The Euler characteristics of $\mathcal{H}_{g,n}$

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Abstract

In this short note, we compute the orbifold and the ordinary Euler characteristic of $\mathcal{H}_{g,n}$, the moduli space of pointed hyperelliptic curves. As a by-product, we obtain an identity involving hypergeometric functions.

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1. Introduction

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of $n$-pointed genus $g$ stable curves. As well known, it has the structure of an analytic orbifold of complex dimension $3g - 3 + n$. The Euler orbifold and ordinary characteristic of $\overline{\mathcal{M}}_{g,n}$ and that of the open set $\mathcal{M}_{g,n}$ are calculated in [3] and [6].

The geometry of $\mathcal{M}_{g,n}$ is quite rich and fascinating, but the topology of its subvarieties is quite unexplored. Recently, some progress on the locus of curves of compact type has been made in [4] and [7].

A remarkable sublocus of the moduli space of curves is given by the subvariety parametrizing hyperelliptic curves, which is usually denoted by $\mathcal{H}_g$. It has the homotopy type of a point for any $g \geq 2$ and it is known to be rational (see [5]). Note that $\mathcal{H}_g$ stays rational for any $g$ whereas $\mathcal{M}_g$ becomes of general type for $g$ big enough. Moreover, $\mathcal{H}_g$ is affine and isomorphic to the quotient of $\mathcal{M}_{0,2g+2}$ under the action of the symmetric group $\mathfrak{S}_{2g+2}$, which permutes the $2g + 2$ marked points. We recall that $\mathcal{M}_{0,2g+2}$ is the moduli space of $(2g + 2)$-pointed (smooth) rational curves.

In this short note we compute the orbifold and the ordinary characteristic of $\mathcal{H}_{g,n}$. The orbifold characteristic is easily computed by comparing it to the orbifold characteristic of the orbifold $\overline{\mathcal{M}}_{0,2g+2}/\mathfrak{S}_{2g+2}$ and, after that, by using the ‘forgetful’ morphism, which is a fibration in the category of (analytic) orbifolds. As for the ordinary Euler characteristic, we first describe a stratification of $\mathcal{H}_{g,n}$ along the lines of the stratification used in [2] for the moduli space $\mathcal{M}_{2,n}$. Each stratum can be described as the covering space of a quotient of the moduli space $\mathcal{M}_{0,n}$ for suitable $n$. In some cases, such a covering map is ramified. We therefore determine the Euler characteristic of the ramification locus. Noticeably, the ordinary Euler characteristic of the hyperelliptic locus can be expressed in terms of the hypergeometric function.
of hypergeometric functions. As a by-product, we obtain an identity involving hypergeometric functions, which was not previously known.

In Section 2 we focus on the orbifold characteristic. In Section 3 we describe the stratification of \( H_{g,n} \) and give formulas for the Euler characteristic. In particular, we deduce that \( H_{g,n} \) has the homotopy type of a point if and only if \( n = 0 \). In [1], cohomological information about the space \( H_{g,n} \) is given via curves counting over finite fields. We focus on the complex case and obtain closed formulas for any \( g \) and any \( n \). Our methods yield an unexpected connection between hyperelliptic curves and hypergeometric functions.

2. The orbifold Euler characteristic

Let \( S_{2g+2} \) the symmetric group of order \((2g+2)!\). There is a natural action of \( S_{2g+2} \) on the moduli space \( M_{0,2g+2} \) that permutes the marked points. Since any hyperelliptic curve is a degree 2 covering of \( \mathbb{P}^1 \), there is a natural map \( \varphi \) from \( H_{g} \) to the quotient orbifold \( M_{0,2g+2}/\mathbb{S}_{2g+2} \). If we regard these two spaces as varieties, the map \( \varphi \) is clearly an isomorphism. Nonetheless, the orbifold Euler characteristic of \( H_{g} \) does not equal that of \( M_{0,2g+2}/\mathbb{S}_{2g+2} \). Indeed, the following holds.

**Proposition 1.** The orbifold Euler characteristic of \( H_{g} \) is given by

\[
\chi(H_{g}) = -\frac{1}{8g(g+1)(2g+1)}.
\]

**Proof.** Let \([\mathbb{P}^1; p_1, \ldots, p_n]\) be a marked curve in \( M_{0,2g+2}/\mathbb{S}_{2g+2} \). Let \( x_0 \) be a point in \( \mathbb{P}^1 \setminus \{p_1, \ldots, p_n\} \) and consider the monodromy representation from \( \pi_1(\mathbb{P}^1 \setminus \{p_1, \ldots, p_n\}, x_0) \) to \( \mathbb{Z}/2\mathbb{Z} \). This representation yields a covering of \( \mathbb{P}^1 \) up to the labeling of the sheets. Moreover, there is a unique possibility which gives the hyperelliptic involution. Thus, the preimage of \([\mathbb{P}^1; p_1, \ldots, p_n]\) under \( \varphi \) is a point in \( H_{g} \) with isotropy group \( \mathbb{Z}/2\mathbb{Z} \). Accordingly, the orbifold Euler characteristic of \( H_{g} \) is given by

\[
\chi(H_{g}) = \frac{1}{2} \chi(M_{0,2g+2}/\mathbb{S}_{2g+2}) = \frac{(-1)^{2g-1}(2g-1)!}{2(2g+2)!},
\]

so the claim follows. \( \square \)

Let \( \pi: H_{g,n} \to H_{g} \) be the morphism which forgets the marked points and stabilizes the target curve. We recall that \( \pi \) is an orbifold fibration; hence it is easy to prove that the following holds.

**Corollary 1.** The orbifold Euler characteristic of \( H_{g,n} \) is given by

\[
(-1)^{n+1} \frac{(2g+n-3)!}{2(2g+2)!}.
\]

3. The Euler–Poincaré characteristic

In this section we first describe a stratification of \( H_{g,n} \) which generalizes the one described in [2].

Let \( \varphi: H_{g,n} \to \overline{M}_{n+2g+2}/\mathbb{S}_{2g+2} \) be the morphism defined as follows. Let \([C]\) be the point parametrizing the \( n \)-pointed hyperelliptic curve \((C; p_1, \ldots, p_n)\). Then, \( \varphi([C]) \in \overline{M}_{n+2g+2}/\mathbb{S}_{2g+2} \) is the quotient rational curve \( C/\tau \), where \( \tau \) denotes the hyperelliptic involution, marked by the (unordered) images of the Weierstrass points of \( C \) and by the (ordered) images of the points \( p_1, \ldots, p_n \). A rational tail is thus added to separate any two marked points when they coalesce. This morphism is clearly non-surjective.

The moduli space \( \overline{M}_{0,n+2g+2}/\mathbb{S}_{2g+2} \) is naturally stratified by topological type. The general element of the strata in \( \varphi(H_{g,n}) \) can be described as follows. Let \( j \) and \( r \) be positive integers such that \( 0 \leq j \leq \min(n, 2g+2) \) and \( 0 \leq r \leq [(n-j)/2] \). Then, there are \( j + r \) two-pointed rational tails which are attached to an \( n + 2g + 2 - j - r \) rational curve. If we pull-back the natural stratification via \( \varphi \), we obtain a stratification on \( H_{g,n} \).
A more explicit description of the strata is given by the quasi-projective subvarieties $U_{j,r}$ defined in the following way:

$$U_{j,r} := \{(C; p_1, \ldots, p_n) \mid \tau(p_i) = p_i, \ i = 1, \ldots, j, \ \tau(p_{j+2i}) = p_{j+2i-1}, \ i = 1, \ldots, r, \ \tau(p_i) \neq p_k \text{ otherwise}\},$$

where $\tau$ is the hyperelliptic involution. In other words, $U_{j,r}$ parametrizes $n$-pointed curves such that $j$ marked points are fixed by $\tau$ and $r$ marked points are exchanged by $\tau$. As remarked in [2], permuting the marked points yield various subvarieties isomorphic to $U_{j,r}$ for fixed $(j, r)$. It is easy to check that there are

$$a_{j,r} = \binom{n}{j} \frac{(n-j)!}{2^r (n-j-2r)! r!}$$

strata isomorphic to $U_{j,r}$. Analogously to the genus two case (see [2]), there exists a map $f_{j,r} : U_{j,r} \to M_{0,n+2g+2-r-j}/\Theta_{2g+2-j}$. If $j = n$ and $r = 0$, then $f_{n,0}$ is an isomorphism; otherwise it is easy to see that it is a degree $2^{n-j-r-1}$ covering of the target space.

**Lemma 1.** The maps $f_{j,r}$ are ramified coverings if and only if $j = 0$ and $r = n - 2$. Let $d$ be a divisor of $2g + 2$. The ramification locus is isomorphic to $M_{0,d+1}/\Theta_d$ for $n = 3$ and to $M_{0,d}/\Theta_d$ for $n = 2, 4$.

**Proof.** Fix a branch point in the target space of $f_{j,r}$, namely a rational curve with $n + 2g + 2 - r - j$ marked points. By definition of the map $f_{j,r}$, any ramification point comes equipped with an automorphism $\sigma$, which induces an automorphism $\sigma'$ of the curve corresponding to the branch point. The automorphism $\sigma'$ fixes two points, so it induces a cyclic map of the smooth rational curve. Hence, if $j > 0$ or $n - r > 2$, $\sigma'$ fixes at least three marked points and is the identity. Accordingly, $\sigma$ is the identity too.

On the other hand, if $j = 0$ and $n - r = 2$, the map $f_{j,r}$ is ramified. By the constraints on $j$ and $r$ there is ramification only in a few cases, namely $n = 2, 3, 4$. We prove the claim in the case $n = 2$, which means $r = 0$. The remaining cases can be dealt with analogously. Any ramification point of $f_{0,0}$ fixes a marked point, say $p_1$, and maps the other one, say $p_2$, to $\tau(p_2)$. Moreover it is easy to check that $\sigma$ and $\tau$ commute. Clearly, $\sigma$ has finite order since it permutes the ramification points of $C$. Suppose that $\sigma$ has order $l$ and denote by $G$ the group generated by $\sigma$. The map $\psi : C/\tau \to C/(G, \tau)$ is a morphism of $\mathbb{P}^1$ onto itself. The points $p_1$ and $p_2$ are total ramification points; by the Riemann–Hurwitz formula they are the only ramification points. Hence the map $\psi$ induces a cyclic automorphism of $\mathbb{P}^1$ onto itself of the form $x \mapsto ax$, where $a^l = 1$ that fixes $0$ and $\infty$. The Weierstrass points of $C$ are grouped into $((2g + 2)/l = d$ orbits. Thus, a ramification point $(C; p_1, p_2)$ corresponds to an element of $M_{0,d}/\Theta_d$, where $d$ is a divisor of $2g + 2$. This holds for $n = 2$ as well as for $n = 4$ since the four marked points form two pairs that are exchanged by the hyperelliptic involution. For $n = 3$ there is just one extra point besides $0$ and $\infty$. \[\square\]

**Remark 1.** In [2] we compute the Euler characteristic of the ramification locus in a slightly different way, which works out to be right when $g = 2$. The approach used in Lemma 1 works for the general case.

**Proposition 2.** The Euler–Poincaré characteristic of $H_{2g,n}$ is equal to $1, 2, 2, 0$ and $-2g$ for $n = 0, 1, 2, 3, 4$, respectively.

**Proof.** For $n = 1$, the Euler characteristic is given by

$$a_{0,0}e(U_{0,0}) + a_{1,0}e(U_{1,0}) = e(M_{0,2g+3}/\Theta_{2g+2}) + e(M_{0,2g+2}/\Theta_{2g+1}) = 2.$$

As for $n = 2$, we have

$$e(H_{2g,2}) = a_{0,0}e(U_{0,0}) + a_{0,1}e(U_{0,1}) + a_{1,0}e(U_{1,0}) + a_{2,0}e(U_{2,0}).$$

By Lemma 1, the Euler characteristic of $U_{0,0}$ is given by

$$e(U_{0,0}) = 2e(M_{0,2g+4}/\Theta_{2g+2}) - 1 = -1.$$

Hence the claim follows.
Theorem 1. Let \( n \) be a positive integer such that \( 5 \leq n \leq 2g + 2 \). Then the Euler-Poincaré characteristic of \( \mathcal{H}_{g,n} \) is given by

\[
e(\mathcal{H}_{g,n}) = -\left(\frac{2g}{2(2g+2)!}\right)^n \left(\frac{(2g-1)!}{(2g+n-3)!}\right) - \left(\frac{2g}{2(2g+2)!}\right)^{n-1} \left(\frac{(2g-1)!}{(2g+n-3)!}\right)
\]

By direct computation, the following holds.

Proposition 3. For \( n \geq 2g + 3 \) we have

\[
e(\mathcal{H}_{g,n}) = (-1)^{n+1} \left(\frac{(2g+n-3)!}{2(2g+2)!}\right).
\]

As a by-product of Theorem 1, Corollary 1 and Maple calculations, we obtain an unexpected numerical identity. In order to state it, we recall basic definitions of special functions. Let \( \gamma a_{j,1}, \ldots, a_{p,1}; b_1, \ldots, b_q; \gamma z \) be the hypergeometric function in the variable \( z \). As usual, denote by \( \Gamma(x) \) the Euler Gamma function. Furthermore, recall the definition of LegendreP\((a, b, z)\), which can be expressed in terms of hypergeometric functions in the following way:

\[
\text{LegendreP}(a, b, z) = \frac{(z + 1)^{b/2} F_1(-a, a + 1; 1 - b, \frac{1-z}{2})}{(z - 1)^{b/2} \Gamma(1 - b)}.
\]
Corollary 2. For any positive integer $g$ and $n$ such that $n \geq 2g + 3$ the following identity holds:

\[
(-1)^{n+1} \frac{(2g + n - 3)!}{2(2g + 2)!} + \frac{(-2)^n n!}{4(2g + 1)!} \left\{ \frac{(2g + 1)!}{2(2g + 2)!} \left( \frac{2g + n - 3}{n - 4} \right) - (2g + 1)! \frac{2g + n - 4}{n - 6} \right\} \\
\times \frac{3}{2} F_2(1, 3 - n/2, 7/2 - n/2; 4, -2g - n + 4; 1) \\
+ \frac{(-2)^n n!}{4(2g + 1)!} \left\{ \frac{2g + n - 2}{n - 1} \right\} - (2g + 1)! \frac{2g + n - 3}{n - 3} \\
+ \frac{(-2)^n n!}{4(2g + 1)!} \left\{ \frac{1}{16} \left( 2g + 1 \right) \right\} - (2g + 1)! \frac{2g + n - 4}{n - 5} \\
\times 3 F_2(1, 3 - n/2, 5/2 - n/2; 3, -2g - n + 4; 1) \\
+ 3 F_2(1, 1 - n/2 + \frac{1}{2} (n - 1/2) - 2g - n + 3 + \frac{1}{2} (n - 1/2); 1) \\
\times 3 F_2(1, 2 - n/2, 5/2 - n/2; 2, -2g - n + 4; 1) \\
+ 3 F_2(1, 2 - n/2 + \frac{1}{2} (n - 2/2), 5/2 - n/2 + \frac{1}{2} (n - 2/2); 2 + \frac{1}{2} (n - 2/2); -2g - n + 4 + \frac{1}{2} (n - 2/2); 1) \\
\times 3 F_2(1, 1 + 2g + \frac{1}{2} (n - j/2), -n + j + 1 + \frac{1}{2} (n - j/2); -1/4) \right\} \\
- \frac{2n}{2} E,
\]

where

\[
E = \sum_{j=3}^{2g+2} \frac{(-1)^j}{2^j j (j - 1)(j - 2)} \left\{ \frac{4^{(n - 2)/2 + j/2} (1)^{(n - 2)/2 - j/2}}{5^{2g}} \left( \frac{2g - 1}{2g + 2 - j} \right) \left( \frac{2g + n - 1 - j}{n - j} \right) \\
\times \frac{\Gamma(-2g - n + 1 + j) \operatorname{LegendreP}(-2g, 2g + n - j, 3/5) + 1}{4 \cdot 2^{2j(n - j)/2}} \left( \frac{1}{(n - j)/2 + 1} \right) \left( \frac{2g + n - 2 - j - (n - j)/2}{2g + 2 - j} \right) \left( \frac{2g + n - 2 - j - (n - j)/2}{n - j - 1 - (n - j)/2} \right) \\
\times 3 F_2(1, 1 + 2g + \frac{1}{2} (n - j/2); -n + j + 1 + \frac{1}{2} (n - j/2); 2 + \frac{1}{2} (n - j/2); -1/4) \right\}.
\]

Finally, we remark the following fact.

Corollary 3. The moduli space $\mathcal{H}_{g,n}$ has the same homotopy type of a point if and only if $n = 0$. 

Proof. The claim is known for $n = 0$. If $n > 0$, the Euler characteristic is never 1, so the claim is completely proved. □

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