(\ell_{-k}[f])^{(m)}(0)\bar{g}^{(j)}(0) + (\ell_{-k}[f])^{(j)}(0)\bar{g}^{(m)}(0)] \\
&\quad + \int_0^{\infty} (\ell_{-k}[f])^{(k)}(x)\bar{g}^{(k)}(x)\mathrm{e}^{-x}\,\mathrm{d}x \\
&= \int_0^{\infty} (\ell_{-k}[f])^{(k)}(x)\bar{g}^{(k)}(x)\mathrm{e}^{-x}\,\mathrm{d}x \quad \text{by (9.5) and (9.6)} \\
&= \int_0^{\infty} [-(x\mathrm{e}^{-x}f^{(k+1)}(x))'\bar{g}^{(k)}(x) + (k+r)f^{(k)}(x)\bar{g}^{(k)}(x)\mathrm{e}^{-x}]\,\mathrm{d}x \quad \text{(see (5.30))}.
\end{aligned}$$

Furthermore, integration by parts yields

$$\begin{aligned}
(T_{k,1}f, g)_k
&= \int_0^{\infty} [-(x\mathrm{e}^{-x}f^{(k+1)}(x))'\bar{g}^{(k)}(x) + (k+r)f^{(k)}(x)\bar{g}^{(k)}(x)\mathrm{e}^{-x}]\,\mathrm{d}x \\
&= [x\mathrm{e}^{-x}f^{(k)}(x)\bar{g}^{(k+1)}(x) - x\mathrm{e}^{-x}f^{(k+1)}(x)\bar{g}^{(k)}(x)]_0^{\infty} \\
&\quad + \int_0^{\infty} [-(x\mathrm{e}^{-x}\bar{g}^{(k+1)}(x))'f^{(k)}(x) + (k+r)\bar{g}^{(k)}(x)f^{(k)}(x)\mathrm{e}^{-x}]\,\mathrm{d}x \\
&= \int_0^{\infty} [-(x\mathrm{e}^{-x}\bar{g}^{(k+1)}(x))'f^{(k)}(x) + (k+r)\bar{g}^{(k)}(x)f^{(k)}(x)\mathrm{e}^{-x}]\,\mathrm{d}x
\end{aligned}$$

since, by Theorem 5.4, part (b), we have

$$\lim_{x\to\infty} x\mathrm{e}^{-x}f^{(k)}(x)\bar{g}^{(k+1)}(x) = \lim_{x\to\infty} x\mathrm{e}^{-x}f^{(k+1)}(x)\bar{g}^{(k)}(x) = 0, \tag{9.7}$$

while the definition of $V_{k+2,-k}$ and Corollary 5.1 gives

$$f^{(k)}(0) = f^{(k+1)}(0) = g^{(k)}(0) = g^{(k+1)}(0) = 0.$$

That is,

$$(T_{k,1}f, g)_k = \int_0^\infty [-(x\mathrm{e}^{-x}\bar{g}^{(k+1)}(x))' f^{(k)}(x) + (k+r)\bar{g}^{(k)}(x) f^{(k)}(x)\mathrm{e}^{-x}]\,\mathrm{d}x. \tag{9.8}$$

On the other hand, a similar computation yields

$$\begin{aligned}
&(f, T_{k,1}g)_k \\
&\quad = \sum_{m=0}^{k-1}\sum_{j=0}^{m} B_{m,j}[f^{(m)}(0)(\ell_{-k}[\bar{g}])^{(j)}(0) + f^{(j)}(0)(\ell_{-k}[\bar{g}])^{(m)}(0)] \\
&\qquad + \int_0^\infty f^{(k)}(x)(\ell_{-k}[\bar{g}])^{(k)}(x)\mathrm{e}^{-x}\,\mathrm{d}x \\
&\quad = \int_0^\infty [-(x\mathrm{e}^{-x}\bar{g}^{(k+1)}(x))' f^{(k)}(x) + (k+r)\bar{g}^{(k)}(x) f^{(k)}(x)\mathrm{e}^{-x}]\,\mathrm{d}x \quad \text{by Corollary 5.1} \\
&\quad = (T_{k,1}f, g)_k
\end{aligned}$$

and hence $T_{k,1}$ is symmetric in $W_{k,1}[0,\infty)$. Moreover, with $f = g$ in (9.8), one integration by parts yields

$$\begin{aligned}
(T_{k,1}f, f)_k &= \int_0^\infty (-(x\mathrm{e}^{-x}\bar{f}^{(k+1)}(x))' f^{(k)}(x) + (k+r)|f^{(k)}(x)|^2\mathrm{e}^{-x})\,\mathrm{d}x \\
&= -x\mathrm{e}^{-x}\bar{f}^{(k+1)}(x) f^{(k)}(x)|_0^\infty + \int_0^\infty (x\mathrm{e}^{-x}|f^{(k+1)}(x)|^2 + (k+r)|f^{(k)}(x)|^2\mathrm{e}^{-x})\,\mathrm{d}x \\
&= \int_0^\infty (x\mathrm{e}^{-x}|f^{(k+1)}(x)|^2 + (k+r)|f^{(k)}(x)|^2\mathrm{e}^{-x})\,\mathrm{d}x \quad \text{from (9.5) and (9.7)} \\
&\geqslant (k+r)\int_0^\infty |f^{(k)}(x)|^2\mathrm{e}^{-x}\,\mathrm{d}x = (k+r)(f,f)_k \ \text{from (8.6)}
\end{aligned}$$

this establishes (9.4). Lastly, it is clear that $\{L_n^{-k}\}_{n=k}^\infty \subset V_{k+2,-k} = \mathscr{D}(T_{k,1})$; since

$$\ell_{-k}[L_n^{-k}](x) = (n+r)L_n^{-k}(x) \quad (x \in \mathbb{R}; n \geqslant k),$$

we see that $\lambda_n = n + r$ is an eigenvalue of $T_{k,1}$ with associated eigenfunction L_n^{-k} for each integer $n \geqslant k$. Lastly, the completeness of $\{L_n^{-k}\}_{n=k}^\infty$ in $W_{k,1}[0,\infty)$ follows from Theorem 8.1 □

Remark 9.1. We remark that the symmetry of $T_{k,1}$ also follows from Lemma 9.1 and the inequality in (9.4). Indeed, there is a well-known result (see [8, Problem 3, p. 535]) that states a densely defined linear operator A, with domain $\mathscr{D}(A)$, in a complex Hilbert space H with inner product $(\cdot,\cdot)_H$ is symmetric if and only if $(Ax,x)_H \in \mathbb{R}$ for all $x \in \mathscr{D}(A)$.

In order to show that $T_{k,1}$ is self-adjoint, we need the following result.

Theorem 9.2. *The operator $T_{k,1}$ is a closed operator in $W_{k,1}[0,\infty)$.*

Proof. Suppose $\{f_n\} \subset \mathscr{D}(T_{k,1}) = V_{k+2,-k} = \mathscr{D}(B_{k,-k})$, where $B_{k,-k}$ is the kth left-definite operator defined in (5.12), satisfies

$$f_n \to f \quad \text{in } W_{k,1}[0,\infty),$$

$$T_{k,1} f_n = \ell_{-k}[f_n] \to g \quad \text{in } W_{k,1}[0,\infty).$$

From Theorem 8.2, the inner products $(\cdot,\cdot)_k$ and $(\cdot,\cdot)_{k,-k}$ are equivalent; consequently, it follows that

$$f_n \to f \quad \text{in } H_{k,-k}, \tag{9.9}$$

$$B_{k,-k} f_n = \ell_{-k}[f_n] \to g \quad \text{in } H_{k,-k}. \tag{9.10}$$

Since $B_{k,-k}$ is self-adjoint, it is a closed operator. Thus, we see from (9.9) and (9.10) that

$$f \in \mathscr{D}(B_{k,-k}) = \mathscr{D}(T_{k,1}),$$

$$B_{k,-k} f = \ell_{-k}[f] = T_{k,1} f = g,$$

that is to say, the operator $T_{k,1}$ is closed. □

The following theorem is well known and can be found, for example, in [6, Theorem 3, p. 173, Theorem 6, p. 184].

Theorem 9.3. *Suppose A is a closed, symmetric operator in a Hilbert space H and suppose $\{f_n\}_{n=1}^{\infty}$ is a complete set of eigenfunctions of A. Then A is self-adjoint.*

It now follows, from this theorem, together with Theorems 8.1, 9.1, and 9.2, that the operator $T_{k,1}$ is self-adjoint in $W_{k,1}[0,\infty)$ and the Laguerre polynomials $\{L_n^{-k}\}_{n=k}^{\infty}$ form a complete orthonormal sequence in this space.

Theorem 9.4. *The operator $T_{k,1}$ is self-adjoint and bounded below by $(k+r)I$ in $W_{k,1}[0,\infty)$. The spectrum of $T_{k,1}$ is simple and discrete and given by*

$$\sigma(T_{k,1}) = \{n + r \mid n \in \mathbb{N}; n \geqslant k\}.$$

The Laguerre polynomials $\{L_n^{-k}\}_{n=k}^{\infty}$ form a complete orthogonal set of eigenfunctions of $T_{k,1}$ in $W_{k,1}[0,\infty)$.

10. The self-adjoint Laguerre operator $T_{k,2}$

Definition 10.1. The operator $T_{k,2} : \mathscr{D}(T_{k,2}) \subset W_{k,2}[0,\infty) \to W_{k,2}[0,\infty)$ is given by

$$\begin{cases} lT_{k,2} f := \ell_{-k}[f], \\ f \in \mathscr{D}(T_{k,2}) := S_{k,2}[0,\infty), \end{cases} \tag{10.1}$$

where $\ell_{-k}[\cdot]$ is the Laguerre expression defined in (2.9) and where $S_{k,2}[0,\infty)$ is defined in (8.2).

We see from the next theorem that this operator $T_{k,2}$ is self-adjoint in $W_{k,2}[0,\infty)$.

Theorem 10.1. *The operator $T_{k,2}$ is self-adjoint and bounded below by rI in $W_{k,2}[0,\infty)$; that is,*

$$(T_{k,2}f,f)_k \geqslant r(f,f)_k \quad (f \in \mathscr{D}(T_{k,2})). \tag{10.2}$$

Furthermore, the Laguerre polynomials $\{L_n^{-k}\}_{n=0}^{k-1}$ are eigenfunctions of $T_{k,2}$ and the spectrum of $T_{k,2}$ is simple and discrete and given by

$$\sigma(T_{k,1}) = \{n+r \mid n=0,1,\ldots,k-1\}. \tag{10.3}$$

Proof. It suffices to show that $T_{k,2}$ is symmetric; since its domain is the Hilbert space $W_{k,2}[0,\infty)$, it follows (see, for example, [8, p. 534]) that $T_{k,2}$ is self-adjoint. Furthermore, it suffices to show that

$$(T_{k,2}L_n^{-k}, L_m^{-k})_k = (L_n^{-k}, T_{k,2}L_m^{-k})_k \quad (0 \leqslant n,m \leqslant k-1). \tag{10.4}$$

Indeed, through linearity on the k-dimensional space $W_{k,2}[0,\infty)$, it would then follow that

$$(T_{k,2}f,g)_k = (f,T_{k,2}g)_k \quad (f,g \in W_{k,2}[0,\infty)),$$

which gives the symmetry of $T_{k,2}$. To this end, from (6.3) and the fact that $\ell_{-k}[L_n^{-k}] = (n+r)L_n^{-k}$, we see that

$$(T_{k,2}L_n^{-k}, L_m^{-k})_k = (L_n^{-k}, T_{k,2}L_m^{-k})_k = (n+r)\delta_{n,m} \quad (0 \leqslant n,m \leqslant k-1). \tag{10.5}$$

Furthermore, using this orthogonality condition (10.5), a calculation shows that if

$$f = \sum_{j=0}^{k-1} c_j L_j^{-k},$$

then

$$(T_{k,2}f,f)_k = \sum_{j=0}^{k-1} |c_j|^2 (j+r) \geqslant r \sum_{j=0}^{k-1} |c_j|^2 = r(f,f)_k,$$

establishing (10.2). Lastly, it is clear that the spectrum of the operator $T_{k,2}$ is as given in (10.3). □

11. The self-adjointness of the Laguerre operator T_k in $W_k[0,\infty)$

We begin by proving the following general result; in this theorem $(\cdot,\cdot)$ and $\|\cdot\|$ denote, respectively, the inner product and norm in a Hilbert space H.

Theorem 11.1. *Suppose H is a Hilbert space with the orthogonal decomposition*

$$H = H_1 \oplus H_2,$$

where H_1 and H_2 are closed subspaces of H. Suppose $A_1 : \mathscr{D}(A_1) \subset H_1 \to H_1$ and $A_2 : \mathscr{D}(A_2) \subset H_2 \to H_2$ are self-adjoint operators. For $f_1 \in \mathscr{D}(A_1)$ and $f_2 \in \mathscr{D}(A_2)$, write

$$f = f_1 + f_2$$

and let $A : \mathscr{D}(A) \subset H \to H$ be the operator defined by

$$Af := A_1 f_1 + A_2 f_2,$$

$$f \in \mathscr{D}(A) := \mathscr{D}(A_1) \oplus \mathscr{D}(A_2).$$

Then A is self-adjoint in H.

Proof. We first show that A is densely defined. To this end, let $f = f_1 + f_2 \in H$, where $f_i \in H_i$, and let $\varepsilon > 0$. Since each A_i $(i = 1, 2)$ is densely defined, there exists $g_i \in \mathscr{D}(A_i)$ such that

$$\|f_i - g_i\| < \frac{\varepsilon}{\sqrt{2}} \quad (i = 1, 2).$$

Let $g = g_1 + g_2$ so $g \in \mathscr{D}(A)$ and

$$\begin{aligned}
\|f - g\|^2 &= (f - g, f - g) \\
&= (f_1 - g_1 + f_2 - g_2, f_1 - g_1 + f_2 - g_2) \\
&= \|f_1 - g_1\|^2 + (f_1 - g_1, f_2 - g_2) + (f_2 - g_2, f_1 - g_1) + \|f_2 - g_2\|^2 \\
&= \|f_1 - g_1\|^2 + \|f_2 - g_2\|^2 \quad \text{since } f_1 - g_1 \perp f_2 - g_2 \\
&< \varepsilon^2.
\end{aligned}$$

We next show that A is symmetric in H. Let $f, g \in \mathscr{D}(A)$ so

$$f = f_1 + f_2, \ g = g_1 + g_2 \quad (f_i, g_i \in \mathscr{D}(A_i), \ i = 1, 2).$$

Then

$$\begin{aligned}
(Af, g) &= (A_1 f_1 + A_2 f_2, g_1 + g_2) \\
&= (A_1 f_1, g_1) + (A_1 f_1, g_2) + (A_2 f_2, g_1) + (A_2 f_2, g_2) \\
&= (A_1 f_1, g_1) + (A_2 f_2, g_2) \quad \text{since } H_1 \perp H_2 \\
&= (f_1, A_1 g_1) + (f_2, A_2 g_2) \quad \text{since each } A_i \text{ is self-adjoint} \\
&= (f_1, A_1 g_1) + (f_1, A_2 g_2) + (f_2, A_1 g_1) + (f_2, A_2 g_2) \\
&= (f_1 + f_2, A_1 g_1 + A_2 g_2) \\
&= (f, Ag).
\end{aligned}$$

This shows that A is symmetric in H; that is, $A \subset A^*$. To show that $A^* \subset A$, let $g \in \mathscr{D}(A^*)$. Write

$$g = g_1 + g_2 \quad (g_i \in H_i),$$

$$A^* g = h_1 + h_2 \quad (h_i \in H_i).$$

Then

$$(Af, g) = (f, A^* g) \quad (f \in \mathscr{D}(A)).$$

If $f = f_1 + f_2$, where $f_i \in \mathscr{D}(A_i)$ $(i = 1, 2)$, then

$$(Af, g) = (A_1 f_1 + A_2 f_2, g_1 + g_2) = (A_1 f_1, g_1) + (A_2 f_2, g_2). \tag{11.1}$$

On the other hand,

$$(f, A^* g) = (f_1, A^* g) + (f_2, A^* g). \tag{11.2}$$

so that

$$(A_1 f_1, g_1) + (A_2 f_2, g_2) = (f_1, A^* g) + (f_2, A^* g). \tag{11.3}$$

Let $f_2 = 0$ so (11.3) reads

$$\begin{aligned}(A_1 f_1, g_1) &= (f_1, A^* g)\\ &= (f_1, h_1) + (f_1, h_2)\\ &= (f_1, h_1).\end{aligned}$$

It follows that

$$g_1 \in \mathscr{D}(A_1^*) = \mathscr{D}(A_1) \quad \text{and} \quad A_1 g_1 = h_1.$$

Similarly, by letting $f_1 = 0$ in (11.3), we see that

$$g_2 \in \mathscr{D}(A_2^*) = \mathscr{D}(A_2) \quad \text{and} \quad A_2 g_2 = h_2.$$

Hence

$$g = g_1 + g_2 \in \mathscr{D}(A_1) \oplus \mathscr{D}(A_2) = \mathscr{D}(A)$$

and

$$Ag = A_1 g_1 + A_2 g_2 = h_1 + h_2 = A^* g.$$

This shows that A is self-adjoint and completes the proof of the theorem. □

We are now in position to make the following definition and to prove the theorem which follows.

Definition 11.1. Let $k \in \mathbb{N}$. For $f_i \in \mathscr{D}(T_{k,i})$, $i = 1, 2$, let $f = f_1 + f_2$. Define the operator $T_k : \mathscr{D}(T_k) \subset W_k[0, \infty) \to W_k[0, \infty)$ by

$$\begin{cases} T_k f := T_{k,1} f_1 + T_{k,2} f_2 = \ell_{-k}[f], \\ f \in \mathscr{D}(T_k) := \mathscr{D}(T_{k,1}) \oplus \mathscr{D}(T_{k,2}), \end{cases} \tag{11.4}$$

where $T_{k,1}$ and $T_{k,2}$ are the self-adjoint operators defined in (9.1) and (10.1), respectively, and where $\ell_{-k}[\cdot]$ is the Laguerre differential expression defined in(2.9).

Theorem 11.2. *The operator T_k, defined in (11.4), is self-adjoint and bounded below by rI in $W_k[0, \infty)$; that is to say,*

$$(T_k f, f)_k \geqslant r(f, f)_k \quad (f \in \mathscr{D}(T_k)). \tag{11.5}$$

The Laguerre polynomials $\{L_n^{-k}\}_{n=0}^{\infty}$ *form a complete set of eigenfunctions of* T_k *and the spectrum of* T_k *is simple and discrete and is given by*

$$\sigma(T_k) = \{n + r \mid n \in \mathbb{N}_0\}. \tag{11.6}$$

Furthermore,

$$\begin{aligned}\mathscr{D}(T_k) = \{f : [0,\infty) \to \mathbb{C} \mid\ & f^{(j)} \in AC_{\text{loc}}[0,\infty)\ (j = 0,1,\ldots,k-1);\\ & f^{(k+j)} \in AC_{\text{loc}}(0,\infty)\ (j = 0,1);\\ & f^{(k+j)} \in L^2((0,\infty); x^j \mathrm{e}^{-x})\ (j = 0,1,2)\}.\end{aligned} \tag{11.7}$$

Proof. The self-adjointness of T_k follows immediately from the general result in Theorem 11.1, together with Theorems 9.4 and 10.1. Furthermore, it is clear from the definition of T_k that, for each $n \in \mathbb{N}_0$, $y = L_n^{-k}$ is an eigenfunction of T_k corresponding to the eigenvalue $\lambda_n = n + r$; standard results show that (11.6) is valid. Recall, from (9.4) and (10.2), that the operators $T_{k,1}$ and $T_{k,2}$ are bounded below by $(k + r)I$ and rI in their respective spaces. Consequently, for $f \in \mathscr{D}(T_k)$, write $f = f_1 + f_2$, where $f_i \in \mathscr{D}(T_{k,i})$ $(i = 1, 2)$. Then

$$\begin{aligned}(T_k f, f)_k &= (T_{k,1} f_1, f_1)_k + (T_{k,2} f_2, f_2)_k\\ &\geqslant (k + r)(f_1, f_1)_k + r(f_2, f_2)_k\\ &\geqslant r((f_1, f_1)_k + (f_2, f_2)_k) = r(f, f)_k,\end{aligned}$$

establishing (11.5). It remains to show that $\mathscr{D}(T_k)$ is given as in (11.7). We remind the reader of the definitions of $\mathscr{D}(T_{k,1})$ (see (9.2)) and $\mathscr{D}(T_{k,2})$ (see (10.1) and (8.2). Let

$$\begin{aligned}\mathscr{D} = \{f : [0,\infty) \to \mathbb{C} \mid\ & f^{(j)} \in AC_{\text{loc}}[0,\infty)\ (j = 0,1,\ldots,k-1);\\ & f^{(k+j)} \in AC_{\text{loc}}(0,\infty)\ (j = 0,1);\\ & f^{(k+j)} \in L^2((0,\infty); x^j \mathrm{e}^{-x})\ (j = 0,1,2)\}.\end{aligned} \tag{11.8}$$

Let $f \in \mathscr{D}(T_k) = \mathscr{D}(T_{k,1}) \oplus \mathscr{D}(T_{k,2})$. Then $f = g_1 + g_2$ for some $g_i \in \mathscr{D}(T_{k,i})$ $(i = 1, 2)$. Clearly g_1 satisfies the conditions in (11.8) and, since g_2 is a polynomial of degree $< k$, g_2 also satisfies the conditions given in (11.8). Consequently,

$$\mathscr{D}(T_k) \subset \mathscr{D}. \tag{11.9}$$

Conversely, suppose $f \in \mathscr{D}$. Write

$$f = h_1 + h_2,$$

where

$$h_1(x) := \left(f(x) - \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} x^j\right) \quad \text{and} \quad h_2(x) := \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} x^j.$$

It is clear that $h_1 \in \mathscr{D}(T_{k,1})$ and $h_2 \in \mathscr{D}(T_{k,2})$ so $f \in \mathscr{D}(T_{k,1}) \oplus \mathscr{D}(T_{k,2}) = \mathscr{D}(T_k)$; hence,

$$\mathscr{D} \subset \mathscr{D}(T_k). \tag{11.10}$$

Combining (11.9) and (11.10), we obtain the claim in (11.7). This completes the proof of this theorem. □

Remark 11.1. Since T_k is self-adjoint and bounded below in $W_k[0,\infty)$, the left-definite theory in [11] asserts the existence of a continuum of left-definite spaces and operators associated with the pair $(W_k[0,\infty), T_k)$. It would be interesting to determine these spaces and operators in a subsequent paper. Indeed, the Laguerre polynomials $\{L_n^{-k}\}_{n=0}^{\infty}$ form a complete orthogonal set in each of these left-definite spaces; furthermore, these polynomials also form a complete set of eigenfunctions of each of the associated left-definite operators.

References

[1] N.I. Akhiezer, I.M. Glazman, Theory of Linear Operators in Hilbert Space, Dover Publications, New York, 1993.
[2] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach Publishers, New York, 1978.
[3] R.S. Chisholm, W.N. Everitt, On bounded integral operators in the space of integrable-square functions, Proc. Roy. Soc. Edinburgh Sect. A 69 (1970/71) 199–204.
[4] R.S. Chisholm, W.N. Everitt, L.L. Littlejohn, An integral operator with applications, J. Inequal. Appl. 3 (1999) 245–266.
[5] W.N. Everitt, K.H. Kwon, L.L. Littlejohn, R. Wellman, On the spectral analysis of the Laguerre polynomials $\{L_n^{-k}(x)\}$ for positive integers, in Spectral theory and computational methods of Sturm–Liouville problems, Lecture Notes in Pure and Applied Mathematics, Vol. 191, Marcel-Dekker Publishers, New York, 1997.
[6] G. Hellwig, Differential Operators of Mathematical Physics, Addison-Wesley Publishers, Palo Alto, 1967.
[7] A.M. Krall, Laguerre polynomial expansions in indefinite inner product spaces, J. Math. Anal. Appl. 70 (1979) 267–279.
[8] E. Kreyszig, Introductory Functional Analysis with Applications, Wiley Classics Library Wiley, New York, 1989.
[9] K.H. Kwon, L.L. Littlejohn, The orthogonality of the Laguerre polynomials $\{L_n^{-k}(x)\}$ for positive integers k, Ann. Numer. Math. 2 (1–4) (1995) 289–303.
[10] K.H. Kwon, L.L. Littlejohn, Sobolev orthogonal polynomials and second-order differential equations, Rocky Mountain J. Math. 28 (2) (1998) 547–594.
[11] L.L. Littlejohn, R. Wellman, A general left-definite theory for certain self-adjoint operators with applications to differential equations, J. Differential Equations 181 (2) (2002) 280–339.
[12] B. Muckenhoupt, Hardy's inequality with weights, Studia Math. T. XLIV (1972) 31–38.
[13] M.A. Naimark, Linear Differential Operators, Vol. II, Frederick Ungar Publishing Co., New York, 1968.
[14] V.P. Onyango-Otieno, The application of ordinary differential operators to the study of classical orthogonal polynomials, Ph.D. Thesis, University of Dundee, Dundee, Scotland, 1980.
[15] E.D. Rainville, Special Functions, Chelsea Publishing Co., New York, 1960.
[16] G. Szego, Orthogonal Polynomials, Vol. 23, 3rd Edition, American Mathematical Society, Colloquium Publications, Providence, RI, 1974.
[17] G. Talenti, Osservazioni sopra una classe di disuguaglianze, Rend. Sem. Mat. Fis. Milano 39 (1969) 171–185.
[18] G. Tomaselli, A class of inequalities, Boll. Un. Mat. Ital. 4 (1969) 622–631.
[19] R. Wellman, Self-adjoint representations of a certain sequence of spectral differential equations, Ph.D. Thesis, Utah State University, Logan, Utah, U.S.A., 1995.