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On the maximal order of numbers in the "factorisatio numerorum" problem

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Abstract

Let $m(n)$ be the number of ordered factorizations of $n \geqslant 1$ in factors larger than 1. We prove that for every $\varepsilon > 0$

$$m(n) < \frac{n^{\rho}}{\exp((\log n)^{1/\rho}/(\log\log n)^{1+\varepsilon})}$$

holds for all integers $n > n_0$, while, for a suitable constant $c > 0$,

$$m(n) > \frac{n^{\rho}}{\exp(c(\log n/\log\log n)^{1/\rho})}$$

holds for infinitely many positive integers n, where $\rho = 1.72864\ldots$ is the positive real solution to $\zeta(\rho) = 2$. We investigate also arithmetic properties of $m(n)$ and the number of distinct values of $m(n)$.

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1. Introduction

Let $m(n)$ be the number of ordered factorizations of a positive integer n in factors bigger than 1. For example, $m(12) = 8$ since we have the factorizations $12, 2\cdot 6, 6\cdot 2, 3\cdot 4, 4\cdot 3, 2\cdot 2\cdot 3, 2\cdot 3\cdot 2$, and $3\cdot 2\cdot 2$. By the definition, $m(1) = 0$ but we will see that in some situations it is useful to set $m(1) = 1$ or $m(1) = 1/2$. Kalmár [13] found the average order of $m(n)$: for $x \to \infty$,

$$M(x) = \sum_{n\leqslant x} m(n) = \phi x^{\rho}\bigl(1 + o(1)\bigr), \tag{1}$$

where $\rho = 1.72864\ldots$ is the positive real solution to $\zeta(\rho) = 2$ and $\phi = 0.31817\ldots$ is given by $\phi = -1/\rho\zeta'(\rho)$. (As usual, $\zeta(s) = \sum_{n\geqslant 1} n^{-s}$.) Further detailed and strong results on the average order of $m(n)$ were obtained by Hwang [9].

In contrast, good bounds on the maximal order of $m(n)$ were lacking. Erdős claimed in the end of his article [4] that there exist positive constants $0 < c_2 < c_1 < 1$ such that

$$m(n) < \frac{n^{\rho}}{\exp((\log n)^{c_2})}$$

holds for all $n > n_0$, while

$$m(n) > \frac{n^{\rho}}{\exp((\log n)^{c_1})}$$

holds for infinitely many n, but he gave no details. To our knowledge, the best proved bounds on the maximal order state that $m(n) < n^{\rho}$ for every $n \geqslant 1$ (Chor, Lemke and Mador [1], a simple proof by induction was recently given by Coppersmith and Lewenstein [3]), and that for any $\varepsilon > 0$ one has $m(n) > n^{\rho-\varepsilon}$ for infinitely many n (Hille [8], [3] gives an explicit construction). (In Lemma 2.4, we strengthen the argument of [1] and show that $m(n) \leqslant n^{\rho}/2$ for every $n \geqslant 1$.)

Here, we come close to determining the maximal order of $m(n)$. We prove that it is, roughly, $n^{\rho}/\exp((\log n)^{1/\rho})$. More precisely, we prove that for every $\varepsilon > 0$,

$$m(n) < \frac{n^{\rho}}{\exp((\log n)^{1/\rho}/(\log\log n)^{1+\varepsilon})}$$

holds for all $n > n_0$ (Theorem 3.1), while

$$m(n) > \frac{n^{\rho}}{\exp(c(\log n)^{1/\rho}/(\log\log n)^{1/\rho})}$$

holds with a certain constant $c > 0$ for infinitely many positive integers n (Theorem 4.1).

The paper is organized as follows. In Section 2, we give auxiliary results, of which Lemma 2.3 on the speed of convergence $\rho_k \to \rho$ (ρ_k is a "finite" counterpart of ρ for $m(n)$ restricted to smooth numbers n with no prime factor exceeding p_k, the kth prime number), and Lemmas 2.4–2.6 giving explicit inequalities for $m(n)$ and $m_k(n)$ ($m_k(n) = m(n)$ if n has no prime factor $> p_k$ and $m_k(n) = 0$ else) may be of independent interest. Section 3 is devoted to the proof of the upper bound. The proof is elementary (uses real analysis only) and is obtained by combining the combinatorial bounds on $m(n)$ in Lemmas 2.4 and 2.5, standard bounds from the theory of prime numbers, and the convergence bound in Lemma 2.3. Section 4 is devoted to the proof of the lower bound. In the first version of this article, still available at [15, version 1], we proved by an elementary approach similar to that in Section 3, with the additional ingredient being Kalmár's asymptotic relation (1), a weaker lower bound that has $(\log n)^{1/\rho}$ in the denominator replaced with the bigger power $(\log n)^{\rho/(\rho^2-1)+o(1)}$. Here, we prove in Section 4 a lower bound with the matching exponent $1/\rho$ of the $\log n$ by a method suggested to us by an anonymous referee. The method works in the complex domain and combines the uniform version of (1) for $m_k(n)$ with error estimates independent on k, bounds on smooth numbers, and again Lemma 2.3. In Section 5, we give further references and comments on the history of $m(n)$ and some related problems. We also investigate arithmetical properties of $m(n)$.

2. Preliminaries and auxiliary results

Let us begin by recalling some notation. For a positive integer n we write $\omega(n)$ and $\Omega(n)$ for the number of distinct prime factors of n and the total number of prime factors of n (including multiplicities), respectively. We use the letters p and q with or without subscripts to denote prime numbers. We put $P(n)$ for the largest prime factor of n. We write log for the natural logarithm. In the complex domain (mainly in Section 4), we use s to denote a generic variable and write σ and τ for its real and imaginary part, respectively, so $s=\sigma+i\tau$, where $i=\sqrt{-1}$. We use the Vinogradov symbols $\ll$ and $\gg$ and the Landau symbols O and o with their usual meanings.

The proof of the following estimate is standard and we omit it.

Lemma 2.1. *If $\delta>\delta_0>1$, then the estimate*

$$\sum_{p>t}\frac{1}{p^\delta}=\frac{(\delta-1)^{-1}}{t^{\delta-1}\log t}+O\left(\frac{1}{t^{\delta-1}(\log t)^2}\right) \tag{2}$$

holds uniformly for $t>2$.

Let p_k be the kth prime. We shall use the well-known asymptotic relations

$$\sum_{p\leqslant x}\log p=x+O(x/\log x)$$

(equivalent to the Prime Number Theorem), and

$$p_k=k\log k+k\log\log k+O(k)$$

(the full asymptotic expansion $p_k=k(\log k+\log\log k-1+\cdots)$ was found by Cipolla [2]). Let $\mathcal{N}_k$ be the set of positive integers (including 1) composed only of the primes $p_1=2,p_2,\ldots,p_k$, and $m_k(n)$ be the number of ordered factorizations of n in factors lying in $\mathcal{N}_k\setminus\{1\}$. We allow $k=\infty$, in which case $p_k=\infty$, $\mathcal{N}_\infty=\mathbb{N}$ is the set of all positive integers, and $m_\infty(n)=m(n)$. Note that, for $k\in\mathbb{N}$, $m_k(n)>0$ iff $n\in\mathcal{N}_k$. Further, if $m_k(n)>0$ then $m_k(n)=m(n)$, and if $n\leqslant p_k$ then $m_k(n)=m(n)$. Let, for complex s with $\sigma>1$ and $k\in\mathbb{N}\cup\{\infty\}$,

$$\zeta_k(s)=\prod_{p\leqslant p_k}\left(1-\frac{1}{p^s}\right)^{-1}=\sum_{n\in\mathcal{N}_k}\frac{1}{n^s},$$

and ρ_k be the positive real solution to $\zeta_k(\rho_k) = 2$. For $k = \infty$, we get the Euler–Riemann zeta function $\zeta(s) = \zeta_\infty(s)$ and the number $\rho = \rho_\infty$. Note that for $k \in \mathbb{N}$ the series for $\zeta_k(s)$ converges absolutely even for $\sigma > 0$. For every s with $\sigma > 1$, we have the convergence $\zeta_k(s) \to \zeta(s)$ as $k \to \infty$. For $k \in \mathbb{N} \cup \{\infty\}$, one has the identity (setting $m_k(1) = 1$ for every k)

$$\sum_{n \geqslant 1} \frac{m_k(n)}{n^s} = \sum_{l \geqslant 0} \big(\zeta_k(s) - 1\big)^l = \frac{1}{2 - \zeta_k(s)},$$

which implies that $m_k(n) = o(n^{\rho_k + \varepsilon})$ for every fixed $\varepsilon > 0$. Our approach to estimating $m(n)$ is based on approximating the "infinite" quantities $m(n)$, ρ, and $\zeta(s)$, with their "finite" counterparts $m_k(n)$, ρ_k, and $\zeta_k(s)$ for $k \in \mathbb{N}$ but $k \to \infty$. We quantify the degrees of approximation in the following two lemmas. The first lemma is obtained by considering the infinite series defining $\zeta_k(s)$ and $\zeta(s)$ and its easy proof is omitted.

Lemma 2.2. *We have*

$$\rho_1 = 1 < \rho_2 = 1.43527\ldots < \rho_3 = 1.56603\ldots < \cdots < \rho = 1.72864\ldots$$

and $\rho_k \to \rho$ as $k \to \infty$. The convergence $\zeta_k(s) \to \zeta(s)$ as $k \to \infty$ is uniform on every complex domain $\sigma > \sigma_0 > 1$ and the same is true for the convergence $\zeta_k'(s) \to \zeta'(s)$ and for all higher derivatives. Also, for every $k \in \mathbb{N} \cup \{\infty\}$, we have $\zeta_k'(\rho_k) < 0$.

We shall use the above lemma to bound various expressions containing ρ_k, $\zeta_k(\rho_k)$, $\zeta_k(s)$, $1/\zeta_k'(\rho_k)$, etc., by constants independent on k.

Lemma 2.3. *The estimate*

$$\rho - \rho_k = \frac{2}{(\rho - 1)|\zeta'(\rho)|} \cdot \frac{1}{k^{\rho-1}(\log k)^\rho}\left(1 + O\left(\frac{\log\log k}{\log k}\right)\right)$$

holds for all $k \geqslant 2$.

Proof. We will assume that $k \geqslant 2$. The equation $\zeta_k(\rho_k)^{-1} = \zeta(\rho)^{-1} = 1/2$ implies that

$$\prod_{2 \leqslant p \leqslant p_k} \left(1 - \frac{1}{p^{\rho_k}}\right) = \prod_{p \geqslant 2} \left(1 - \frac{1}{p^\rho}\right).$$

Taking logarithms and regrouping, we get

$$\sum_{2 \leqslant p \leqslant p_k} \left(\log\left(1 - \frac{1}{p^\rho}\right) - \log\left(1 - \frac{1}{p^{\rho_k}}\right)\right) = -\sum_{p > p_k} \log\left(1 - \frac{1}{p^\rho}\right).$$

The left side satisfies, by Lagrange's Mean-Value Theorem (the derivative of the function $x \mapsto \log(1 - 1/p^x)$ is $(\log p)/(p^x - 1)$),

$$\sum_{2\leqslant p\leqslant p_k} \log\left(1-\frac{1}{p^{\rho}}\right)-\log\left(1-\frac{1}{p^{\rho_k}}\right)=(\rho-\rho_k)\sum_{2\leqslant p\leqslant p_k}\frac{\log p}{p^{\sigma_p}-1}$$
$$>(\rho-\rho_k)(\log 2)/3 \qquad (3)$$

for some numbers $\sigma_p \in (\rho_k, \rho) \subset (1.4, 1.8)$. The right side is

$$\begin{aligned}
-\sum_{p>p_k}\log\left(1-\frac{1}{p^{\rho}}\right)&=\sum_{p>p_k}\frac{1}{p^{\rho}}+O\left(\sum_{p>p_k}\frac{1}{p^{2\rho}}\right)\\
&=\frac{(\rho-1)^{-1}}{p_k^{\rho-1}\log(p_k)}\left(1+O\left(\frac{1}{\log p_k}\right)\right)\\
&=\frac{(\rho-1)^{-1}}{k^{\rho-1}(\log k)^{\rho}}\left(1+O\left(\frac{\log\log k}{\log k}\right)\right), \qquad (4)
\end{aligned}$$

where we used Lemma 2.1 and the fact that $p_k = k(\log k + O(\log\log k))$. We get immediately that

$$\rho-\rho_k \ll \frac{1}{k^{\rho-1}(\log k)^{\rho}}. \qquad (5)$$

To do better, we return to (3) and write

$$\frac{\log p}{p^{\sigma_p}-1}=\frac{\log p}{p^{\rho}-1}\left(1+\frac{p^{\sigma_p}}{p^{\sigma_p}-1}\left(p^{\rho-\sigma_p}-1\right)\right).$$

We have $1 \leqslant p^{\sigma_p}/(p^{\sigma_p}-1) \leqslant 2$ and, using (5),

$$p^{\rho-\sigma_p}-1 \leqslant \exp\left((\rho-\rho_k)\log p_k\right)-1 \ll (\rho-\rho_k)\log p_k \ll \frac{1}{k^{\rho-1}(\log k)^{\rho-1}}.$$

Hence, the right side of (3) equals

$$\begin{aligned}
(\rho-\rho_k)\sum_{2\leqslant p\leqslant p_k}\frac{\log p}{p^{\sigma_p}-1}&=(\rho-\rho_k)\left(1+O\left(k^{1-\rho}(\log k)^{1-\rho}\right)\right)\sum_{2\leqslant p\leqslant p_k}\frac{\log p}{p^{\rho}-1}\\
&=(\rho-\rho_k)\left(1+O\left(k^{-1/2}\right)\right)\sum_{2\leqslant p\leqslant p_k}\frac{\log p}{p^{\rho}-1}.
\end{aligned}$$

Equating the right sides of (3) and (4), we get the relation

$$(\rho-\rho_k)\sum_{2\leqslant p\leqslant p_k}\frac{\log p}{p^{\rho}-1}=\frac{(\rho-1)^{-1}}{k^{\rho-1}(\log k)^{\rho}}\left(1+O\left(\frac{\log\log k}{\log k}\right)\right).$$

All is left to notice is that

$$\frac{|\zeta'(\rho)|}{\zeta(\rho)} = \sum_{p \geqslant 2} \frac{\log p}{p^\rho - 1} = \sum_{p \leqslant p_k} \frac{\log p}{p^\rho - 1} + \sum_{p > p_k} \frac{\log p}{p^\rho - 1}$$
$$= \sum_{p \leqslant p_k} \frac{\log p}{p^\rho - 1} + O\big(k^{-1/2}\big),$$

where the last estimate follows again from Lemma 2.1 via the fact that $\log p \ll p^{1/10}$:

$$\sum_{p > p_k} \frac{\log p}{p^\rho - 1} \ll \sum_{p > p_k} \frac{1}{p^{\rho - 0.1}} \ll \frac{1}{p_k^{\rho - 1.1} \log p_k} < k^{-1/2}.$$

The claimed estimate now follows. □

In the next three lemmas, we prove combinatorial inequalities involving $m_k(n)$ and $m(n)$. In the first lemma, we slightly improve the result from [1, Theorem 5] that $m_k(n) < n^{\rho_k}$ for every $n \geqslant 1$. The second lemma is crucial for obtaining bounds of the type $m(n) = o(n^\rho)$. The third lemma gives some lower estimates on $m(n)$.

Lemma 2.4. *For every $k \in \mathbb{N} \cup \{\infty\}$ and $n \geqslant 1$ (with $m_k(1) = 0$),*

$$m_k(n) \leqslant \frac{1}{2} n^{\rho_k}.$$

Proof. For every $r, s \geqslant 1$ we have (now setting $m_k(1) = 0$),

$$m_k(rs) \geqslant 2 m_k(r) m_k(s). \tag{6}$$

To show this inequality, we assume that $r, s \geqslant 2$ (for $r = 1$ or $s = 1$ it holds trivially) and consider the set X of all pairs (u, v) where u (v) is an ordered factorization of r (s) in factors lying in $\mathcal{N}_k \setminus \{1\}$, and the set Y of the same factorizations of rs. If u is $r = d_1 \cdot d_2 \cdot \ldots \cdot d_i$ and v is $s = e_1 \cdot e_2 \cdot \ldots \cdot e_j$, we define the factorizations of rs

$$F((u, v)) = d_1 \cdot d_2 \cdot \ldots \cdot d_i \cdot e_1 \cdot e_2 \cdot \ldots \cdot e_j,$$
$$G((u, v)) = d_1 \cdot d_2 \cdot \ldots \cdot d_{i-1} \cdot (d_i e_1) \cdot e_2 \cdot \ldots \cdot e_j.$$

The inequality (6) follows from the fact that the mappings F and G are injections from X to Y which moreover have disjoint images. We leave a simple verification of this fact to the reader.

Suppose now that $m_k(n_0) > n_0^{\rho_k}/2$ for some $n_0 \geqslant 2$. Then, for some small $\delta > 0$ we have that

$$m_k(n_0) > \frac{(1+\delta)}{2} n_0^{\rho_k}.$$

By repeated applications of inequality (6), we have that for each positive integer i

$$m_k\big(n_0^{2^i}\big) \geqslant 2\big(m_k\big(n_0^{2^{i-1}}\big)\big)^2 \geqslant 2^{1+2} m_k\big(n_0^{2^{i-2}}\big)^4 \geqslant \cdots \geqslant 2^{1+2+\cdots+2^{i-1}} m_k(n_0)^{2^i}$$
$$> \frac{(1+\delta)^{2^i}}{2} n_0^{2^i \rho_k}.$$

Let i be so large such that $(1+\delta)^{2^i} > 2$. Put $n_1 = n_0^{2^i}$. Then the above inequality implies that $m_k(n_1) > n_1^{\rho_k+\varepsilon}$ for some small $\varepsilon > 0$. Then, again by repeated applications of (6), we have $m_k(n_1^{2j}) \geqslant (n_1^{2j})^{\rho_k+\varepsilon}$ for every $j = 1, 2, \ldots$, which is in contradiction with $m_k(n) = o(n^{\rho_k+\varepsilon})$. $\square$

Lemma 2.5. *Suppose that $q_1, \ldots, q_k$ are primes, not necessarily distinct, such that the product $q_1 q_2 \cdots q_k$ divides n. Then, with $m(1) = 1$,*

$$m(n) < \bigl(2\Omega(n)\bigr)^k \cdot m(n/q_1 q_2 \cdots q_k). \tag{7}$$

Proof. It suffices to prove only the case $k = 1$; i.e., the inequality

$$m(n) < 2\Omega(n) \cdot m(n/p), \tag{8}$$

where p is a prime dividing n, because the general case follows easily by iteration. Let X be the set of all pairs (u, i) where u is an ordered factorization of n/p (in parts bigger than 1), and i is an integer satisfying $1 \leqslant i \leqslant 2r + 1$, where r is the number of parts in u. Let Y be the set of all ordered factorizations of n in parts bigger than 1. We shall define a surjection F from X onto Y. This will prove (8) because $r \leqslant \Omega(n/p) = \Omega(n) - 1$, and therefore for every u we have $2r + 1 < 2\Omega(n)$ pairs (u, i), and so

$$m(n) = |Y| \leqslant |X| < 2\Omega(n) \cdot m(n/p).$$

For $(u, i) \in X$, where u is $n/p = d_1 \cdot d_2 \cdot \ldots \cdot d_r$, we define $j = i - r$ and set $F((u, i))$ to be the factorization

$$n = d_1 \cdot \ldots \cdot d_{i-1} \cdot (p d_i) \cdot d_{i+1} \cdot \ldots \cdot d_r,$$

if $1 \leqslant i \leqslant r$ and

$$n = d_1 \cdot \ldots \cdot d_{j-1} \cdot p \cdot d_j \cdot \ldots \cdot d_r,$$

if $r + 1 \leqslant i \leqslant 2r + 1$ (for $j = 1$, p is the first part, and for $j = r + 1$ it is the last one). It is clear that F is a surjection. $\square$

Lemma 2.6. *If $n_1, n_2, \ldots, n_k$ are positive integers such that for no $i \neq j$ we have $n_i \mid n_j$, then*

$$m(n_1 n_2 \cdots n_k) \geqslant k! \cdot m(n_1) m(n_2) \cdots m(n_k).$$

This implies that for every $n \geqslant 1$ we have

$$m(n) \geqslant \omega(n)! \cdot 2^{\Omega(n)-\omega(n)} \quad \text{and} \quad m(n) \geqslant 2^{\Omega(n)-1}.$$

Proof. Let X be the set of all k-tuples $(u_1, u_2, \ldots, u_k)$, where u_i is an ordered factorization of n_i in parts bigger than 1 and let Y be the set of these factorizations for $n_1 n_2 \cdots n_k$. For every permutation σ of $1, 2, \ldots, k$, we define a mapping $F_\sigma : X \to Y$ by

$$F_\sigma((u_1, u_2, \ldots, u_k)) = u_{\sigma(1)} \cdot u_{\sigma(2)} \cdot \ldots \cdot u_{\sigma(k)},$$

i.e., we concatenate factorizations u_i in the order prescribed by σ. It is clear that each F_σ is an injection. Suppose that $F_\sigma((u_1, u_2, \dots, u_k)) = F_\tau((v_1, v_2, \dots, v_k))$ for some permutations σ, τ and factorizations u_i and v_i. It follows that $u_{\sigma(1)}$ is an initial segment of $v_{\tau(1)}$ or vice versa, and hence $n_{\sigma(1)}$ divides $n_{\tau(1)}$ or vice versa. This implies that $\sigma(1) = \tau(1)$ and $u_{\sigma(1)} = v_{\tau(1)}$. Applying the same argument, we obtain that $\sigma(j) = \tau(j)$ and $u_{\sigma(j)} = v_{\tau(j)}$ also for $j = 2, \dots, k$. Thus $\sigma = \tau$ and $u_j = v_j$ for $j = 1, 2, \dots, k$. We have proved that the $k!$ mappings F_σ have mutually disjoint images. Therefore

$$k! m(n_1) m(n_2) \cdots m(n_k) = k!|X| \leqslant |Y| = m(n_1 n_2 \cdots n_k).$$

If $n = q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k}$ is the prime factorization of n, applying the first inequality to the k numbers $n_i = q_i^{a_i}$ and using that $m(p^a) = 2^{a-1}$, we obtain

$$m(n) \geqslant k! \prod_{i=1}^{k} 2^{a_i - 1} = k! \cdot 2^{\Omega(n) - k},$$

which is the second inequality. Using that $k!/2^k \geqslant 1/2$ for every $k \geqslant 1$, we get the third inequality. □

Note that $m(n) \geqslant 2^{\Omega(n)-1}$ is tight for every $n = p^a$.

3. The upper bound

We prove the following upper bound on the maximal order of $m(n)$.

Theorem 3.1. *We have*

$$m(n) < \frac{n^\rho}{\exp((\log n)^{1/\rho} / (\log \log n)^{1+o(1)})}$$

as $n \to \infty$.

Proof. Let $\varepsilon > 0$ be given. To bound $m(n)$ from above, we split the integers $n > 0$ in two groups, those with $\omega(n) \leqslant k$ and those with $\omega(n) > k$, which we shall treat by different arguments; the optimal value of the parameter $k = k(n)$ will be selected in the end of the proof.

The case $\omega(n) \leqslant k$. Let $n = q_1^{a_1} q_2^{a_2} \cdots q_r^{a_r}$, $r \leqslant k$, be the prime decomposition of n where $q_1 < q_2 < \cdots < q_r$. We denote by $\bar{n}$ the number obtained from n by replacing q_i in the decomposition by p_i, the ith prime. Then $\bar{n} \leqslant n$. From the fact that $m(n)$ depends only on the exponents a_i and from Lemma 2.4, we get

$$m(n) = m(\bar{n}) = m_r(\bar{n}) < \bar{n}^{\rho_r} \leqslant n^{\rho_k}.$$

Thus, by Lemma 2.3,

$$\begin{aligned} m(n) &< n^{\rho_k} \\ &= n^\rho \exp\bigl(-(\rho - \rho_k) \log n\bigr) \\ &= n^\rho \exp\left(-\bigl(c + o(1)\bigr) \frac{\log n}{k^{\rho-1} (\log k)^\rho}\right), \end{aligned} \tag{9}$$

where $c = 2(\rho - 1)^{-1} |\zeta'(\rho)|^{-1} > 0$.

The case $\omega(n) > k$. Let $l(n)$ be the product of some k distinct prime factors of n; then $l(n) \geqslant p_1 p_2 \cdots p_k$, the product of the k smallest primes. We have the estimates

$$\sum_{p \leqslant p_k} \log p = p_k + O(p_k/\log p_k) = k\log k + k\log\log k + O(k),$$

and

$$2\Omega(n) \leqslant (2/\log 2)\log n < 3\log n.$$

By Lemmas 2.4, 2.5 and the above estimates,

$$\begin{aligned} m(n) &< \big(2\Omega(n)\big)^k m\big(n/\ell(n)\big) < (3\log n)^k \frac{n^\rho}{\ell(n)^\rho} \\ &\leqslant (3\log n)^k \frac{n^\rho}{(p_1\cdots p_k)^\rho} \\ &= n^\rho \exp\big(-k\big(\rho\log k + \rho\log\log k - \log\log n + O(1)\big)\big). \end{aligned} \tag{10}$$

To determine the best upper bound on $m(n)$, we begin with k in the form $k = k(n) = (\log n)^{\alpha+o(1)}$ where $\alpha \in (0,1)$ is a constant. Necessarily $\alpha \geqslant 1/\rho$, for else the argument of exp in (10) is eventually positive and we get a useless bound. It follows that the optimum is $\alpha = 1/\rho$, when the arguments of both exps in (9) and (10) are $-(\log n)^{1/\rho+o(1)}$, provided that

$$\rho\log k + \rho\log\log k - \log\log n + O(1) > c' > 0 \tag{11}$$

for all sufficiently large n. Now we set, more precisely,

$$k = k(n) = \left\lfloor \frac{(\log n)^{1/\rho}}{(\log\log n)^d} \right\rfloor$$

with a constant $d > 0$. With this k, the function in (11) becomes $\rho(1-d+o(1))\log\log\log n + O(1)$, and we see that condition (11) is satisfied for $d < 1$ (for $d > 1$ the argument of the exp in (10) is again eventually positive). With this k, the arguments of the exps in (9) and (10) are, respectively,

$$-\frac{(\log n)^{1/\rho}}{(\log\log n)^{1+(\rho-1)(1-d)+o(1)}} \quad\text{and}\quad -\frac{(\log n)^{1/\rho}}{(\log\log n)^{d+o(1)}}.$$

Setting $d = 1 - \varepsilon/(2(\rho-1))$, we obtain the stated bound with $1+\varepsilon+o(1)$ for the exponent of $\log\log n$. Since $\varepsilon > 0$ was arbitrary, letting n tend to infinity we get the desired estimate. □

4. The lower bound

We prove the following lower bound on the maximal order of $m(n)$.

Theorem 4.1. *There exists a constant $c > 0$ such that the inequality*

$$m(n) > \frac{n^\rho}{\exp(c(\log n/\log\log n)^{1/\rho})}$$

holds for infinitely many integers $n > 0$.

We shall see that it is possible to take $c = 3.02$. We begin with explaining the effective Ikehara–Ingham theorem on Dirichlet series. We then apply it to $1/(2 - \zeta_k(s))$ to obtain an asymptotic relation for the average order of $m_k(n)$ with an error estimate independent on k. Finally, combining this relation with an estimate on the density of smooth numbers, we obtain Theorem 4.1. For the background on Dirichlet series, we refer to Tenenbaum [27].

Suppose that $(a_n)_{n\geqslant 1}$ is a sequence of non-negative real numbers with the summatory function

$$A(t) = \sum_{n\leqslant e^t} a_n,$$

and the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \int_{0-}^{\infty} e^{-st}\,\mathrm{d}A(t).$$

Suppose that $F(s)$ converges for $\sigma > a > 0$. We may assume that a is the abscissa of (absolute) convergence; then, by the Phragmén–Landau theorem, a is a singularity of $F(s)$. The effective Ikehara–Ingham theorem, proved by Tenenbaum [27] (who used the method of Ganelius [5]), extracts an asymptotic relation for $A(x)$ as $x \to \infty$ from the local behavior of $F(s)$ near a and, moreover, it provides an explicit estimate of the error term in terms of the regularity of $F(s)$ on the vertical segments $a + \sigma + i\tau$, $-T \leqslant \tau \leqslant T$, as $\sigma \to 0+$. We quote the theorem verbatim from Tenenbaum [27, p. 234].

Theorem 4.2 *("Effective" Ikehara–Ingham). Let $A(t)$ be a non-decreasing function such that the integral*

$$F(s) := \int_0^{\infty} e^{-st}\,\mathrm{d}A(t)$$

converges for $\sigma > a > 0$. Suppose that there exist constants $c \geqslant 0$, $\omega > -1$, such that the function

$$G(s) := \frac{F(s+a)}{s+a} - \frac{c}{s^{\omega+1}} \quad (\sigma > 0)$$

satisfies

$$\eta(\sigma, T) := \sigma^{\omega} \int_{-T}^{T} \left|G(2\sigma + i\tau) - G(\sigma + i\tau)\right| \mathrm{d}\tau = o(1) \quad (\sigma \to 0+) \tag{12}$$

for each fixed $T > 0$. Then we have

$$A(x) = \left\{ \frac{c}{\Gamma(\omega+1)} + O\big(\rho(x)\big) \right\} e^{ax} x^{\omega} \quad (x \geqslant 1), \tag{13}$$

with

$$\rho(x) := \inf_{T \geqslant 32(a+1)} \big\{ T^{-1} + \eta(1/x, T) + (Tx)^{-\omega-1} \big\}.$$

Furthermore, the implicit constant in (13) *depends only on a, c, and ω. An admissible choice for this constant is*

$$52 + 1652c(a+1)(\omega+1) + 69c\big(1 + (\omega+1)e^{1-\omega}(\omega+1)^{\omega+2}\big)/\Gamma(\omega+1).$$

Note that for a meromorphic $F(s)$ with a simple pole at $s = a$ (so $\omega = 0$), the condition (12) is satisfied iff $F(s)$ has on the line $\sigma = a$ no other poles.

We shall apply Theorem 4.2 to the functions

$$F(s) = F_k(s) = \sum_{n \geqslant 1} \frac{m_k(n)}{n^s} = \frac{1}{2 - \zeta_k(s)}$$

for $k \geqslant 2$, $a = \rho_k$, $c = c_k = -1/\rho_k \zeta_k'(\rho_k)$, and $\omega = 0$. It is not hard to prove (we do this in the next proposition) that ρ_k is the only pole of $F_k(s)$ on $\sigma = \rho_k$ when $k \geqslant 2$ (this is not true for $k = 1$) and thus, by Theorem 4.2,

$$\sum_{n \leqslant x} m_k(n) = \big(c_k + o(1)\big) x^{\rho_k} \quad (x \to \infty)$$

for each fixed $k \geqslant 2$. (In contrast, $\sum_{n \leqslant x} m_1(n) = 2^r - 1$, where $2^r \leqslant x < 2^{r+1}$.) To get a good lower bound on $m(n)$, we have to strengthen this by obtaining uniformity in k of the error term $o(1)$. This follows from Theorem 4.2, once we prove that for $F(s) = F_k(s)$ the condition (12) is satisfied uniformly in k.

Proposition 4.3. *Let, for $k \geqslant 2$,*

$$G_k(s) = \frac{F_k(s + \rho_k)}{s + \rho_k} - \frac{c_k}{s} = \frac{1}{(2 - \zeta_k(s + \rho_k))(s + \rho_k)} - \frac{c_k}{s}$$

and $T > 0$ be arbitrary but fixed. Then

$$\lim_{\sigma \to 0+} \int_{-T}^{T} \big| G_k(2\sigma + i\tau) - G_k(\sigma + i\tau) \big| \, \mathrm{d}\tau = 0$$

uniformly in $k \geqslant 2$; that is, the condition (12) *holds uniformly in k.*

Proof. Let $t(\sigma) = \sigma^{1/5}$; any function $t(\sigma) > 0$ satisfying, as $\sigma \to 0+$, that $t(\sigma) \to 0$ and $\sigma / t(\sigma)^4 \to 0$ would do in our argument. For every fixed $T > 0$, we bound the integrand by a quantity that depends only on σ and not on τ and $k \geqslant 2$, and that goes to 0 as $\sigma \to 0+$; this will prove the statement. We manage to do this by splitting $[-T, T]$ in two ranges, $t(\sigma) \leqslant |\tau| \leqslant T$ and $|\tau| \leqslant t(\sigma)$, in which we apply different arguments.

The range $t(\sigma) \leqslant |\tau| \leqslant T$. Denoting by γ the horizontal segment with endpoints $\sigma + i\tau$ and $2\sigma + i\tau$, we have the bound

$$\bigl|G_k(2\sigma + i\tau) - G_k(\sigma + i\tau)\bigr| = \left|\int_\gamma G_k'(z)\,\mathrm{d}z\right| \leqslant \sigma \bigl|G_k'(s_0)\bigr|,$$

where s_0 is some point lying on γ. The derivative of $G_k(s)$ equals

$$G_k'(s) = \frac{(s+\rho_k)\zeta_k'(s+\rho_k) + \zeta_k(s+\rho_k) - 2}{(2-\zeta_k(s+\rho_k))^2 (s+\rho_k)^2} + \frac{c_k}{s^2}.$$

We bound the numerators and denominators of this expression. As for the numerators, by Lemma 2.2, there is a constant $c = c(T) > 0$ depending only on T such that

$$\bigl|(s+\rho_k)\zeta_k'(s+\rho_k) + \zeta_k(s+\rho_k) - 2\bigr|, \quad |c_k| < c$$

holds for every $k \geqslant 2$ and s with $0 < \sigma < 1$ and $|\tau| \leqslant T$. For the second denominator, we have, in our range and for $0 < \sigma < 1$,

$$\frac{\sigma}{|s_0|^2} \leqslant \frac{\sigma}{\sigma^2 + t(\sigma)^2} = \frac{\sigma^{3/5}}{\sigma^{8/5} + 1} < \sigma^{3/5}.$$

We bound the first denominator. Clearly, $|s + \rho_k|^2 \geqslant \rho_k^2 > 1$ for every s with $\sigma > 0$. For every $k \geqslant 2$ and every s with $0 < \sigma < 1$ and any τ, we have

$$\bigl|2 - \zeta_k(s+\rho_k)\bigr| \geqslant \mathrm{Re}\bigl(2 - \zeta_k(s+\rho_k)\bigr) = \sum_{\substack{n \geqslant 1 \\ P(n) \leqslant p_k}} \frac{1}{n^{\rho_k + \sigma}} \bigl(n^\sigma - \cos(\tau \log n)\bigr)$$

and, consequently (recall that $k \geqslant 2$ and $1 < \rho_k < 2$),

$$\bigl|2 - \zeta_k(s+\rho_k)\bigr|^2 > \left(\frac{2 - \cos(\tau\log 2) - \cos(\tau \log 3)}{27}\right)^2 =: h(\tau).$$

Since $2^\alpha = 3$ holds for no rational α, $h(\tau) = 0$ only for $\tau = 0$. The function $h(\tau)$ is continuous, increasing in a right neighborhood of 0, and even and $h(\tau) \sim \beta\tau^4$ as $\tau \to 0$ for a constant $\beta > 0$. Thus, there is a constant $\beta_1 = \beta_1(T) < 1$ depending only on T such that if $0 < \sigma < \beta_1$, then the minimum of $h(\tau)$ on $[t(\sigma), T]$ is attained at $t(\sigma)$ and $h(t(\sigma)) > \beta t(\sigma)^4/2$. Hence, in our range and for $0 < 2\sigma < \beta_1$,

$$\frac{\sigma}{|2 - \zeta_k(s_0+\rho_k)|^2 \cdot |s_0 + \rho_k|^2} < \frac{2\sigma}{\beta t(\sigma)^4} = \frac{2\sigma^{1/5}}{\beta}.$$

Taking together all estimates, we have in our range and for $0 < \sigma < \beta_1/2$ that

$$\left|G_k(2\sigma + i\tau) - G_k(\sigma + i\tau)\right| \leqslant \sigma\left|G_k'(s_0)\right| < c\left(2\sigma^{1/5}/\beta + \sigma^{3/5}\right),$$

which is the required bound.

The range $|\tau| \leqslant t(\sigma)$. We prove that there is an absolute constant $\delta > 0$ such that for every $k \geqslant 2$ and s with $|s| < \delta$ we have the expansion

$$G_k(s) = d_k + O(s),$$

where d_k is a constant and the constant implicit in O is absolute. (We need independence on k both for the constant in $O(s)$ and for the domain of validity of the error estimate.) Then if $0 < \sigma < \delta^5/32$ and $|\tau| \leqslant t(\sigma)$, both numbers $\sigma + i\tau$ and $2\sigma + i\tau$ satisfy $|s| < \delta$, and we have the bound

$$\left|G_k(2\sigma + i\tau) - G_k(\sigma + i\tau)\right| = O\left(|\sigma + i\tau| + |2\sigma + i\tau|\right) = O\left(\sigma^{1/5}\right)$$

with absolute constants in the Os, which is the required bound.

We begin with the origin-centered closed disc $B = B(0, 0.1)$; the point of the radius 0.1 is only that $\rho_2 - 0.1 > 1$. We define functions $f_k(s)$ by

$$f_k(s) = \frac{\zeta_k(s + \rho_k) - 2 - s\zeta_k'(\rho_k) - s^2\zeta_k''(\rho_k)/2}{s^3}.$$

Let a_k be the maximum value taken by $|\zeta_k(s)|$ on the circle $|s - \rho_k| = 0.1$. By the maximum modulus principle ($f_k(s)$ is holomorphic on B), for every $s \in B$ we have

$$\left|f_k(s)\right| \leqslant 10^3\left(a_k + 2 + 10^{-1}\zeta_k'(\rho_k) + 10^{-2}\zeta_k''(\rho_k)/2\right).$$

Thus, by Lemma 2.2, there is an absolute constant $M > 0$ such that

$$\left|f_k(s)\right| < M$$

holds for every $s \in B$ and every $k \geqslant 2$. We rewrite $\zeta_k(s + \rho_k) = 2 + s\zeta_k'(\rho_k) + s^2\zeta_k''(\rho_k)/2 + s^3 f_k(s)$ as

$$\begin{aligned}
\frac{1}{(2 - \zeta(s + \rho_k))(s + \rho_k)} &= -\frac{1}{s\rho_k\zeta_k'(\rho_k)} \times \frac{1}{1 + s/\rho_k} \\
&\quad \times \frac{1}{1 + s\zeta_k''(\rho_k)/2\zeta_k'(\rho_k) + s^2 f_k(s)/\zeta_k'(\rho_k)} \\
&= -\frac{1}{s\rho_k\zeta_k'(\rho_k)} \times \frac{1}{1 + s/\rho_k} \times \frac{1}{1 + sb_k + s^2h_k(s)}.
\end{aligned}$$

It follows, by Lemma 2.2 and the bound $|f_k(s)| < M$ valid on B, that there is a δ, $0 < \delta < 0.1$, such that $|s/\rho_k| < 1/2$ and $|sb_k + s^2h_k(s)| < 1/2$ whenever $|s| < \delta$ and $k \geqslant 2$. Using the estimate $(1 + s)^{-1} = 1 - s + O(s^2)$, valid for $|s| < 1/2$, and Lemma 2.2, we obtain for $k \geqslant 2$ and $|s| < \delta$ the expansion

$$\frac{1}{(2-\zeta_k(s+\rho_k))(s+\rho_k)} = \frac{c_k}{s}\left(1-\frac{s}{\rho_k}+O\big(s^2\big)\right)\left(1-s\frac{\zeta_k''(\rho_k)}{2\zeta_k'(\rho_k)}+O\big(s^2\big)\right)$$
$$= \frac{c_k}{s} - c_k\left(\frac{1}{\rho_k}+\frac{\zeta_k''(\rho_k)}{2\zeta_k'(\rho_k)}\right)+O(s),$$

where $c_k = -1/\rho_k\zeta_k'(\rho_k)$ and the constants in the Os are absolute. Now the required expansion $G_k(s) = d_k + O(s)$ (valid for $|s| < \delta$ and with an absolute constant in the O) is immediate. □

Corollary 4.4. *There is a constant $\beta_2 > 2$ such that for every $x > \beta_2$ and every $k \geqslant 2$ we have*

$$\sum_{\substack{n\leqslant x\\ P(n)\leqslant p_k}} m(n) = \sum_{n\leqslant x} m_k(n) > x^{\rho_k}/5.$$

Proof. By Theorem 4.2 and Proposition 4.3, there is a function $e(x) > 0$ such that $e(x) \to 0$ as $x \to \infty$, and for every $x \geqslant 1$ and every $k \geqslant 2$ we have

$$\left|\sum_{n\leqslant x} m_k(n) - c_k x^{\rho_k}\right| < e(x)x^{\rho_k}.$$

The sequence of $c_k = -1/\rho_k\zeta_k'(\rho_k)$, $k = 1, 2, \ldots$, monotonically decreases and converges to $c_\infty = \phi = -1/\rho\zeta'(\rho) > 0.3$. Thus, if x is big enough so that $e(x) < 0.1$, then the sum $\sum_{n\leqslant x} m_k(n)$ must be bigger than $0.2x^{\rho_k}$. □

We now proceed to the proof of Theorem 4.1. We denote, as usual,

$$\Psi(x, y) = \#\big\{n \leqslant x\colon\ P(n) \leqslant y\big\}.$$

By Corollary 4.4, for every $k \geqslant 2$ and $x > \beta_2$ there exists an $n_0 \leqslant x$ such that

$$\Psi(x, p_k)m(n_0) > \frac{x^{\rho_k}}{5} = \frac{x^{\rho}}{5\exp((\rho-\rho_k)\log x)}.$$

We select $k = k(x)$ so that it satisfies

$$k = (\log x)^{\alpha+o(1)}$$

as $x \to \infty$, for some absolute constant $\alpha \in (0, 1)$ (we make our choice of k more precise later). Then

$$p_k = \big(1 + o(1)\big)k\log k = (\log x)^{\alpha+o(1)}.$$

A theorem due to de Bruijn (see Theorem 2 in Tenenbaum's book [27, p. 359]), shows that

$$\log\big(\Psi(x, p_k)\big) = \big(1 + o(1)\big)Z,$$

where

$$\begin{aligned} Z &= \frac{\log x}{\log p_k}\log\left(1+\frac{p_k}{\log x}\right)+\frac{p_k}{\log p_k}\log\left(1+\frac{\log x}{p_k}\right)\\ &= \frac{p_k}{\log p_k}\bigl(1+o(1)\bigr)+\frac{p_k}{\log p_k}\log\left(1+\frac{\log x}{p_k}\right)\\ &= \bigl(1+o(1)\bigr)k(\log\log x-\log k). \end{aligned}$$

By Lemma 2.3,

$$\rho-\rho_k=\frac{c_1+o(1)}{k^{\rho-1}(\log k)^{\rho}},$$

where $c_1=2/((\rho-1)|\zeta'(\rho)|)$. Substituting both estimates in the lower bound on $\Psi(x,p_k)m(n_0)$, we get (absorbing the 5 in the denominator in the $o(1)$ terms),

$$m(n_0)>\frac{x^{\rho}}{\exp(c_1(1+o(1))\frac{\log x}{k^{\rho-1}(\log k)^{\rho}}+(1+o(1))k(\log\log x-\log k))}.$$

This suggests to choose k so that both terms in the argument of the exponential,

$$\frac{\log x}{k^{\rho-1}(\log k)^{\rho}}\quad\text{and}\quad k(\log\log x-\log k),$$

are of the same order of magnitude. This occurs when $\alpha=1/\rho$, more precisely when

$$k=\bigl\lfloor d(\log x)^{1/\rho}(\log\log x)^{-(\rho+1)/\rho}\bigr\rfloor$$

with any constant $d>0$, because then

$$\frac{\log x}{k^{\rho-1}(\log k)^{\rho}}=d^{1-\rho}\rho^{\rho}\bigl(1+o(1)\bigr)\left(\frac{\log x}{\log\log x}\right)^{1/\rho},$$

and

$$k(\log\log x-\log k)=\bigl(1-\rho^{-1}\bigr)d\bigl(1+o(1)\bigr)\left(\frac{\log x}{\log\log x}\right)^{1/\rho}.$$

Thus, for this selection of k,

$$m(n_0)>\frac{x^{\rho}}{\exp((c+o(1))(\frac{\log x}{\log\log x})^{1/\rho})},$$

where $c>0$ is a constant depending only on the choice of d. The lower bound eventually increases monotonically to infinity, and we conclude that there exist infinitely many numbers n_0 satisfying

$$m(n_0)>\frac{n_0^{\rho}}{\exp((c+o(1))(\frac{\log n_0}{\log\log n_0})^{1/\rho})}.$$

The proof of Theorem 4.1 is complete.

It is not difficult to find the optimal value of d; it yields the value

$$c = \big(\rho^{\rho+1}c_1\big)^{1/\rho} = \left(\frac{2\rho^{\rho+1}}{(\rho-1)|\zeta'(\rho)|}\right)^{1/\rho} \approx 3.01091.$$

5. Historical remarks and arithmetical properties of $m(n)$

We continue with a survey of some previous results on $m(n)$. We restrict our attention only to works dealing directly with this quantity. There are many other variants of factorization counting functions (with restrictions on factors, counting unordered factorizations, etc.), and for a survey on these we refer the reader to Knopfmacher and Mays [16].

Kalmár proved in [14] that the error term $o(1)$ in (1) is

$$O\big(\exp(-\alpha \log\log x \cdot \log\log\log x)\big), \quad \text{for any } \alpha < \frac{1}{2(\rho-1)\log 2} \approx 0.98999.$$

Ikehara devoted three papers to the estimates of $M(x)$. In [10], he gave weak bounds of the type $M(x) > x^{\rho-\varepsilon}$ on a sequence of x tending to infinity, and $M(x) < x^{\rho+\varepsilon}$ for all large enough x. In the review of [10], Kalmár pointed out a gap in the proof and sketched a correct argument. In [11], Ikehara gave a proof of (1) with an error bound $O(\exp(q \log\log x))$ for some constant $q < 0$, which is slightly weaker than Kalmár's result. Finally, in [12], he succeeded to get a stronger error bound

$$O\big(\exp\big(-\alpha(\log\log x)^{\gamma}\big)\big), \quad \text{for } 0 < \alpha < 1/2 \text{ and any } \gamma < 4/3.$$

Hwang [9] obtained an improvement of Ikehara's last bound by replacing 4/3 with 3/2.

Rieger proved in [23], besides other results, that for all positive integers k, l with $(k,l) = 1$ one has

$$\sum_{n \leqslant x,\, n \equiv l\,(k)} m(n) = \frac{1+o(1)}{\varphi(k)} M(x) = \frac{-1}{\varphi(k)\rho\zeta'(\rho)} \cdot x^{\rho}\big(1+o(1)\big).$$

Warlimont investigated in [28] variants of $m(n)$ counting ordered factorizations with distinct parts and with coprime parts and estimated their summatory functions. Hille in [8] proved that $m(n) = O(n^{\rho})$ and that $m(n) > n^{\rho-\varepsilon}$ for infinitely many n. We already mentioned in Section 1 the remark of Erdős on $m(n)$ in [4] and we mentioned (and improved) the result of Chor, Lemke and Mador [1] that $m(n) < n^{\rho}$ for all n. Other elementary and constructive proofs of the bounds $m(n) \leqslant n^{\rho}$ and $\limsup_n m(n)/n^{\rho-\varepsilon} = \infty$ were recently given by Coppersmith and Lewenstein [3].

We now turn to recurrences and explicit formulas. The recurrence $m(1) = 1$ and

$$m(n) = \sum_{d \mid n,\, d<n} m(d) \quad \text{for } n > 1 \tag{14}$$

is immediate from fixing the first part in a factorization. If we set $m^*(1) = 1/2$ and $m^*(n) = m(n)$ for $n > 1$, then $2m^*(n) = \sum_{d \mid n} m(d)$ holds for all $n \geqslant 1$. By Möbius inversion, $m(n) =$

$2\sum_{d|n}\mu(d)m^*(n/d)$ for all $n \geqslant 1$. For $n = q_1^{a_1}q_2^{a_2}\cdots q_r^{a_r} > 1$, this can be rewritten as the recurrence formula

$$m(n) = 2\left(\sum_i m\left(\frac{n}{q_i}\right) - \sum_{i<j} m\left(\frac{n}{q_i q_j}\right) + \cdots + (-1)^{r-1} m\left(\frac{n}{q_1 q_2 \cdots q_r}\right)\right), \tag{15}$$

in which we must set $m(1) = 1/2$. Formulas (14) and (15) are from Hille's paper [8]. In fact, (15) is stated there incorrectly with $m(1) = 1$, as was pointed out by Kühnel [17] and Sen [24].

Clearly, $m(p^a) = 2^{a-1}$ because ordered factorizations of p^a in parts > 1 are in bijection with (additive) compositions of a in parts > 0. If $p \neq q$ are primes and $a \geqslant b \geqslant 0$ are integers, we have the formula

$$m\left(p^a q^b\right) = 2^{a+b-1}\sum_{k=0}^{b}\binom{a}{k}\binom{b}{k}2^{-k}$$

that was derived in [1] and before by Sen [24] and MacMahon [21]. In particular,

$$m\left(p^a q\right) = (a+2)2^{a-1} \quad \text{and} \quad m\left(p^a q^2\right) = \left(a^2 + 7a + 8\right)2^{a-2}. \tag{16}$$

In general, for $n = q_1^{a_1}q_2^{a_2}\cdots q_r^{a_r}$, and $a = a_1 + a_2 + \cdots + a_r$, MacMahon [21] derived the formula

$$m\left(q_1^{a_1}q_2^{a_2}\cdots q_r^{a_r}\right) = \sum_{j=1}^{a}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}\prod_{k=1}^{r}\binom{a_k + j - i - 1}{a_k}.$$

A more complicated summation formula for $m(q_1^{a_1}q_2^{a_2}\cdots q_r^{a_r})$ but involving only non-negative summands was obtained by Kühnel in [17,18]. Let $d_k(n)$ be the number of solutions of $n = n_1 n_2 \cdots n_k$, where $n_i \geqslant 1$ are positive integers; so $d_2(n)$ is the number of divisors of n. Sklar [25] mentions the formula

$$m(n) = \sum_{k=1}^{\infty}\frac{d_k(n)}{2^{k+1}}. \tag{17}$$

Somewhat surprisingly, $m(n)$ has an additive definition in terms of integer partitions. We say that a partition $(1^{a_1}, 2^{a_2}, \ldots, k^{a_k})$ of n is *perfect*, if for every $m < n$ there is exactly one k-tuple $(b_1, \ldots, b_k)$, $0 \leqslant b_i \leqslant a_i$ for all i, such that $(1^{b_1}, 2^{b_2}, \ldots, k^{b_k})$ is a partition of m. MacMahon [19] proved the identity

$$m(n) = \#\text{ perfect partitions of } (n-1).$$

For example, since $m(12) = 8$, we have 8 perfect partitions of 11, namely $(1^2, 3, 6)$, $(1, 2^2, 6)$, $(1^5, 6)$, $(1, 2, 4^2)$, $(1^3, 4^2)$, $(1^2, 3^3)$, $(1, 2^5)$, and (1^{11}).

In the conclusion of the survey of previous results, we should remark that from an enumerative point of view it is natural to consider $m(n)$ as a function of the partition $\lambda = (a_1, a_2, \ldots, a_k)$ of $\Omega(n)$, where $n = q_1^{a_1}q_2^{a_2}\cdots q_k^{a_k}$ with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k$, rather than n. Then $m(\lambda)$ is defined as the number of ways to write $\lambda = v_1 + v_2 + \cdots + v_t$, where each v_i is a k-tuple of non-negative integers, the order of summands matters, and no v_i is a zero vector. So $m(\lambda)$ is naturally understood as the number of k-dimensional compositions of λ. This approach was pursued by MacMahon in his memoirs [19–21] (see also [22]).

The sequence

$$\big(m(n)\big)_{n\geqslant 1} = (1, 1, 1, 2, 1, 3, 1, 4, 2, 3, 1, 8, 1, 3, 3, 8, 1, 8, 1, 8, 3, 3, 1, 20, 2, \ldots)$$

forms entry A074206 of the database [26]. Continuing the sequence a little further, we notice that $m(48) = 48$ and that $n = 48 = 2^4 \cdot 3$ is the smallest $n > 1$ such that $m(n) = n$. The first formula in (16) produces infinitely many n with this property: setting $n = 2^{2q-2}q$ with a prime $q > 2$, we get $m(n) = n$. We record this observation as follows:

Proposition 5.1. *There exist infinitely many positive integers n such that $m(n) = n$.*

This result was obtained independently also by Knopfmacher and Mays [16].

We look at periodicity properties of the numbers $m(n)$. The recurrence (15) implies easily the following result.

Proposition 5.2. *The number $m(n)$ is odd if and only if n is square-free.*

It would be interesting to characterize the behavior of $m(n)$ with respect to other moduli besides 2. In the next proposition, we give a partial result in this direction. Recall that an integer valued function $f(n)$ defined on the set of positive integers is called *eventually periodic* modulo k if there exist integers n_0 and T such that $f(n) \equiv f(n+T) \pmod{k}$ for all $n > n_0$. We show that $m(n)$ is not eventually periodic modulo k by proving a stronger result that $m(n)$ is not eventually constant modulo k on any infinite arithmetic progression with coprime difference and first term.

Proposition 5.3. *The function $m(n)$ is not eventually constant modulo k, where $k \geqslant 2$, on any infinite arithmetic progression $n \equiv A \pmod{K}$, $K \geqslant 2$, with coprime A and K.*

Proof. By Dirichlet's theorem, this arithmetic progression contains infinitely many prime numbers and therefore $m(n) = 1$ for infinitely many $n \equiv A \pmod{K}$. We select a prime q not dividing K and an integer z (coprime with K) such that $qz \equiv A \pmod{K}$. Since there are infinitely many prime numbers congruent to z modulo K, there are also infinitely many $n \equiv A \pmod{K}$ of the form qp, where p is a prime. Thus, there are infinitely many $n \equiv A \pmod{K}$ with $m(n) = 3$. Because $1 \not\equiv 3 \pmod{k}$ for $k > 2$, we are done if $k > 2$. For $k = 2$, $m(n) \equiv 1 \pmod{2}$ for infinitely many $n \equiv A \pmod{K}$ as before. As we noted, $m(n)$ is even iff n is not square-free. It follows that $m(n) \equiv 0 \pmod{2}$ for infinitely many $n \equiv A \pmod{K}$ as well, which settles the case $k = 2$. □

For $(A, K) > 1$, Proposition 5.3 in general does not hold. For example, by Proposition 5.2 we have $n \equiv 4 \pmod{8} \Rightarrow m(n) \equiv 0 \pmod{2}$ and therefore $m(n)$ is constantly 0 modulo 2 on the progression $n \equiv 4 \pmod{8}$.

Recall now that a sequence $(f(n))_{n\geqslant 1}$ is holonomic if there exist positive integer polynomials $g_0, \ldots, g_k$, not all zero, such that

$$g_k(n)f(n+k) + g_{k-1}(n)f(n+k-1) + \cdots + g_0(n)f(n) = 0 \quad \text{for all } n \geqslant 1. \tag{18}$$

Proposition 5.4. *The sequence $m(n)$ is not holonomic.*

Proof. Dividing (18) by one of the (non-zero) coefficients g_j with the largest degree, we obtain the relation

$$f(n+j) = \sum_{0\leqslant i\leqslant k, i\neq j} h_i(n) f(n+i),$$

where the h_i's are rational functions such that each $h_i(x)$ goes to a finite constant c_i as $x \to \infty$ (we may even assume that $|c_i| \leqslant 1$ for every i). Hence there is a constant $C > 0$ (depending only on k and the polynomials g_i), such that

$$\bigl|f(n)\bigr| \leqslant C \max\bigl\{\bigl|f(n+i)\bigr|\colon -k \leqslant i \leqslant k,\ i \neq 0\bigr\} \quad \text{for every } n \geqslant k+1.$$

We show that $(m(n))_{n\geqslant 1}$ violates this property.

We fix two integers $k, a \geqslant 1$ with the only restriction that a is coprime to each of the numbers $1, 2, \ldots, k$. It is an easy consequence of the Fundamental Lemma of the Combinatorial Sieve (see [6]) that there is a constant $K > 0$ depending only on k so that

$$\Omega\bigl((an-k)(an-k+1)\cdots(an-1)(an+1)\cdots(an+k)\bigr) \leqslant K$$

holds for infinitely many integers $n \geqslant 1$. For each of these n's, the $2k$ values $m(an+i)$, $-k \leqslant i \leqslant k$ and $i \neq 0$, are bounded by a constant (depending only on k) while the value $m(an)$ is at least $m(a)$ and can be made arbitrarily large by an appropriate selection of a. This contradicts the above property of holonomic sequences. □

Remark 5.5. The above proof can be adapted in a straightforward way to show that other number theoretical functions such as $\omega(n)$, $\Omega(n)$ and $\tau(n)$, where $\tau(n)$ is the number of divisors of n, are not holonomic.

We present two more estimates related to the function $m(n)$.

Proposition 5.6. *The estimate*

$$\#\bigl\{m(n)\colon n \leqslant x\bigr\} \leqslant \exp\bigl(\pi\sqrt{2/\log 8}\bigl(1+o(1)\bigr)(\log x)^{1/2}\bigr)$$

holds as $x \to \infty$.

Proof. Because $m(n)$ depends only on the partition $a_1 + \cdots + a_k = \Omega(n)$, where $n = q_1^{a_1}\cdots q_k^{a_k}$ ($q_1, \ldots, q_k$ are distinct primes and $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k > 0$ are integers), we have that

$$\#\bigl\{m(n)\colon n \leqslant x\bigr\} \leqslant p(1) + p(2) + \cdots + p(r) \leqslant rp(r),$$

where $p(n)$ denotes the number of partitions of n and $r = \max_{n\leqslant x} \Omega(n)$. The result follows from $r \leqslant \log x/\log 2$ and the classic asymptotic relation $p(n) \sim \exp(\pi\sqrt{2n/3})/(4n\sqrt{3})$ due to Hardy and Ramanujan [7]. □

We show that the same bound on the number of distinct values of $m(n)$ holds when the condition $n \leqslant x$ is replaced with $m(n) \leqslant x$.

Proposition 5.7. *The estimate*

$$\#\{m(n)\colon\ m(n) \leqslant x,\ n \geqslant 1\} \leqslant \exp\bigl(\pi\sqrt{2/\log 8}\bigl(1+o(1)\bigr)(\log x)^{1/2}\bigr)$$

holds as $x \to \infty$.

Proof. As in Proposition 5.6, we have

$$\#\{m(n)\colon\ m(n) \leqslant x,\ n \geqslant 1\} \leqslant p(1) + p(2) + \cdots + p(r) \leqslant rp(r),$$

where now $r = \max_{m(n)\leqslant x} \Omega(n)$. By the third inequality in Lemma 2.6, $2^{r-1} = 2^{\Omega(n)-1} \leqslant m(n) \leqslant x$ for some n. Thus, $r \leqslant 1 + \log x/\log 2$, and the result follows as in the proof of Proposition 5.6 using the asymptotics of $p(n)$. □

In conclusion we mention some research directions. It would be nice to gain more information on the modular behavior of $m(n)$. There is still a small gap between our bounds on the maximal order—can one lower the exponent $1 + o(1)$ in Theorem 3.1 to $1/\rho + o(1)$? What can be said about the structure of highly factorable numbers, i.e., numbers satisfying $m(n) > m(u)$ for all u, $1 \leqslant u < n$?

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