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Journal of Combinatorial Theory, Series A 113 (2006) 1621–1628

Journal of Combinatorial Theory Series A

www.elsevier.com/locate/jcta

Maximal sets of numbers not containing $k+1$ pairwise coprimes and having divisors from a specified set of primes

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Received 10 August 2005

Available online 15 June 2006

Abstract

We find the formula for the cardinality of a maximal set of integers from $\{1, \ldots, n\}$ which does not contain $k+1$ pairwise coprimes and each integer has a divisor from a specified set of r primes. We also find the explicit formula for this set when $r = k+1$.

Keywords: Greatest common divisor; Coprimes; Squarefree numbers; Number theoretical extremal problems

1. Introduction and main results

Let $\mathbb{P} = \{p_1 < p_2 < \cdots\}$ be the set of primes and $\mathbb{N}$ be the set of natural numbers. Denote $\mathbb{N}(n) = \{1, \ldots, n\}$, $\mathbb{P}(n) = \mathbb{P} \cap \mathbb{N}(n)$. For $a, b \in \mathbb{N}$ denote the greatest common divisor of a and b by (a, b). Let also $S(n, k)$ be the family of sets $A \subset \mathbb{N}(n)$ of integers not containing $k+1$ coprimes. Define

$$f(n,k) = \max_{A \in S(n,k)} |A|.$$

In [1] the following was proved.

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0097-3165/$ – see front matter
doi:10.1016/j.jcta.2006.03.015

Theorem 1. *For all sufficiently large* n

$$f(n,k) = \left|\mathbb{E}(n,k)\right|,$$

where

$$\mathbb{E}(n,k) = \left\{a \in \mathbb{N}(n)\colon\ a = up_i,\ \textit{for some } i = 1,\dots,k\right\}. \tag{1}$$

Let now $\mathbb{Q} = \{q_1 < q_2 < \cdots < q_r\} \subset \mathbb{P}$ be a finite set of primes and $R(n,\mathbb{Q}) \subset S(n,1)$ be such a family of sets of positive integers such that for arbitrary $a \in A \in R(n,\mathbb{Q})$, $(a, \prod_{j=1}^{r} q_j) > 1$. In [2] the following was proved.

Theorem 2. *Let* $n \geqslant \prod_{j=1}^{r} q_j$, *then*

$$f(n,\mathbb{Q}) \triangleq \max_{A \in R(n,\mathbb{Q})} |A| = \max_{1 \leqslant t \leqslant r} \left|M(2q_1,\dots,2q_t,q_1\cdots q_t) \cap \mathbb{N}(n)\right|, \tag{2}$$

where $M(B)$ *is the set of multiples of the set of integers* B.

In [2] was also stated the problem of finding a maximal set of positive integers from $\mathbb{N}(n)$ which satisfies the conditions of Theorems 1 and 2 simultaneously, i.e., to find a set A without $k+1$ coprimes and such that each element of this set has a divisor from $\mathbb{Q}$. This paper is devoted to the solution of this problem. In our work we use the methods from paper [1].

Denote by $R(n,k,\mathbb{Q}) \subset S(n,k)$ the family of sets of positive integers with the property that an arbitrary $a \in A \in R(n,k,\mathbb{Q})$ has a divisor from $\mathbb{Q}$. For given s and $\mathbb{T} = \{r_1 < r_2 < \cdots\} = \mathbb{P} - \mathbb{Q}$ let $F(n,k,s,\mathbb{Q}) \subset R(n,k,\mathbb{Q})$ be the family of sets of squarefree positive numbers such that for arbitrary $a \in A \in F(n,k,s,\mathbb{Q})$ we have $(r_i,a) = 1$, $i > s$. For given s, r the cardinality of the family $F(n,k,s,\mathbb{Q})$ and the cardinalities of $A \in F(n,k,s,\mathbb{Q})$ are bounded from above as $n \to \infty$.

Next we formulate our main result, which in some sense extends both, Theorems 1 and 2.

Theorem 3. *Let* $\mathbb{Q} \neq \emptyset$. *Then for sufficiently large* n *the following relation is valid*

$$\varphi(n,k,\mathbb{Q}) \triangleq \max_{A \in R(n,k,\mathbb{Q})} |A| = \max_{F \in F(n,k,s-1,\mathbb{Q})} \left|M(F) \cap \mathbb{N}(n)\right|, \tag{3}$$

where s *is the minimal integer which satisfies the inequality* $r_s > r$.

We have the following important

Corollary 1. *If* $r = k+1$, *then*

$$\varphi(n,k,\mathbb{Q}) = \left|M(q_1,\dots,q_k) \cap \mathbb{N}(n)\right|. \tag{4}$$

This corollary gives the solution of obtaining an explicit formula for $\varphi(n,k,\mathbb{Q})$ in the first nontrivial case (since if $r \leqslant k$, then trivially $M(q_1,\dots,q_r) \cap \mathbb{N}(n)$ is a maximal set).

2. Proofs

Let us remind the definition of left pushing which the reader can find in [2]. For arbitrary

$$a = up_j^{\alpha}, \quad p_i < p_j,\ (p_i p_j, u) = 1,\ \alpha > 0 \text{ and } p_j \notin \mathbb{Q} \text{ or } p_i, p_j \in \mathbb{Q} \tag{5}$$

define

$$L_{i,j}(a,\mathbb{Q}) = p_i^{\alpha} u.$$

If a is not of the form (5), we set $L_{i,j}(a,\mathbb{Q}) = a$. For $A \subset \mathbb{N}$ denote

$$L_{i,j}(a,A,\mathbb{Q}) = \begin{cases} L_{i,j}(a,\mathbb{Q}), & L_{i,j}(a,\mathbb{Q}) \notin A, \\ a, & L_{i,j}(a,\mathbb{Q}) \in A. \end{cases}$$

Finally set

$$L_{i,j}(A,\mathbb{Q}) = \{L_{i,j}(a,A,\mathbb{Q});\ a \in A\}.$$

We say that A is left compressed if for arbitrary $i < j$

$$L_{i,j}(A,\mathbb{Q}) = A.$$

It can be easily seen that every finite $A \subset \mathbb{N}$, after finite number of left pushing operations, can be made left compressed,

$$|L_{i,j}(A,\mathbb{Q})| = |A|$$

and if $A \in R(n,k,\mathbb{Q})$, then $L_{i,j}(A,\mathbb{Q}) \in R(n,k,\mathbb{Q})$.

If we denote by $O(n,k,\mathbb{Q}) \subset R(n,k,\mathbb{Q})$ the family of sets achieving the maximum in (3) and if $C(n,k,\mathbb{Q}) \subset R(n,k,\mathbb{Q})$ is the family of left compressed sets from $R(n,k,\mathbb{Q})$, then it follows that $O(n,k,\mathbb{Q}) \cap C(n,k,\mathbb{Q}) \neq \emptyset$. Next we assume that $A \in C(n,k,\mathbb{Q}) \cap O(n,k,\mathbb{Q})$.

For arbitrary $a \in A$ we have the decomposition $a = a^1 a^2$, where $a^1 = r_{i_1}^{\alpha_1} \cdots r_{i_f}^{\alpha_f}$, $r_i < r_j$, $i < j$, $a^2 = q_{j_1}^{\beta_1} \cdots q_{j_\ell}^{\beta_\ell}$; $q_{j_m} < q_{j_s}$, $m < s$, $\alpha_j, \beta_j > 0$. If $a = r_{i_1}^{\alpha_1} \cdots r_{i_f}^{\alpha_f} q_{j_1}^{\beta_1} \cdots q_{j_\ell}^{\beta_\ell} \in A$, $\alpha_j, \beta_j > 0$, then $\bar{a} = r_{i_1} \cdots r_{i_f} q_{j_1} \cdots q_{j_\ell} \in A$ as well and also $\hat{a} = ua \in A$ for all $u \in \mathbb{N}$: $ua \leqslant n$. Consider all squarefree numbers $A^* \subset A$ and for given a^2 the set of all a^1 such that $a^1 a^2 \in A^*$. This set is the ideal generated by division (we omit for a moment the restriction $\bar{a} \leqslant n$). The set of minimal elements from this ideal we denote by $P(a^2, A^*)$. It follows that $(A \in O(n,k,\mathbb{N}))$,

$$A = M(\{a^1 a^2;\ a^1 \in P(a^2, A^*)\}) \cap \mathbb{N}(n).$$

For each a^2 we order $\{a_1^1 < a_2^1 < \cdots\} = P(a^2, A^*)$ colexicographically according to their decompositions $a_i^1 = r_{i_1} \cdots r_{i_f}$. Let ρ be maximal integer such that r_ρ divides a_i^1 for which $a_i^1 a^2 \in A^*$ for some a^2. Then from the left compressedness of the set $B \subset A$ of elements $b = b^1 b^2 \leqslant n$, $(b^1, \prod_{j=1}^r q_j) = 1$ such that $b^2 = q_{j_1}^{\beta_1} \cdots q_{j_\ell}^{\beta_\ell}$, $\beta_j > 0$ and $a_i^1 | b^1$, $a_j^1 \nmid b^1$, $j < i$ is exactly the set

$$B(a) = \left\{u \leqslant n\colon u = r_{i_1}^{\alpha_1} \cdots r_{i_f}^{\alpha_f} r_\rho^{\alpha_\rho} q_{j_1}^{\beta_1} \cdots q_{j_\ell}^{\alpha_\ell} F;\ \alpha_i, \beta_i > 0,\ \left(F, \prod_{j=1}^{\rho} r_j \prod_{j=1}^{r} q_j\right) = 1\right\}.$$

Denote

$$P^\rho(a^2, A^*) = \{a^1 a^2\colon a^1 \in P(a^2, A^*)\colon (a^1, r_\rho) = r_\rho\},$$

$$P_s^\rho(A^*) = \left\{a^1 a^2\colon a^1 \in P^\rho(a^2, A^*) \text{ and } a^2, \right.$$
$$\left. \text{is such that } (a^2, q_s) = q_s,\ \left(a^2, \prod_{j=1}^{s-1} q_j\right) = 1\right\}$$

and

$$L^\rho(a^2) = \bigcup_{a \in P^\rho(a^2, A^*)} B(a).$$

Then the set $\bigcup_{s=1}^r P_s^\rho(A^*)$ is exactly the set $\bigcup_{a^2} P^\rho(a^2, A^*)$ of numbers from A^* which are divisible by r_ρ. Since each $a \in P(a^2, A^*)$ for all a^2 has divisor from $\mathbb{Q}$, it follows that for some $1 \leqslant s \leqslant r$

$$\left| \bigcup_{a \in P_s^\rho(A^*)} B(a) \right| \geqslant \frac{1}{r} \left| \bigcup_{a^2} L^\rho(a^2) \right|. \tag{6}$$

Next for this s we define the transformation

$$\bar{P}(a^2, A^*) = \left(P(a^2, A^*) - P^\rho(a^2, A^*)\right) \cup R_s^\rho(a^2, A^*),$$

where

$$R_s^\rho(a^2, A^*) = \left\{v \in \mathbb{N};\ v r_\rho \in P_s^\rho(a^2, A^*)\right\},$$
$$P_s^\rho(a^2, A^*) = \left\{a = a^1 a^2 \in P_s^\rho(A^*)\right\}.$$

It is easy to see that

$$\bigcup_{a^2} \bar{P}(a^2, A^*) \subset S(n, k, \mathbb{Q}).$$

Next we prove that if $r_\rho > r$, then

$$\left| M\left(\bigcup_{a^2} \bar{P}(a^2, A^*) \right) \cap \mathbb{N}(n) \right| > |A| \tag{7}$$

which is a contradiction to the maximality of A.

For $a \in R_s^\rho(a^2, A^*)$, $a = r_{i_1} \cdots r_{i_f} q_{j_1} \cdots q_{j_\ell}$, $r_{i_1} < \cdots < r_{i_f} < r_\rho$, $q_{j_1} \cdots q_{j_\ell} = a^2$, $j_1 = s$, $q_{j_1} < g_{j_2} < \cdots < q_{j_\ell}$ denote

$$D(a) = \left\{ v \in \mathbb{N}(n)\colon\ v = r_{i_1}^{\alpha_1} \cdots r_{i_f}^{\alpha_f} q_{j_1}^{\beta_1} \cdots q_{j_\ell}^{\beta_\ell} T,\ \alpha_j, \beta_j \geqslant 1,\ \left(T, \prod_{j=1}^{\rho-1} r_j \prod_{j=1}^{r} q_j \right) = 1 \right\}.$$

It can be easily seen that

$$D(a) \cap D(a') = \emptyset, \quad a \neq a'$$

and

$$M\left(\bigcup_{a^2} \left(P(a^2, A^*) - P^\rho(a^2, A^*) \right) \right) \cap D(a) = \emptyset.$$

Thus from (6) it follows that to prove (7) it is sufficient to show that for large $n > n_0$ and $r_\rho > r$

$$\left|D(a)\right| > r \left|B(a r_\rho)\right|. \tag{8}$$

To prove (8) we consider three cases.

First case. $n/(a r_\rho) \geqslant 2$ and $\rho > \rho_0$.

From (15) follows that

$$\begin{aligned}\left|B(ar_\rho)\right| &\leqslant c_2 \sum_{\alpha_i,\alpha,\beta_i\geqslant 1} \frac{n}{r_{i_1}^{\alpha_1}\cdots r_{i_f}^{\alpha_f} r_\rho^{\alpha_\rho} q_{j_1}^{\beta_1}\cdots q_{j_\ell}^{\beta_\ell}} \prod_{j=1}^{\rho}\left(1-\frac{1}{r_j}\right)\prod_{j=1}^{r}\left(1-\frac{1}{q_j}\right)\\ &= c_2 \frac{n}{(r_{i_1}-1)\cdots(r_{i_f}-1)(r_\rho-1)(q_{j_1}-1)\cdots(q_{j_\ell}-1)}\\ &\quad\times\prod_{j=1}^{\rho}\left(1-\frac{1}{r_j}\right)\prod_{j=1}^{r}\left(1-\frac{1}{q_j}\right). \end{aligned} \tag{9}$$

At the same time

$$\bar{D}(a) \triangleq \left\{v\in\mathbb{N}(n);\ v = r_{i_1}\cdots r_{i_f} q_{j_1}\cdots q_{j_\ell} F_1,\ \left(F_1, \prod_{j=1}^{\rho-1} r_j \prod_{j=1}^{r} q_j\right)=1\right\}\subset D(a)$$

and using (15) we obtain the inequalities

$$\left|D(a)\right| \geqslant \left|\bar{D}(a)\right| \geqslant c_1 \frac{n}{r_{i_1}\cdots r_{i_f} q_{j_1}\cdots q_{j_\ell}} \prod_{j=1}^{\rho-1}\left(1-\frac{1}{r_j}\right)\prod_{j=1}^{r}\left(1-\frac{1}{q_j}\right). \tag{10}$$

Thus from (9), (10) it follows that

$$\begin{aligned}\frac{|D(a)|}{|B(ar_\rho)|} &\geqslant \frac{c_1}{c_2} r_\rho \frac{(r_{i_1}-1)\cdots(r_{i_f}-1)}{r_{i_1}\cdots r_{i_f}} \prod_{j\in\{j_1,\dots,j_\ell\}}\left(1-\frac{1}{q_j}\right)\\ &\geqslant \frac{c_1}{c_2}\prod_{j=1}^{f}\left(1-\frac{1}{r_j}\right) r_\rho \prod_{j=1}^{r}\left(1-\frac{1}{q_j}\right) > r.\end{aligned}$$

The last inequality follows from (14).

Second case. $n/(ar_\rho) \geqslant 2$, $\rho < \rho_0$.

Then we apply relations (18) and obtain the inequalities

$$\begin{aligned}\left|B(ar_\rho)\right| &< (1+\epsilon)\frac{n}{(r_{i_1}-1)\cdots(r_{i_f}-1)(r_\rho-1)(q_{j_1}-1)\cdots(q_{j_\ell}-1)}\\ &\quad\times\prod_{j=1}^{\rho}\left(1-\frac{1}{r_j}\right)\prod_{j=1}^{r}\left(1-\frac{1}{q_j}\right),\end{aligned}$$

$$\left|D(a)\right| > (1-\epsilon)\frac{n}{(r_{i_1}-1)\cdots(r_{i_f}-1)(q_{j_1}-1)\cdots(q_{j_\ell}-1)}\prod_{j=1}^{\rho-1}\left(1-\frac{1}{r_j}\right)\prod_{j=1}^{r}\left(1-\frac{1}{q_j}\right).$$

From these inequalities it follows that

$$\frac{|D(a)|}{|B(ar_\rho)|} > \frac{1-\epsilon}{1+\epsilon} r_\rho > r.$$

Here the last inequality is valid for sufficiently small ϵ, because $r_\rho > r$.

Last case. $1 \leqslant n/(ar_\rho) < 2$.

In this case $|B(ar_\rho)| = 1$. Let $r_{i_1} \cdots r_{i_f} r_\rho q_{j_1} \cdots q_{j_\ell} = B(ar_\rho)$. Then we choose $r_g > (q_{j_1})^r$ and $n > \prod_{j=1}^{g} r_j \prod_{j=1}^{r} q_j$. We have $r_\rho > r_g$. Indeed, otherwise

$$n > \prod_{j=1}^{g} r_j \prod_{j=1}^{r} q_j > 2 \prod_{j=1}^{\rho} r_j \prod_{j=1}^{r} q_j > 2ar_\rho$$

which is a contradiction to our case.

Hence

$$\left\{r_{i_1} \cdots r_{i_f} q_{j_1} \cdots q_{j_\ell}, r_{i_1} \cdots r_{i_f} q_{j_1}^2 \cdots q_{j_\ell}, \ldots, r_{i_1} \cdots r_{i_f} q_{j_1}^r \cdots q_{j_\ell}, r_{i_1} \cdots r_{i_f} q_{j_1} \cdots q_{j_\ell} r_\rho\right\} \subset D(a).$$

Thus in this case also $|D(a)| > r = r|B(ar_\rho)|$.

From the above follows that for sufficiently large $n > n_0(\mathbb{Q})$ for all $a \in R_s^\rho(a^2, A^*)$ inequality (8) is valid and taking into account (6) we obtain (7). This is a contradiction to the maximality of A. Hence the maximal $r_i \in \mathbb{P} - \mathbb{Q}$ which appears as a divisor of some $a \in \bigcup_{a^2} P(a^2, A^*)$ such that $M(A^*) \cap \mathbb{N}(n) \in O(n, k, \mathbb{Q})$ satisfies the condition $r_\rho \leqslant r$. This inequality implies the statement of the theorem.

To prove the corollary note that for $\mathbb{Q} = \{q_1 < \cdots < q_k < q_{k+1}\}$

$$M(q_1, q_2, \ldots, q_k) \cap \mathbb{N}(n) \in R(n, k, \mathbb{Q}).$$

From the left compressedness of A it follows that if $q_i \in A$, then $q_j \in A$, $j \leqslant i$. Assume at first that $k > 1$. Let $q_1, \ldots, q_t \in A$, $q_{t+1} \notin A$. Then $q_i q_j$ belongs to A for all $t < i < j \leqslant k + 1$. Next we should maximize (over the choice of $a_{ij} \in \mathbb{N} - M(\mathbb{Q})$) the value

$$\left|M(q_i a_{ij},\ i = t+1, \ldots, k+1) \cap \mathbb{N}(n)\right|$$

such that

$$Z \triangleq \{q_i a_{ij},\ i = t+1, \ldots, k+1\} \subset S(n, k-t, \mathbb{Q}). \tag{11}$$

Completely repeating the proof of the theorem one can show that each a_{ij} can be chosen in such a way that for each i, j a_{ij} is the product of some primes $r_m \in \mathbb{P} - \mathbb{Q}$ such that $r_m \leqslant k - t + 1$. Then it can be easily seen that for arbitrary $t < k$

$$r_{k-t} > k - t + 1 \tag{12}$$

except the cases $r_2 = 3$ and/or $r_1 = 2$, when equality holds in (12).

Thus if (12) is valid, then we can only increase the volume of Z if we choose

$$Z = \{q_i r_j,\ i = t+1, \ldots, k+1,\ j = 1, \ldots, k-t-1\}.$$

But in this case

$$Z \in S(n, k-t-1, \mathbb{Q})$$

and we only increase Z by choosing

$$Z = \{q_{t+1}, q_i r_j,\ i = t+2, \ldots, k+1,\ j = 1, \ldots, k-t-1\}.$$

Continuing this process we arrive at the following three cases:

$$A = \begin{cases} M(q_1, \dots, q_{k-2}, q_{k-1}q_k, q_{k-1}q_{k+1}, q_kq_{k+1}, \\ \quad q_ir_j,\ i = k-1, k, k+1,\ j = 1, 2) \cap \mathbb{N}(n), & r_2 = 3, \\ M(q_1, \dots, q_{k-1}, q_kq_{k+1}, q_kr_1, q_{k+1}r_1) \cap \mathbb{N}(n), & r_1 = 2,\ r_2 > 3, \\ M(q_1, \dots, q_k) \cap \mathbb{N}(n), & \text{otherwise.} \end{cases} \tag{13}$$

Now by comparing the densities (see (16)) of the sets in the right-hand side of (13) we prove that indeed a maximum cardinality among these three possibilities for large n has the set $M(q_1, \dots, q_k) \cap \mathbb{N}(n)$.

It is enough to calculate the contribution of the last three primes q_{k-1}, q_k, q_{k+1} to the corresponding densities. These contributions to the three sets are respectively

$$d_1 = \left(\frac{2}{3}\left(\frac{1}{q_{k-1}} + \frac{1}{q_k} + \frac{1}{q_{k+1}}\right) - \frac{1}{3}\left(\frac{1}{q_{k-1}q_k} + \frac{1}{q_{k-1}q_{k+1}} + \frac{1}{q_kq_{k+1}}\right)\right)\prod_{j=1}^{k-2}\left(1 - \frac{1}{q_j}\right),$$

$$d_2 = \left(\frac{1}{q_{k-1}} + \left(\frac{1}{2q_k} + \frac{1}{2q_{k+1}}\right)\left(1 - \frac{1}{q_{k-1}}\right)\right)\prod_{j=1}^{k-2}\left(1 - \frac{1}{q_j}\right),$$

$$d_3 = \left(\frac{1}{q_{k-1}} + \frac{1}{q_k} - \frac{1}{q_{k-1}q_k}\right)\prod_{j=1}^{k-2}\left(1 - \frac{1}{q_j}\right).$$

It is an easy exercise to show that $d_3 > d_1, d_2$. Thus the third case gives us the maximal set (for sufficiently large n).

The case $k = 1$ can be proved by comparing densities of the sets $M(q_1)$ and $M(q_1r_1, q_2r_1, q_1q_2)$ (it also follows from Theorem 2).

The corollary is proved.

Open problems. It would be interesting to know whether it is possible to find a bound on ρ which depends only on k but not on $\mathbb{Q}$? As it can be seen from (1) and (2) this can be done in the case $\mathbb{Q} = \emptyset$ and $k = 1$.

Another question is whether in some cases the optimal F satisfying $M(F) \cap \mathbb{N}(n) \in O(n, k, \mathbb{Q})$ should contain an $a \in F$, whose decomposition into primes contains more than one element from $\mathbb{P} - \mathbb{Q}$?

3. Auxiliary facts

Statement 1. *We have*

$$p_t \prod_{j=1}^{t}\left(1 - \frac{1}{p_j}\right) \overset{t\to\infty}{\longrightarrow} \infty. \tag{14}$$

This statement is a simple consequence of the following property of primes (see, for example, [4, Theorem 13.13]):

$$\prod_{p\in\mathbb{P}(t)}\left(1 - \frac{1}{p}\right) \overset{t\to\infty}{\sim} \frac{e^{-C}}{\log t},$$

where C is the Euler constant.

Statement 2. *If*

$$\phi(x,y) = \left|\left\{a \leqslant x\colon \left(a, \prod_{p_j<y} p_j\right) = 1\right\}\right|,$$

then for some constants c_1, c_2 and all x, y; $x \geqslant 2y \geqslant 4$,

$$c_1 x \prod_{p_j<y}\left(1 - \frac{1}{p_j}\right) \leqslant \phi(x,y) \leqslant c_2 x \prod_{p_j<y}\left(1 - \frac{1}{p_j}\right). \tag{15}$$

The proof of this statement one can find in [3].

Define the dB density of $B \subset \mathbb{N}$ as the limit (if it exists)

$$dB = \lim_{n\to\infty} \frac{|B \cap \mathbb{N}(n)|}{n}. \tag{16}$$

It can be easily seen that the density of the set

$$B = \left\{b = p_{i_1}^{\alpha_1} \cdots p_{i_m}^{\alpha_m} F,\ \alpha_i \geqslant 1,\ \left(F, \prod_{s=1}^{f} p_{j_s}\right) = 1\right\} \tag{17}$$

is equal to

$$\sum_{\alpha_j \geqslant 1} \frac{1}{p_{i_1}^{\alpha_1} \cdots p_{i_m}^{\alpha_m}} \prod_{s=1}^{f}\left(1 - \frac{1}{p_{j_s}}\right) = \frac{1}{(p_{i_1}-1)\cdots(p_{i_m}-1)} \prod_{s=1}^{f}\left(1 - \frac{1}{p_{j_s}}\right)$$

and for a fixed number of $B_j,\ j = 1,\dots,c$ of the form (17) for sufficiently large $n > n(\epsilon)$ we have

$$\left|B_j \cap \mathbb{N}(n)\right| = (1 \pm \epsilon)\frac{n}{(p_{i_1}-1)\cdots(p_{i_m}-1)} \prod_{s=1}^{f}\left(1 - \frac{1}{p_{j_s}}\right), \tag{18}$$

where p_{i_j}, p_{j_s}, m, f can be different for different j.

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