On the notion of cycles in hypergraphs

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\textbf{ABSTRACT}

The notion of hypergraph cyclicity is crucial in numerous fields of application of hypergraph theory (e.g. in computer science, in relational database theory and constraint programming). Surprisingly, while this notion has been well studied during last thirty years, no relevant definition of cycles in hypergraphs has been proposed by the community. In this paper, we propose a definition of cycles in hypergraphs, \(\alpha\)-cycle based on the same principle in graph theory, meaning that a hypergraph is acyclic if it does not contain an \(\alpha\)-cycle. This result completes the theory of the mostly used notion of hypergraph acyclicity, the \(\alpha\)-acyclicity.

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1. Introduction

The concept of graph acyclicity is defined in a natural way thanks to the notion of cycles in graph theory. Yet, it possesses numerous generalizations in hypergraph theory. We can find a first definition of cycles in the works of Berge \cite{2}. The notion of cyclicity has been particularly studied in relational database theory where several acyclicity definitions have been proposed, such as \(\alpha\)-acyclicity \cite{1}, \(\beta\)-acyclicity and \(\gamma\)-acyclicity \cite{5}. The most studied one is the \(\alpha\)-acyclicity, which has numerous applications in relational databases for query optimization \cite{1}, as well as in artificial intelligence and constraint programming \cite{4}.

\textbf{Definition 1.} A hypergraph \(H\) is a pair \(H = (X, E)\) where \(X\) is a set of vertices and \(E\) a family of subsets of \(X\), called hyperedges.

A hypergraph whose hyperedge arity is 2 (each hyperedge contains only two vertices) is a graph.

\textbf{Definition 2.} Let \(V\) be a subset of \(X\). The set of partial hyperedges induced by \(V\) in \(H\) is \(E' = \{e'|e = e \cap V, e \in E\} - \{\emptyset\}\).

\(E'\) is called a vertex-generated set. From this point forward, we will only consider reduced hypergraphs (i.e. no hyperedge is included in another).

\textbf{Definition 3.} Let us consider \(V \subset X\). The sub-hypergraph of \(H\) induced by \(V\) is the hypergraph \(H[V] = (V, E')\) with \(E'\), the set of partial hyperedges induced by \(V\) in \(H\).

\textbf{Definition 4.} A path between two hyperedges \(e_i\) and \(e_j\) of \(H\) is a sequence of hyperedges \((e_{u_1}, e_{u_2}, \ldots, e_{u_K})\) of \(H\) such that:

- \(e_{u_1} = e_i\) and \(e_{u_K} = e_j\)
- \(\forall u, 1 \leq u \leq K - 1, e_{u_1} \cap e_{u+1} \neq \emptyset\).
**Definition 5.** A hypergraph \( H \) is connected if there is a path between each pair of hyperedges.

Fig. 1 depicts a connected, reduced hypergraph.

**Definition 6.** \( CC \subset E \) is a connected component of \( H \) if there is a connected hypergraph and there is no set \( CC' \) such that \( CC \subset CC' \) and \( H[V'] \), with \( V' = \bigcup_{e \in CC'} e \), is a connected hypergraph.

From this point forward, we will consider only connected hypergraphs.

**Definition 7.** Let \( E \) be a connected, reduced set of partial hyperedges, \( e_1 \) and \( e_2 \) two elements of \( E \) and \( q = e_1 \cap e_2 \). \( q \) is an articulation of \( E \) if its removal from all hyperedges of \( E \) disconnects this set.

In Fig. 1, \( q = e_4 \cap e_6 = \{x_{12}, x_{13}\} \) is an articulation of \( E \).

**Definition 8 (I1).** A hypergraph \( H = (X, E) \) is \( \alpha \)-acyclic if every set of partial hyperedges being connected, reduced, induced by a subset of vertices, and admitting no articulation, is trivial (contains only one element).

The hypergraph \( H \) presented in Fig. 1 is not \( \alpha \)-acyclic because the set of partial hyperedges \( \{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_5, x_6\}, \{x_4, x_6, x_8\} \) is connected, reduced, not trivial, and does not admit an articulation. On the contrary, the hypergraph of Fig. 2 is \( \alpha \)-acyclic. This definition admits several equivalent formulations presented in [1]. We restate here two of them that are important for our purpose.

**Definition 9.** A jointree of a hypergraph \( H = (X, E) \) is a tree \( T \) whose nodes are the hyperedges of \( H \) and such that if a vertex \( x \in X \) belongs to two hyperedges \( e_i \) and \( e_j \), it is contained in all nodes of the unique path of \( T \) connecting \( e_i \) to \( e_j \).

So, the set of nodes containing \( x \) induces a connected sub-tree of \( T \). Fig. 3 presents a jointree of the hypergraph given in Fig. 2.
Definition 10. $H$ satisfies the running intersection property if there exists an ordering $\sigma$ on hyperedges such that for all $2 \leq \sigma(i) \leq |E|$, there exists $\sigma(i)<\sigma(i')$ such that $e_{\sigma(i)} \cap \bigcup_{\sigma(i')<\sigma(i)} e_{\sigma(i')} \subset e_{\sigma(i')}$.

The intersection of a hyperedge with the ones preceding it in the ordering is contained in one of these hyperedges.

The hypergraph in Fig. 2 satisfies the running intersection property. With the ordering $\sigma = (e_2, e_3, e_1, e_5, e_4, e_6)$, the intersection of a hyperedge with the ones preceding it in the ordering is contained in one of them.

Theorem 1 ([1]). $H$ is $\alpha$-acyclic iff:
- $H$ admits a jointree
- $H$ satisfies the running intersection property

Note that we call a hypertree a connected hypergraph that is $\alpha$-acyclic.

Surprisingly, the definition of $\alpha$-acyclicity in hypergraphs in terms of cycles has not been really studied. Most of the equivalent definitions are based on articulations in hypergraphs or cycles in a graph representation of hypergraphs related to connections of hyperedges. We have found only one definition proposed in the literature, in [8]. This definition of cycles in hypergraphs gives an equivalence between $\alpha$-acyclicity and the absence of cycles.

Definition 11 ([8]). Let $H = (X, E)$ be a hypergraph. A cycle of $H$ is a sequence of hyperedges $(e_1, e_2, \ldots, e_n)$ satisfying the following conditions:
- $e_{n+1} = e_1$
- $\forall 2 \leq j \leq K - 2$ and $\forall e \in E$, $(S_{j-1} \cup S_j \cup S_{j+1}) \setminus e \neq \emptyset$, with $S_j = e_j \cap e_{j+1}, \forall 1 \leq j \leq K - 1$.

This definition, which is really simple, includes sequences of hyperedges called pseudo-cycles, because they do not require $\alpha$-acyclicity of hypergraphs. Indeed, pseudo-cycles do not satisfy the running intersection property of hypergraphs, which is equivalent to the $\alpha$-acyclicity one (Theorem 1). Nevertheless, it is possible to show that the existence of a pseudo-cycle implies the existence of an "essential cycle" for the property of $\alpha$-acyclicity.

2. Cyclicality on graphical minimal representation of hypergraphs

We propose, here, a definition of a cycle for hypergraphs, called an $\alpha$-cycle, which allows us to define $\alpha$-cyclicality. To introduce them, we must recall the definition of minimal intergraphs and some important properties we need to prove our central theorem.

Definition 12 ([2]). $R = (E, B)$, with $B = \{e_i, e_j \mid e_i, e_j \in E, e_i \neq e_j$ and $e_i \cap e_j \neq \emptyset\}$, is called the line graph of $H$.

The vertices of the line graph of $H$ are hyperedges of $H$, and there is an edge between two vertices of $R$ if their intersection is not empty. $R$ represents the intersections in $H$.

Definition 13 ([3]). Let $R = (E, B)$ be the line graph of $H$. A graph $G = (E, A)$ is an intergraph of $H$, if $A \subseteq B$ and $\forall e_i, e_j \in E$ such that $e_i \cap e_j \neq \emptyset$, there is a path $(e_i = e_{u_1}, e_{u_2}, \ldots, e_{u_p} = e_j)$ such that $\forall 1 \leq k < p, e_i \cap e_j \subseteq e_{u_k} \cap e_{u_{k+1}}$.

The set of intergraphs of $H$ will be denoted $\mathcal{I}(H)$.

Definition 14. An intergraph $G = (E, A)$ of $H$ is minimal if $\forall A' \subseteq A, G = (E, A')$ is not an intergraph of $H$. $\mathcal{I}_m(H)$ denotes the set of minimal intergraphs of $H$.

The condition on minimal intergraphs is similar to the one on jointrees. The minimal intergraphs of a hypergraph satisfy a property related to their number of edges.

Theorem 2 ([7]). If $G = (E, A) \in \mathcal{I}_m(H)$ and $G' = (E, A') \in \mathcal{I}_m(H)$, then $|A| = |A'|$. The number of edges in the minimal intergraphs is an invariant for hypergraphs. As a consequence, we can remark that an intergraph whose number of edges is equal to the number of edges of a minimal intergraph is necessarily minimal. Moreover, there is equivalence between the $\alpha$-acyclicity of hypergraphs and the acyclicity of their minimal intergraphs.
Let $S$ contain two hyperedges. Fig. 4 is a minimal intergraph of this hypergraph. Since it is acyclic, the graph of $S$ is $\alpha$-acyclic.

3. A new definition of cycles in hypergraphs

Now, we define the $\alpha$-neighboring hyperedge and the $\alpha$-path notions in a hypergraph $H = (X, E)$. These definitions are based on minimal intergraphs.

Definition 15. Let $e_u$ and $e_v$ be hyperedges such that $e_u \cap e_v \neq \emptyset$. We call a sequence of neighborhood between $e_u$ and $e_v$, a sequence $(e_u \equiv e_1, e_2, \ldots, e_p \equiv e_v)$ such that $P > 2$ and $e_u \cap e_j \subset e_j \cap e_{j+1}$, for $j = 1, \ldots, P - 1$. We call an elementary sequence of neighborhood between $e_u$ and $e_v$, a sequence of neighborhood between $e_u$ and $e_v$, $(e_{i_1}, e_{i_2}, \ldots, e_{i_q})$ such that $\forall e_{i_k}, e_{i_{k+1}}$, with $1 \leq l < l_0$, $1 \leq b < l_0$ and $e_u \neq e_v$, such that $e_u \cap e_{i_{b+1}} \subset e_u \cap e_{i_{b+1}}$.

Two hyperedges are $\alpha$-neighboring if there is no other path (a sequence of neighborhood) allowing to go from one to the other apart from the trivial $(e_u, e_v)$ one. For the case where there is a sequence of neighborhood between two hyperedges $e_u$ and $e_v$, there is no minimal intergraph of $H$ that contains an edge between $e_u$ and $e_v$.

Definition 16. Let $e_u$ and $e_v$ be two hyperedges of $H$ such that $e_u \cap e_v \neq \emptyset$. $e_u$ and $e_v$ are $\alpha$-neighboring if there is no sequence of neighborhood between them.

Thus, $e_u$ and $e_v$ are $\alpha$-neighboring if there is a minimal intergraph of $H$ that contains an edge between $e_u$ and $e_v$. For example, the hypergraph given in Fig. 2 contains two hyperedges $e_1$ and $e_2$ with a non-empty intersection that are not $\alpha$-neighboring because $(e_1, e_3, e_2)$ is a sequence of neighborhood connecting $e_1$ and $e_2$. The minimal intergraph given in Fig. 5 of the hypergraph of Fig. 2 does not contain the edge $(e_1, e_2)$.

Theorem 4. Every sequence of neighborhood between two hyperedges $e_u$ and $e_v$ of $H$ $(e_u \cap e_v \neq \emptyset)$ contains an elementary sequence of neighborhood between $e_u$ and $e_v$.

Proof. Let $S_1 = (e_{i_1}, e_{i_2}, \ldots, e_{i_p})$ be a sequence of neighborhood between $e_u$ and $e_v$. If $S_1$ is not elementary, then there exist two hyperedges $e_{i_a}, e_{b_{l}}$ such that $l_0, l_0 < l_0, e_u \neq e_{i_a}$ and $e_u \cap e_{i_{b_{l}+1}} \subset e_{i_{b_{l}}} + e_{i_{b_{l}+1}}$. Consider $q = e_u \cap e_{i_{b_{l}+1}}$. $q \subset e_{i_a}$ and $q \subset e_{i_{b_{l}} + 1}$, so $q \subset e_{i_{b_{l}} + 1}$ and $e_u \cap e_{i_{b_{l}} + 1} \subset e_{i_{b_{l}}} + e_{i_{b_{l}+1}}$. Suppose that $l_0 = b_{l+1}$. We can define a new sequence $S_2$ based on $S_1$. $S_2 = (e_{i_1}, \ldots, e_{i_a}, e_{i_{b_{l}+1}}, \ldots, e_{i_p})$ is equal to $S_1$ from $e_{i_a}$ to $e_{i_p}$. This latter is followed by $e_{i_{b_{l}+1}}$, and the remaining $S_2$ is equal to $S_1$ from $e_{i_{b_{l}+1}}$ to $e_{i_p}$. If $l_{b_{l}+1} < i_0$, then $e_{i_{b_{l}+1}}$ is before $e_{i_0}$ in $S_2$. If $S_2$ is elementary, the result holds.
Otherwise, there exist two hyperedges $e_i, e_j$ such that $i_1, i_2 < i_3, e_i \neq e_j$ and $e_i \cap e_{i+1} = e_j \cap e_{j+1}$. In this case, we can repeat the same work as the one realized for $S_1$. Since a hyperedge is withdrawn, at each step, from $S_1$ to build $S_{i+1}$, the last element of this sequence is an elementary sequence of neighborhood between $e_u$ and $e_v$. $\square$

**Corollary 1.** Let $e_u$ and $e_v$ be two hyperedges of $H$, such that $e_u \cap e_v \neq \emptyset$. If $e_u$ and $e_v$ are not $\alpha$-neighboring then there exists an elementary sequence of neighborhood between them.

**Proof.** $e_u$ and $e_v$ being not $\alpha$-neighboring, there exists a sequence of neighborhood between $e_u$ and $e_v$. This sequence contains an elementary sequence of neighborhood between $e_u$ and $e_v$ (from preceding theorem). $\square$

**Definition 17.** An $\alpha$-path in $H$ is a sequence of hyperedges $(e_1, \ldots, e_k)$ such that $\forall j, 1 \leq j < k, e_j$ and $e_{j+1}$ are $\alpha$-neighboring.

This notion of $\alpha$-path preserves the property of connection related to the usual definition of path.

**Definition 18.** $H$ is $\alpha$-connected if there exists an $\alpha$-path connecting every pair of hyperedges in $H$.

**Theorem 5.** Let $e_u$ and $e_v$ be two hyperedges of $H$ that are not $\alpha$-neighboring. If $H$ is connected, then there exists an $\alpha$-path between $e_u$ and $e_v$ whose intersections between consecutive hyperedges strictly contain $e_u \cap e_v$.

**Proof.** We have two possibilities: $e_u \cap e_v \neq \emptyset$ and $e_u \cap e_v = \emptyset$.

Let us consider the first one. $H$ being connected while $e_u$ and $e_v$ are not $\alpha$-neighboring, there exists an elementary sequence of neighborhood $S_{i_1}$ such that the intersections between two consecutive hyperedges strictly contain $e_u \cap e_v$ (Corollary 1). We build an $\alpha$-path between $e_u$ and $e_v$ from this sequence of neighborhood. There are two cases.

First case: the consecutive elements of $S_{i_1}$ are two by two $\alpha$-neighboring, so $S_{i_1}$ is an $\alpha$-path between $e_u$ and $e_v$. Moreover, the intersections between consecutive hyperedges strictly contain $e_u \cap e_v$.

Second case: there exists in $S_{i_1}$ at least one pair of consecutive hyperedges that are not $\alpha$-neighboring. We are going to construct from $S_{i_1}$ a new elementary sequence of neighborhood such that these hyperedges are no longer consecutive. Let $(e_{i_1}, e_{i_2})$ be the first pair (w.r.t. the ordering of hyperedges) of these hyperedges. There exists an elementary sequence of neighborhood $S_{i_2}$ whose intersections between consecutive hyperedges strictly contain $q_1 = e_{i_1} \cap e_{i_2}$. Since $q_2 = e_{i_2} \cap e_{i_3} \subseteq e_{i_2} \cap e_{i_1}$, the intersections of consecutive hyperedges in $S_{i_1, i_2}$ strictly contain $q_2$. So, we modify $S_{i_1}$ by putting the sequence $S_{i_2}$ between $e_{i_2}$ and $e_{i_1}$. However, in an elementary sequence of neighborhood, we cannot have an intersection between consecutive hyperedges contained in another. To ensure this property, we need to modify $S_{i_1}$ again. Since $S_{i_1}$ was an elementary sequence of neighborhood such as $S_{i_1, i_2}$, an intersection $e_{i_1} \cap e_{i_2}$ of two consecutive hyperedges of our current sequence contains another $e_{i_1} \cap e_{i_2}$ only if $e_{i_1}$ and $e_{i_2}$ are in $S_{i_1}$ and $e_{i_1}$ and $e_{i_2}$ are in $S_{i_1, i_2}$. Suppose that $e_{i_1} \cap e_{i_2} \subset e_{i_1} \cap e_{i_2}$. Since $e_{i_2} \cap e_{i_3} \subseteq e_{i_2} \cap e_{i_1}$, $e_{i_1} \cap e_{i_3} \subset e_{i_1} \cap e_{i_2}$. However, in an elementary sequence of neighborhood, therefore, this is not possible. Thus, $e_{i_1} \cap e_{i_2} | e_{i_1} \cap e_{i_2}$. We transform $S_{i_1}$ by coming directly from $e_{i_1}$ to $e_{i_2}$ (if $e_{i_1} \cap e_{i_3}$ to $e_{i_2}$, otherwise) since $e_{i_1} \cap e_{i_2} \subset e_{i_1} \cap e_{i_2}$. We obtain, by so doing, a new elementary sequence of neighborhood joining $e_u$ to $e_v$. Then, we proceed in the same way with the following consecutive hyperedges of $S_{i_1}$ that were not $\alpha$-neighboring. Finally, we obtain an elementary sequence $S_{i_1}$ connecting $e_u$ and $e_v$. If consecutive elements of $S_{i_1}$ are two by two $\alpha$-neighboring, then $S_{i_1}$ is an $\alpha$-path between $e_u$ and $e_v$ whose consecutive hyperedge intersections strictly contain $e_u \cap e_v$. Otherwise, there exists at least one pair of consecutive hyperedges $e_{i_2}$ and $e_{i_3}$ that are not $\alpha$-neighboring. We proceed like previously to build a new sequence $S_{i_2}$, and so on. Furthermore, it is important to note that $e_{i_2}$ and $e_{i_3}$ are in $S_{i_2}$ and were not in $S_{i_1}$ since all not $\alpha$-neighboring consecutive hyperedges of $S_{i_1}$ were already treated. Thus, there are two no $\alpha$-neighboring consecutive hyperedges of $S_{i_1}$ (let us suppose $e_{i_1}$ and $e_{i_2}$) without lack of generality such that $(e_{i_1}$ and $e_{i_2}$) are consecutive hyperedges of its elementary sequence of neighborhood. As a consequence, $(e_{i_1} \cap e_{i_2}) \subseteq (e_{i_2} \cap e_{i_3}) \subseteq (q_1 \subseteq q_2)$. We claim that this sequence of $S_k$ is finite. Suppose it is not. In each $S_k$, there exists at least one pair of consecutive hyperedges $(e_{i_k}, e_{i_{k+1}})$ that are not $\alpha$-neighboring. $(e_{i_k}, e_{i_{k+1}})$ is located in $S_k \setminus S_{k-1} \cup q_{k-1} \subseteq q_k$. Thus sequence of $q_k$ is also infinite. But, this sequence is strictly growing and the number of vertices belonging to $H$ is finite, thus this sequence cannot be infinite. Therefore, its last element is an $\alpha$-path between $e_u$ and $e_v$ whose intersections between consecutive hyperedges strictly contain $e_u \cap e_v$.

Now, we consider the case $e_u \cap e_v = \emptyset$.

Since $H$ is connected, there is a sequence of hyperedges $(e_1 = e_u, e_2, \ldots, e_K = e_v)$ of non-empty intersections that connects $e_u$ and $e_v$. For all $1 \leq i < K$, if $e_i$ and $e_{i+1}$ are $\alpha$-neighboring then $(e_i, e_{i+1})$ is an $\alpha$-path between $e_i$ and $e_{i+1}$. Otherwise, there is an $\alpha$-path between $e_i$ and $e_{i+1}$ strictly containing $e_{i} \cap e_{i+1}$ thanks to the first part of this proof. Thus, the sequence of hyperedges containing the $\alpha$-path between $e_u = e_1$ and $e_v$, followed by the one between $e_i$ and $e_{i+1}$, and so on, until the one between $e_{k-1}$ and $e_k = e_v$, is an $\alpha$-path between $e_u$ and $e_v$, which strictly contains $e_u \cap e_v = \emptyset$. $\square$

**Corollary 2.** $H$ is connected if it is $\alpha$-connected.

**Proof.** Suppose that $H$ is $\alpha$-connected. Let $e_u$ and $e_v$ be two hyperedges of $H$. There exists an $\alpha$-path connecting $e_u$ and $e_v$. Now, the consecutive hyperedges of an $\alpha$-path have a non-empty intersection. Thus, this $\alpha$-path is a sequence of hyperedges of non-empty intersections that connects $e_u$ and $e_v$, $H$ is connected.
Theorem 5 allows us to conclude that there exists an \( \alpha \)-cycle in \( H \). So, \( H \) is \( \alpha \)-acyclic.

Now, suppose that \( H \) is connected. Let \( e_a \) and \( e_b \) be two hyperedges of \( H \). If they are \( \alpha \)-neighboring, \((e_a, e_b)\) is an \( \alpha \)-path connecting them. Else, Theorem 5 allows us to conclude that there exists an \( \alpha \)-path connecting \( e_a \) and \( e_b \). \( H \) is then \( \alpha \)-connected. \( \square \)

**Definition 19.** An \( \alpha \)-cycle in \( H \) is an \( \alpha \)-path \( (e_{i_1}, e_{i_2}, \ldots, e_{i_P}) \) such that \( P > 3 \), \( e_{i_1} = e_{i_P} \), \( \exists 1 \leq a < b < P \), \( e_a \cap e_{i_{b+1}} \subset e_a \cap e_{i_{b+1}} \). If there are no \( a \) and \( b \), \( 1 \leq a < b \leq K \) such that \( e_{a_1} \cap e_{a_{b+1}} \subset e_{a_1} \cap e_{a_{b+1}} \), then \((e_{a_1}, e_{a_2}, \ldots, e_{a_P})\) is an \( \alpha \)-cycle of \( H \). Otherwise, we have \( e_{a_1} \cap e_{a_{b+1}} \subset e_{a_1} \cap e_{a_{b+1}} \). \((e_{a_1}, e_{a_2}, \ldots, e_{a_P}, e_{a_1})\) is not an \( \alpha \)-cycle of \( H \), but we can build a smaller cycle in another minimal intergraph.

In the example depicted in Fig. 7, we suppose \( a, a + 1, b \) and \( b + 1 \) are all different (otherwise, the method is simpler and is given farther). There exists necessarily a path connecting \( e_a \) and \( e_{b+1} \) and containing their intersection. We suppose this path contains the edge \( \{e_{a_1}, e_{b_{a+1}}\} \) (dotted lines). Replacing the edge \( \{e_{a_1}, e_{b_{a+1}}\} \) by \( \{e_{a_2}, e_{b_{a+2}}\} \) preserves properties related to connection because \( e_{a_1} \) and \( e_{b_{a+1}} \) are connected by the path \( (e_{a_1}, e_{a_2}, e_{b_{a+1}}) \), which contains \( e_{a_1} \cap e_{b_{a+1}} \). So we always have an intergraph with the same number of edges thus it is minimal. Moreover, \((e_{a_2}, e_{a_3}, \ldots, e_{a_{b+2}}, e_{b_1})\) is a cycle of this minimal intergraph with fewer elements than the first one.

If the path does not contain the edge \( \{e_{a_{b_1}}, e_{a_{b+1}}\} \) (as depicted in Fig. 8), replacing the edge \( \{e_{a_{b_1}}, e_{a_{b+1}}\} \) by \( \{e_{a_{b_2}}, e_{a_{b+2}}\} \) preserves properties related to connection and the number of edges. The intergraph is minimal and contains \((e_{a_{b+1}}, e_{a_{b+2}}, \ldots, e_{a_{b+2}}, e_{a_{b+1}})\), which is a cycle with fewer elements.

Now, if \( e_{a_{b+1}} = e_{b_1} \) (Fig. 9), we compute a minimal intergraph by replacing the edge \( \{e_{a_{b_1}}, e_{a_{b+1}}\} \) by \( \{e_{a_{b+1}}, e_{b_1}\} \). This intergraph contains the cycle \( (e_{a_1}, e_{a_2}, e_{b_1}, e_{b_2}, e_{a_1}) \), which has fewer nodes.

If \( e_{a_{b+1}} = e_{b_1} \) (Fig. 10), we compute a minimal intergraph by replacing the edge \( \{e_{a_{b_1}}, e_{a_{b+1}}\} \) by \( \{e_{a_{b+1}}, e_{b_1}\} \). This intergraph contains the cycle \( (e_{a_1}, e_{a_2}, e_{b_2}, \ldots, e_{b_1}, e_{a_1}) \), which has fewer nodes.

In all these previous cases, either the new cycle induces an \( \alpha \)-cycle or it can be reduced. The number of elements in the first cycle being finite, this procedure could be repeated only a finite number of times. The last step gives an \( \alpha \)-cycle in \( H \).
The following theorem establishes more formally this equivalence between $\alpha$-acyclicity of a hypergraph and existence of $\alpha$-cycle.

**Theorem 6.** $H$ is $\alpha$-acyclic iff it does not contain an $\alpha$-cycle.

**Proof.** 1. We start by showing that if $H$ is $\alpha$-acyclic then it does not contain an $\alpha$-cycle.

Suppose that $H$ is $\alpha$-acyclic.

We suppose that $H$ contains an $\alpha$-cycle: $(e_{i_1}, e_{i_2}, \ldots, e_{i_p})$. We will prove that there exists a minimal intergraph of $H$ containing the cycle $(e_{i_1}, e_{i_2}, \ldots, e_{i_p})$. For all $G = (E, A) \in I_m(H)$, there is a path $(e_{i_a} = e_{i_1}, e_{i_2}, \ldots, e_{i_L} = e_{i_{i_{p+1}}})$ containing the intersection $e_{i_a} \cap e_{i_{a+1}}$. Since $e_{i_a}$ and $e_{i_{a+1}}$ are consecutive elements of the $\alpha$-cycle, they are $\alpha$-neighboring. Thus, there exists $b$, $1 \leq b < L$ such that $e_{i_a} \cap e_{i_{b+1}} = e_{i_b} \cap e_{i_{b+1}}$ (else this path would be a sequence of neighborhood between $e_{i_a}$ and $e_{i_{i_{p+1}}}$). Moreover, $e_{i_b}$ and $e_{i_{b+1}}$ are not consecutive elements of the $\alpha$-cycle $(e_{i_1}, e_{i_2}, \ldots, e_{i_p})$ because by definition no intersection between two consecutive elements of an $\alpha$-cycle is included in another.

Consider the graph $G'$ such that $G' = (E, A')$ with $A' = (A \setminus \{\{e_{i_b}, e_{i_{b+1}}\}\} \cup \{e_{i_a}, e_{i_{a+1}}\})$. In $G'$, properties related to connection are preserved because $e_{i_b}$ and $e_{i_{b+1}}$ are connected by the path $(e_{i_b}, e_{i_{b-1}}, \ldots, e_{i_2}, e_{i_1}, e_{i_{b+1}}, e_{i_{b-1}}, \ldots, e_{i_{i_{p+1}}})$. Thus $G' \in I(H)$. Moreover, $G'$ has the same number of edges as $G$, thus $G' \in I_m(H)$. We have built a minimal intergraph $G'$ of $H$ in which $e_{i_a}$ and $e_{i_{a+1}}$ are neighboring for each $1 \leq a < P$. 

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**Fig. 8.** Construction of a smaller cycle in a minimal intergraph (2/4).

**Fig. 9.** Construction of a smaller cycle in a minimal intergraph (3/4).

**Fig. 10.** Construction of a smaller cycle in a minimal intergraph (4/4).
Let $(e_{u_1}, e_{u_2}, \ldots, e_{u_k})$ be a cycle of $G$. Now $H$ being $\alpha$-acyclic, it cannot admit a cyclic minimal intergraph (Theorem 3). Thus $H$ does not contain an $\alpha$-cycle.

2. Now, we will prove that if $H$ does not contain an $\alpha$-cycle, then $H$ is $\alpha$-acyclic. For that, we will show that if $H$ is $\alpha$-acyclic, then it contains an $\alpha$-cycle.

Suppose that $H$ is $\alpha$-acyclic. We know that a minimal intergraph $G = (E, A)$ of $H$ is cyclic.

Let $(e_{u_1}, e_{u_2}, \ldots, e_{u_k} = e_{u_1})$ be a cycle of $G$. Thus we have $K \geq 3$. Moreover, $\forall b, 1 \leq b \leq K$, $e_{u_b} \cap e_{u_{b+1}} \neq \emptyset$. In addition, since $G$ is minimal, $\forall c, 1 \leq c \leq L$, $e_{u_c} \cap e_{u_{c+1}} \subseteq e_{u_i} \cap e_{u_{i+1}}$. In that case, the graph $G' = (E, A')$, with $A' = A \setminus \{\{e_{u_b}, e_{u_{b+1}}\}\}$, would be an intergraph of $H$. This would contradict the assumption that $G$ is minimal. Then, we can say that there does not exist a neighborhood between $e_{u_b}$ and $e_{u_{b+1}}$. Thus they are $\alpha$-neighboring.

We show now that the cycle $(e_{u_1}, e_{u_2}, \ldots, e_{u_k})$ induces existence of an $\alpha$-cycle in $H$. We have two cases.

(a) There are no $a$ and $b, 1 \leq a < b \leq K$ such that $e_{u_a} \cap e_{u_b} \subseteq e_{u_b} \cap e_{u_{b+1}}$. Thus $(e_{u_1}, e_{u_2}, \ldots, e_{u_k}, e_{u_1})$ is an $\alpha$-cycle of $H$.

(b) There exist $a$ and $b, 1 \leq a < b \leq K$, such that $e_{u_a} \cap e_{u_{a+1}} \subseteq e_{u_b} \cap e_{u_{b+1}}$. In this case, $K \geq 3$. Indeed, if $K = 3$, the cycle contains uniquely $(e_{u_1}, e_{u_2}, e_{u_2}, e_{u_1})$. Since an intersection $e_{u_b} \cap e_{u_{b+1}}$ is contained in another, then the third intersection also contains $e_{u_a} \cap e_{u_{a+1}}$. The edge $\{e_{u_a}, e_{u_{a+1}}\}$ is thus redundant since there exists a path connecting $e_{u_a}$ and $e_{u_{a+1}}$ that connects their intersection. This contradicts the hypothesis that $G$ is minimal. Thus, necessarily $K \geq 4$. Moreover, we cannot have $u_{b+1} = u_b$ and $u_{b+1} = u_1$: it would be the same intersection. Now, we try to build an $\alpha$-cycle for the case where there exists an intersection between elements of the cycle $(e_{u_1}, e_{u_2}, \ldots, e_{u_k}, e_{u_1})$ that is included in another. So, if $u_{a+1} = u_b$ then $u_{b+1} \neq u_a$. In the same way, if $u_{b+1} = u_a$ then $u_{a+1} \neq u_b$.

i. In a first time, suppose that $u_{a+1} = u_b$, thus $u_{b+1} \neq u_a$.

The edge $(e_{u_b}, e_{u_a})$ is not in $G$ because, in the opposite case, $(e_{u_a}, e_{u_{a+1}}, e_{u_b}, e_{u_a})$ would be a cycle in $G$ with $K = 3$. This is impossible since we proved that $K \geq 4$. Consider the graph $G' = (E, A')$, with $A' = (A \setminus \{\{e_{u_a}, e_{u_{a+1}}\}\}) \cup \{\{e_{u_b}, e_{u_a}\}\}$. $e_{u_a}$ and $e_{u_{a+1}}$ are connected by the path $(e_{u_a}, e_{u_{a+1}}, e_{u_b}, e_{u_{a+1}} = e_{u_a})$, which contains their intersection. We preserve the properties related to connection, thus $G'$ is an intergraph. Furthermore, since $G'$ contains the same number of edges as $G$, a minimal intergraph, it is minimal. In the graph $G'$, $(e_{u_{a+1}}, e_{u_{a+2}}, \ldots, e_{u_b}, e_{u_a})$ is a cycle. Indeed, it contains all the elements of $(e_{u_a}, e_{u_2}, \ldots, e_{u_{a+1}}, e_{u_b})$, except $e_{u_{a+1}} = e_{u_b}$. Since $K \geq 4$, $(e_{u_a+1}, e_{u_{a+2}}, \ldots, e_{u_b}, e_{u_{a+1}})$ contains at least three different elements. We have defined a cycle of a minimal intergraph $G'$ that possesses fewer elements than our first cycle in $G$.

ii. Suppose now that $u_{b+1} \neq u_a$.

Two cases are possible: $e_{u_{b+1}} = u_b$ or $u_{b+1} = u_a$. If $u_{b+1} = u_a$, this cycle is symmetric with the one where $u_{b+1} = u_b$ and $u_{a+1} = u_a$, and it can be solved in the same way.

Suppose that $u_{b+1} \neq u_a$.

The two paths $(e_{u_a}, e_{u_{a+1}}, \ldots, e_{u_b}, e_{u_{a+1}})$ and $(e_{u_b}, e_{u_{b+1}}, \ldots, e_{u_{a+1}}, e_{u_a})$ on the elements of the cycle $(e_{u_a}, e_{u_2}, \ldots, e_{u_{a+1}})$.

We consider two cases:

A. first case, $e_{u_a} \cap e_{u_{a+1}}$ is included along one of these two paths. We suppose that this path is $e_{u_{a+1}} = (e_{u_{a+1}}, e_{u_{a+2}}, \ldots, e_{u_b}, e_{u_{a+1}})$. Consider the graph $G' = (E, A')$, with $A' = (A \setminus \{\{e_{u_a}, e_{u_{a+1}}\}\}) \cup \{\{e_{u_b}, e_{u_{a+1}}\}\}$. $e_{u_a}$ and $e_{u_{a+1}}$ are connected by the path $(e_{u_{a+1}}, \ldots, e_{u_b}, e_{u_{a+1}}, e_{u_b})$, which contains $e_{u_a} \cap e_{u_{a+1}}$. We preserve properties related to connection, thus $G'$ is an intergraph. Furthermore, the edge $(e_{u_{b+1}}, e_{u_{a+1}})$ is not in $G$. Otherwise, the edge $(e_{u_a}, e_{u_{a+1}})$ would be redundant because of the existence of the path $e_{u_{a+1}}, \ldots, e_{u_b}, e_{u_{a+1}}$, which contains $e_{u_a} \cap e_{u_{a+1}}$. This would contradict the hypothesis that $G$ is minimal. Thus $G'$ has the same number of edges as $G$, which is a minimal intergraph. So, we deduce that $G'$ is minimal. In this minimal intergraph, $(e_{u_a}, e_{u_{a+1}}, \ldots, e_{u_b}, e_{u_{a+1}})$ is a cycle that contains fewer elements than the cycle $(e_{u_1}, e_{u_2}, \ldots, e_{u_{a+1}} = e_{u_a})$ of $G$. Indeed, it contains at least three different elements: $e_{u_a}$, $e_{u_{b+1}}$ and the elements of the path $(e_{u_{a+1}}, \ldots, e_{u_b})$ that has at least one element different from $e_{u_{a+1}}$ and $e_{u_{b+1}}$ because these hyperedges are not neighboring in $G$. Moreover, $e_{u_{b+1}}$ does not belong to this new cycle.

If $(e_{u_{b+1}}, e_{u_b}, \ldots, e_{u_b})$ is the path containing $e_{u_a} \cap e_{u_{b+1}}$, we build in the same way a cycle in a minimal intergraph $G'$ such that this cycle contains fewer elements than $(e_{u_1}, e_{u_2}, \ldots, e_{u_{a+1}})$.

B. second case, $e_{u_a} \cap e_{u_{a+1}}$ is not included along one of the two paths $(e_{u_a}, e_{u_{a+1}}, \ldots, e_{u_b}, e_{u_{a+1}})$ and $(e_{u_b}, e_{u_{b+1}}, \ldots, e_{u_{a+1}})$. However, there exists necessarily a path $(e_{u_1} = e_{u_1}, e_{u_1}, \ldots, e_{u_a} = e_{u_{a+1}})$ that connects $e_{u_a}$ and $e_{u_{a+1}}$ and contains their intersection. There are now two cases to consider.

• First case, this path contains the edge $(e_{u_a}, e_{u_{a+1}})$. Consider the graph $G' = (E, A')$, with $A' = (A \setminus \{\{e_{u_a}, e_{u_{a+1}}\}\}) \cup \{\{e_{u_a}, e_{u_{a+1}}\}\}$. We preserve properties related to connection because $e_{u_a}$ and $e_{u_{a+1}}$ are connected by the path $(e_{u_{a+1}}, \ldots, e_{u_b} = e_{u_{a+1}}, e_{u_a})$, which contains $e_{u_a} \cap e_{u_{a+1}}$. $G'$ is thus an intergraph. Moreover, having the same number of edges as $G$, $G'$ is minimal. As previously, $(e_{u_a}, e_{u_{a+1}}, \ldots, e_{u_b}, e_{u_a})$ is a cycle of $G$ that contains fewer elements than the cycle $e_{u_1}, e_{u_2}, \ldots, e_{u_a} = (e_{u_a})$ of $G$. Indeed, $(e_{u_a}, e_{u_{a+1}}, \ldots, e_{u_{b+1}}, e_{u_a})$ contains $e_{u_a}$ and at least another element. Because, in the opposite case, $e_{u_a} \cap e_{u_{b+1}}$ would be included along the path joining $e_{u_a}$ and $e_{u_{b+1}}$: this would contradict the hypothesis that $e_{u_a} \cap e_{u_{b+1}}$ is not included along this path.
• Second case, the path \((e_u, e_{v_1}, e_{v_2}, \ldots, e_{v_L} = e_u + 1)\) does not contain the edge \(\{e_{u}, e_{u+1}\}\). Consider the graph \(G' = (E, A')\), with \(A' = (A \setminus \{\{e_u, e_{u+1}\}\}) \cup \{\{e_{u+1}, e_{u+1}\}\}\). We preserve properties related to connection because \(e_u\) and \(e_{u+1}\) are connected by the path \((e_u = e_{v_1}, e_{v_2}, \ldots, e_{v_L} = e_{u+1}, e_{u+1})\), which contains \(e_u \sqcap e_{u+1}\). Therefore, \(G'\) is an intergraph that has the same number of edges as \(G\). Thus it is minimal. Moreover, \((e_{u+1}, e_{u+2}, \ldots, e_{u_k}, e_{u_{k+1}}, e_{u_{k+1}})\) is a cycle of \(G'\) that contains fewer elements than the cycle \((e_{u_1}, e_{u_2}, \ldots, e_{u_k}, e_{u_1})\) of \(G\) (\(e_u\) does not appear in this cycle).

We have shown that for all cases, either the cycle \((e_{u_1}, e_{u_2}, \ldots, e_{u_k}, e_{u_1})\) induces the existence of an \(\alpha\)-cycle in \(H\) or the existence of a cycle whose length is strictly smaller in another minimal intergraph. For the case where it induces a smaller cycle, we can repeat the operation on the latter. The number of elements in the first cycle being finite, this procedure could be repeated only a finite number of times. The last step allows us to deduce the existence of an \(\alpha\)-cycle in \(H\). □

Furthermore, in the case of graphs, an \(\alpha\)-cycle is a cycle.

Our goal here was to define explicitly the concept of cycles in hypergraphs. Indeed, while the concept of cyclicity is important in this field, only acyclicity has been studied by the community, independently from the notion of cycle. This is because acyclicity of hypergraphs is very important in many domains of computer science as relational database theory, constraint programming and probabilistic reasoning (topology of Bayesian networks). More precisely, this work was motivated by the need for a formal tool defining explicitly the concept of cycle to facilitate the management of acyclic constraint networks (acyclic hypergraphs) for solving constraint satisfaction problems [6]. In this framework, it is important to identify the \(\alpha\)-neighborhood of a hyperedge in order to compute incrementally join trees of an acyclic hypergraph. This was not possible with the previous definitions of acyclicity or the one given in [8].

References