Fuzzy Proximities and Totally Bounded Fuzzy Uniformities

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0. INTRODUCTION

After the classical papers of Zadeh [18] and Chang [3], many concepts of general topology have been extended to fuzzy set theory: in this paper we are interested to develop the study of fuzzy uniformities, fuzzy proximities, and of the connexions between such structures.

Fuzzy uniformities have been introduced by Lowen in [9] and, with slight modifications, in [10] and by Hutton in [5]; the two approaches are quite different; however, the one proposed by Hutton seems to suit much better to fuzzy set theory, also because it allows to generalize a lot of results in the most proper way. Therefore in this paper we shall deal with Hutton fuzzy uniformities.

The concept of fuzzy proximity used up to now (see, e.g., [7, 8, 15]) is quite unsatisfactory: indeed its “fuzzyness” is rather poor since these fuzzy proximities are in a canonical 1–1 correspondence with the usual proximities (see Remark 2.6): moreover the open sets of the induced topologies are crisp and, though every Lowen fuzzy uniformity induces a fuzzy proximity, this correspondence cannot work well since the two structures do not give the same fuzzy topology. For these reasons we propose a new definition of fuzzy proximity; it differs from the old one only in Axiom P5: in short we could say that we privilege complementation with respect to intersection (an agreement between the two operations being problematic in fuzzy set theory).

We point out that, as a consequence, a fuzzy set may be “far” from itself; however, this fact ought not to amaze anyone since it may happen just for “pale” sets which, in a certain sense, resemble the empty set.

Nevertheless, this modification enables us to associate a topology in a completely different way; moreover, every fuzzy uniformity induces a fuzzy proximity and, vice versa, we succeed to construct a fuzzy uniformity starting with a given fuzzy proximity: and the induced topologies do not change at any step. This implies that the topologies which admit a fuzzy proximity are exactly the completely regular ones. Moreover, we prove that there exists a 1–1 correspondence between fuzzy proximity spaces and a
subclass of fuzzy uniform spaces: it turns out that this subclass consists precisely of those uniform spaces which we call "totally bounded," and the correspondence is functorial. Totally bounded fuzzy uniform spaces form a reflective full subcategory of the category of fuzzy uniform spaces. These results, and others related to them, are the natural extensions to fuzzy set theory of well-known classical theorems.

Other nice extensions of classical results are provided by Theorems 3.5 and 5.10: in the former we show that a fuzzy normal space admits a canonical fuzzy proximity, in the latter we prove that the usual fuzzy uniformity on the fuzzy unit interval is totally bounded.

At last we remark that it is possible to define a different kind of fuzzy proximity, which accords with Lowen uniformities in a satisfactory way: this will be the matter of a forthcoming paper.

1. Notations and Preliminaries

Throughout this paper, \((L, \vee, \wedge, ',)\) will be a (complete) completely distributive lattice with order reversing involution ' (= complementation).

Given a set \(X\), any element of \(L^X\) is called "fuzzy set" and will be denoted by small Greek letters, such as \(\gamma, \mu, \nu, \rho, \sigma, \tau\).

0 and 1 denote the infimum and the supremum of \(L\), respectively; if \(Y\) is a subset of \(X\), we use the same letter \(Y\) to indicate the element of \(L^X\) so defined:

\[
\begin{align*}
  f(x) &= 1 & \text{if } x \in Y, \\
  f(x) &= 0 & \text{otherwise};
\end{align*}
\]

for \(a \in L, x \in X\), \(ax\) denotes the element of \(L^X\) which takes the value \(a\) at the point \(x\) and 0 elsewhere; \(ax\) is said to be a fuzzy point and \(x\) is its support; put also \(1x = x\). If \(\mu \in L^X\), we say that \(ax\) belongs to \(\mu\), or that \(ax\) is a fuzzy point of \(\mu\) if \(a \leq \mu(x)\).

\(L^X\) inherits a structure of lattice with order reversing involution in a natural way, by defining \(\vee, \wedge, ',\) pointwise (same notations of \(L\) are used).

If \(f: X \to Y\) is a function, and \(\mu, v\) belong to \(L^X, L^Y\) respectively, as usual we put: \(f^\vee(v)(x) = v(f(x)) = (v \circ f)(x)\) for \(x \in X\);

\[
\begin{align*}
  f(\mu)(y) &= \sup\{\mu(x): x \in X, f(x) = y\} & \text{for } y \in Y;
\end{align*}
\]

clearly \(f^\vee(v) \in L^X, f(\mu) \in L^Y\) and we have easily

\[
\begin{align*}
  f(f^\vee(v)) &= v \wedge f(X) & \text{and } f(\vee) \mu) \geq \mu.
\end{align*}
\]

Moreover observe that \(f^\vee\) preserves complementation, arbitrary unions and arbitrary intersections and that \(f(\bigvee_{i \in I} \mu_i) = \bigvee_{i \in I} f(\mu_i)\).
A fuzzy topological space is a pair \((X, \mathcal{F})\) where \(\mathcal{F} \subseteq L^1\) contains the constants \(0\) and \(1\), and is closed under finite intersections and arbitrary unions. The elements of \(\mathcal{F}\) are called open and their complements closed.

Given a fuzzy topological space \((X, \mathcal{F})\), a fuzzy set \(\mu \in L^X\) is said to be a \(\mathcal{F}\)-neighborhood (simply neighborhood if no confusion may arise) of any point \(x\) if there exists \(\nu \in \mathcal{F}\) such that \(ax \leq \nu \leq \mu\); clearly a fuzzy set is open if and only if it is a neighborhood of any of its points; interior and closure of fuzzy sets are defined in the usual way.

If \((X, \mathcal{F}), (Y, \mathcal{G})\) are fuzzy topological spaces, a function \(f: X \to Y\) is said to be continuous if \(f^{-1}(\mathcal{G}) \in \mathcal{F}\) for every \(\mathcal{G} \in \mathcal{G}\).

Since we shall always deal with "fuzzy" concepts, for the sake of brevity, we shall sometimes write "f. topology" or simply "topology" instead of "fuzzy topology," similarly for fuzzy uniformities, fuzzy proximities and so on; when we shall refer to the classical case, we shall write it explicitly, using words such as "usual" or "classical."

We use the definition of fuzzy uniform space given by Hutton in [5]. Denote by \(\mathcal{U}\) the set of maps \(U: L^X \to L^X\) which satisfy:

\[
U(\emptyset) = \emptyset \quad \text{(A1)}
\]
\[
U(\mu) \supseteq \mu \quad \text{(A2)}
\]
\[
U \left( \bigvee_{i \in I} \mu_i \right) = \bigvee_{i \in I} U(\mu_i) \quad \text{for } \mu, \mu_i \in L^X. \quad \text{(A3)}
\]

As in [5], if \(U, V\) belong to \(\mathcal{U}\), define \(U \land V\) to be the infimum of \(U\) and \(V\) in \(\mathcal{U}\), which turns out to satisfy

\[
(U \land V)(\mu) = \bigwedge_{\mu_1 \lor \mu_2 = \mu} (U(\mu_1) \lor V(\mu_2));
\]

moreover define

\[
U^{-1}(\mu) = \inf\{\rho : U(\rho') \leq \mu'\};
\]

an element \(U\) such that \(U = U^{-1}\) is called symmetric.

A fuzzy uniformity on \(X\) is a subset \(\mathcal{U}\) of \(\mathcal{U}\) such that

\[
\mathcal{U} \neq \emptyset, \quad \text{(U1)}
\]
\[
U \in \mathcal{U} \text{ and } U \leq V \in \mathcal{U}, \text{ implies } V \in \mathcal{U}, \quad \text{(U2)}
\]
\[
U, V \in \mathcal{U} \text{ implies } U \land V \in \mathcal{U}, \quad \text{(U3)}
\]
\[
U \in \mathcal{U} \text{ implies there exists } V \in \mathcal{U} \text{ such that } V \circ V \leq U, \quad \text{(U4)}
\]
\[
U \in \mathcal{U} \text{ implies } U^{-1} \in \mathcal{U}. \quad \text{(U5)}
\]
Subbasis and basis of a uniformity get the obvious significance (see also [5, p. 563]); it is also clear what we mean when we say that a f. uniformity is finer or coarser than another one.

Clearly (U5) may be replaced by

\[ \mathcal{U} \] has a basis of symmetric elements. \hspace{1cm} (U5')

Given a function \( f: X \to Y \), for any \( V: L^Y \to L^Y \), following [5], we define \( f^\ast(V): L^X \to L^X \) by the position:

\[ f^\ast(V)(\mu) = f^\ast(V(f(\mu))) \quad \text{for any} \quad \mu \in L^X. \]

It is easy to verify that if \( V \) satisfies (A1)-(A3), then \( f^\ast(V) \) satisfies (A1)-(A3), too.

If \( (X, \mathcal{U}), (Y, \mathcal{V}) \) are uniform spaces, a function \( f: X \to Y \) is said to be a uniform map if for every \( V \in \mathcal{V} \), the element \( f^\ast(V) \) belongs to \( \mathcal{U} \).

Hutton has shown that any fuzzy uniformity \( \mathcal{U} \) induces a fuzzy topology \( \mathcal{F}_\mathcal{U} \) putting

\[ \mu \in \mathcal{F}_\mathcal{U} \quad \text{if and only if} \quad \mu = \sup\{\rho \in L^X: U(\rho) \leq \mu \text{ for some } U \in \mathcal{U}\}; \]

moreover every uniform map from \( (X, \mathcal{U}) \) to \( (Y, \mathcal{V}) \) is continuous equipping \( X \) and \( Y \) with the induced fuzzy topologies.

Later on we shall need the following:

1.1 Proposition. Let \( (X, \mathcal{U}), (Y, \mathcal{V}) \) be uniform spaces, \( f: X \to Y \) a function, and \( \mathcal{J} \) a subbasis of \( \mathcal{V} \). Then \( f \) is a uniform map if and only if \( f^\ast(S) \in \mathcal{U} \) for every \( S \in \mathcal{J} \).

Proof: The "only if" part is trivial.

For the converse, clearly it is enough to show that, if \( S_1, S_2 \) belong to \( \mathcal{J} \), then \( f^\ast(S_1 \wedge S_2) \) belongs to \( \mathcal{U} \); namely, we show that

\[ f^\ast(S_1 \wedge S_2) = f^\ast(S_1) \wedge f^\ast(S_2). \]

First observe that the first member of the equality is less than or equal to the second one. For the other inequality we have, for \( \mu \in L^X \) and \( x \in X \),

\[ (f^\ast(S_1) \wedge f^\ast(S_2))(\mu)(x) = \bigwedge_{\mu_1 \vee \mu_2 = \mu} (S_1(f(\mu_1)) \vee S_2(f(\mu_2)))(f(x)), \]

and

\[ f^\ast(S_1 \wedge S_2)(\mu)(x) = (S_1 \wedge S_2)(f(\mu))(f(x)) = \bigwedge_{v_1 \vee v_2 = f(\mu)} (S_1(v_1) \vee S_2(v_2))(f(x)); \]
now notice that if \( v_1 \lor v_2 = f(\mu) \), we have

\[
(f^+(v_1) \land \mu) \lor (f^+(v_2) \land \mu) = (f^+(v_1) \lor f^+(v_2)) \land \mu = f^+(f(\mu)) \land \mu = \mu;
\]

moreover for \( i = 1, 2 \),

\[
f(f^+(v_i) \land \mu)(y) = \sup \{ (f^+(v_i) \land \mu)(x) : f(x) = y \}
\]

\[
= \sup \{ v_i(f(x) \land \mu(x)) : f(x) = y \}
\]

\[
= v_i(y) \land \sup \{ \mu(x) : f(x) = y \} = v_i(y) \land f(\mu)(y) = v_i(y).
\]

Hence if one takes \( \mu_i = f^+(v_i) \land \mu \), one gets \( \mu_1 \lor \mu_2 = \mu \) and \( f(\mu_i) = v_i \), and the conclusion follows. 1

2. Fuzzy Proximities

2.1 Definition. A fuzzy proximity on a set \( X \) is a function \( \delta : L^X \times L^X \to \{0, 1\} \) which satisfies, for any \( \mu, v, \rho \in L^X \), the following conditions:

\[
\begin{align*}
\delta(0, 1) &= 0, \quad \text{(P1)} \\
\delta(\mu, \rho) &= \delta(\rho, \mu), \quad \text{(P2)} \\
\delta(\mu, \rho) \lor \delta(v, \rho) &= \delta(\mu \lor v, \rho), \quad \text{(P3)} \\
\text{if } \delta(\mu, \rho) &= 0, \text{ there exists } \gamma \in L^X \text{ such that} \\
\delta(\mu, \gamma) &= 0, \quad \delta(\rho, \gamma') = 0 \\
\delta(\mu, \rho) &= 0 \text{ implies } \mu \preceq \rho'. \quad \text{(P5)}
\end{align*}
\]

The pair \( (X, \delta) \) is said to be a fuzzy proximity space.

If \( \delta(\mu, \rho) = 0 \) we say that \( \mu \) and \( \rho \) are far; otherwise we say that they are proximal.

(P1)–(P4) are the natural extensions of the classical case; (P5) needs some comment since in [7] the analogous axiom was formulated in a different manner: anyway its role will become clear later on: but now we point out that in the case \( L = \{0, 1\} \), (P5) means exactly that if two subsets intersect, then they are proximal. In the case \( L = [0, 1] = I \), (P5) means that \( \mu \) and \( \rho \) are proximal whenever there exists \( x \in X \) such that: \( \mu(x) + \rho(x) > 1 \).

2.2 Definition. Let \( (X, \delta) \), \( (Y, \eta) \) be fuzzy proximity spaces. A function
$f : X \rightarrow Y$ is a proximity map if one of the following equivalent conditions holds:

(a) for every $v, \sigma \in L^X$, $\eta(v, \sigma) = 0$ implies $\delta(f^*(v), f^*(\sigma)) = 0$,

(b) for every $\mu, \rho \in L^X$, $\delta(\mu, \rho) = 1$ implies $\eta(f(\mu), f(\rho)) = 1$.

To see that conditions (a) and (b) are equivalent, one may use part (i) of the next lemma; part (ii) will be used later on.

2.3 Lemma. Let $(X, \delta)$ be a fuzzy proximity space.

(i) If $\delta(\mu, \rho) = 0$, $\mu \geq v$, $\rho \geq \sigma$, then $\delta(v, \sigma) = 0$.

(ii) If $\delta(\mu_i, \rho_i) = 0$ for $i = 1, \ldots, n$, then $\delta(\bigwedge_{i=1, \ldots, n} \mu_i, \bigwedge_{i=1, \ldots, n} \rho_i) = 0$.

Proof. Use (P3) to prove (i); (i) and (P3) to prove (ii).

Clearly the set of all the proximitics on a given set $X$ can be equipped with a partial order by defining $\delta_1$ finer than $\delta_2$ (or $\delta_2$ coarser than $\delta_1$) if the identity of $X$ is a proximity map from $(X, \delta_1)$ to $(X, \delta_2)$.

Our aim is now to define and investigate the fuzzy topology induced by a fuzzy proximity.

Take a proximity space $(X, \delta)$ and, for any $\mu \in L^X$, put

$$\text{int}(\mu) = \sup\{\rho : \delta(\mu, \rho) = 0\};$$

we shall write indifferently $\mu$ or int$(\mu)$.

2.4 Proposition. The function $\text{int} : L^X \rightarrow L^X$ satisfies the interior axioms; namely, we have, for $\mu, \rho \in L^X$,

$$\text{int}(1) = 1,$$

$$\text{int}(\mu) \leq \mu,$$  \hspace{1cm} (12)

$$\text{int}(\text{int}(\mu)) = \text{int}(\mu),$$ \hspace{1cm} (13)

$$\text{int}(\mu \land \rho) = \text{int}(\mu) \land \text{int}(\rho).$$ \hspace{1cm} (14)

Proof. (11) and (12) follow trivially from (P1) and (P5), respectively.

(13) Clearly $\text{int}(\text{int}(\mu)) \leq \text{int}(\mu)$; now take $\rho$ such that $\delta(\rho, \mu') = 0$. By (P4) there exists $\gamma$ such that $\delta(\rho, \gamma') = 0$ and $\delta(\gamma, \mu') = 0$; hence $\rho \leq \text{int}(\gamma)$, $\gamma \leq \text{int}(\mu)$ and $\text{int}(\gamma) \leq \text{int}(\text{int}(\mu))$ because int is monotone; therefore $\rho \leq \text{int}(\mu)$ for every $\rho$ such that $\delta(\rho, \mu') = 0$, so that $\text{int}(\text{int}(\mu)) \geq \text{int}(\mu)$.

(14) Trivially $\text{int}(\mu \land \rho) \leq \text{int}(\mu) \land \text{int}(\rho)$. 
For the converse, observe that in a completely distributive lattice [2, p. 119], the infinite distributive law holds, hence we have

\[
\text{int}(\mu) \land \text{int}(\rho) = \sup \{ v : \delta(v, \mu') = 0 \} \land \sup \{ \sigma : \delta(\sigma, \rho') = 0 \}
\]

\[
= \sup \{ v \land \sigma : \delta(v, \mu') = 0 = \delta(\sigma, \rho') \}
\]

\[
\subseteq \sup \{ r : \delta(r, \mu' \lor \rho') = 0 \}
\]

\[
- \sup \{ r : \delta(r, (\mu \land \rho)') = 0 \} - \text{int}(\mu \land \rho).
\]

2.5 DEFINITION. The f. topology induced by the f. proximity \( \delta \) is denoted by \( \mathcal{E}_\delta \) and consists of all the fuzzy sets \( \mu \in L^X \) such that \( \mu = \text{int}(\mu) \).

Clearly the closure of \( \mu \) in the topology \( \mathcal{E}_\delta \), denoted by \( \text{cl} \mathcal{E}_\delta(\mu) \), or \( \bar{\mu} \), is given by \( \text{int}(\mu')' \).

We remark that if \( L = I \), then \( \mu \) is a \( \mathcal{E}_\delta \)-neighborhood of \( ax \) if and only if for every \( b < a \) we have \( \delta(bx, 1 - \mu) = 0 \).

2.6 Remark. (1) If \( (X, \delta) \) is a classical proximity space, for any \( \mu \in L^X \), put: \( \text{coz}(\mu) = \{ x \in X : \mu(x) > 0 \} \) and define

\[
\hat{\delta}(\mu, \rho) = 0 \quad \text{if and only if} \quad \text{coz}(\mu) \delta \text{coz}(\rho).
\]

It is easy to verify that \( \hat{\delta} \) is a fuzzy proximity, and the \( \mathcal{E}_\delta \)-open fuzzy sets are exactly the characteristic functions of the sets which are open in the topology induced by \( \delta \).

(2) Clearly the fuzzy proximities introduced by Katsaras satisfy conditions (P1)-(P5), and the \( \hat{\delta} \) of the example above is a Katsaras proximity. Furthermore, given a Katsaras proximity \( \eta \), it is easy to prove that there exists a classical proximity \( \delta \) such \( \hat{\delta} = \eta \): indeed for \( A, B \) subsets of \( X \), put

\[
A \delta B \quad \text{if and only if} \quad A \eta B;
\]

to prove that \( \delta \) is a (usual) proximity and that \( \hat{\delta} = \eta \), consider the fact that for every \( \mu, \rho \in I^X \) we have that the closure of \( \mu \) introduced by Katsaras (denoted by \( \bar{\mu} \) in this example) is a characteristic function and that

\[
\mu \eta \rho \quad \text{if and only if} \quad \bar{\mu} \eta \bar{\rho}
\]

\[
\text{coz}(\mu) \eta \text{coz}(\rho) \text{ if and only if} \quad \text{coz}(\mu) \delta \text{coz}(\rho) \text{ if and only if} \quad \hat{\delta}(\mu, \rho) = 1 \quad \text{(for the first equivalence, see [7, p. 103])}.
\]

The essence of the speech is that the Katsaras proximities are in a canonical 1–1 correspondence with the usual proximities.

2.7 PROPOSITION. Let \( (X, \delta) \), \( (Y, \eta) \) be f. proximity spaces. If \( f : X \to Y \) is
a proximity map, then it is continuous equipping $X$ and $Y$ with the induced f.
topologies.

Proof. Let $v$ be a $\mathcal{F}_\mathfrak{V}$-open set, i.e., $v = \sup \{ \sigma : \eta(\sigma, v') = 0 \}$; hence $f^*(v) = \sup \{ f^*(\sigma) : \eta(\sigma, v') = 0 \} \leq \sup \{ \rho : \delta(\rho, (f^*(v))' = 0 \}$, i.e., $f^*(v) = \text{int}(f^*(v))$
is a $\mathcal{F}_\mathfrak{V}$-open set. 

2.8. PROPOSITION. Let $\delta$ be a fuzzy proximity on $X$. Then,

(i) $\delta(\mu, \rho) = 0$ if and only if $\delta(\bar{\mu}, \rho) = 0$,

(ii) $\bar{\mu} = \sup \{ v : \delta(\mu, \rho) = \delta(v, \rho) \text{ for every } \rho \in L^X \}$.

Proof. (i) The “if” part is trivial; for the converse, take $\gamma$ such that $\delta(\gamma', \mu) = 0 = \delta(\mu', \gamma)$; hence $\gamma' \leq \text{int}(\mu')$ so that $\gamma \geq (\text{int}(\mu'))' = \bar{\mu}$ and $\delta(\rho, \bar{\mu}) = 0$.

(ii) By (i) we get that $\bar{\mu} \leq \sup \{ v : \delta(\mu, \rho) = \delta(v, \rho) \text{ for every } \rho \in L^X \}$; then take $v \leq \bar{\mu}$ such that $\delta(\mu, \rho) = \delta(v, \rho)$ for every $\rho \in L^X$ and put $\tau = \bar{\mu} \vee v$; observe that $\tau \geq \bar{\mu}$ and $\delta(\tau, \rho) = \delta(\mu, \rho)$ for every $\rho \in L^X$. Since $\tau' \leq (\bar{\mu})' - \text{int}(\mu')$, by the definition of int there exists $\sigma \leq \tau'$ such that $\delta(\mu, \sigma) = 0$, while (P5) implies $\delta(\tau, \sigma) = 1$, a contradiction. 

3. FUZZY PROXIMITIES AND SEPARATION AXIOMS

We collect here some definitions which we shall use: some of them are the
usual ones (see, e.g., [4, 5, 12, 16]); however we remark that the Hausdorff
axiom has had a hard life in fuzzy set theory since many authors proposed
different definitions (e.g., [14, 12, 17]); one of these definitions, given in [1],
is rather strange, since it implies that every fuzzy set is open, as one can
easily check.

3.1 DEFINITIONS. A fuzzy topological space $(X, \mathcal{F})$ is said to be

(1) $T_1$ if every fuzzy point is closed;

(2) Hausdorff or $T_2$ if, given fuzzy points $ax, by$ such that $ax \leq (by)'$
$\text{(iff } by \leq (ax)' \text{)},$ there exist f. open sets $\mu, \rho$ such that: $ax \leq \mu,$ $by \leq \rho$ and $\mu \leq \rho'$:

(3) regular if for every fuzzy point $ax$ and f. closed set $\sigma$ such that $ax \leq \sigma'$, there exist f. open sets $\mu, \rho$ such that $ax \leq \mu,$ $by \leq \rho$ and $\mu \leq \rho'$;

(4) completely regular [5] if for any f. open set $\mu$ there exist a
collection $\mu_i$ and continuous functions $f_i : X \rightarrow I(L)$ such that $\sup \{ \mu_i \} = \mu$ and $\mu_i(x) \leq f_i(x)(1-) \leq f_i(x)(0+) \leq \mu(x)$ for $x \in X$;

(5) normal [4] if for every pair of f. closed sets $\sigma, \tau$ such that $\sigma \leq \tau'$, there exist f. open sets $\mu, \rho$ such that $\sigma \leq \mu \leq \rho' \leq \tau'$.
The previous definitions coincide with the usual ones in the case
$L = \{0, 1\}$ and the next proposition shows that they behave as one expects
(which provides the first justification for our definition of Hausdorff space).

3.2 **Proposition.** (i) $T_2$ implies $T_1$.

(ii) regular $T_1$ implies $T_2$.

(iii) normal $T_1$ implies regular.

(iv) normal $T_1$ implies completely regular.

**Proof.** (i), (ii), and (iii) are immediate by definitions; (iv) is a consequence of [4, Theorem 1].

3.3 **Definition.** A proximity $\delta$ is said to be separated if the axiom (P5)
is replaced by

given fuzzy points $ax, by$, we have $\delta(ax, by) = 0$ iff $ax \leq (by)'$. (P5')

Trivially (P1)-(P4), (P5') imply (P1)-(P5) and we can state the following:

3.4 **Proposition.** The fuzzy topology induced by a separated proximity $\delta$ is $T_2$.

**Proof.** $ax \leq (by)'$ implies, by (P5'), $\delta(ax, by) = 0$; hence there exists $\gamma$ such that $\delta(ax, \gamma') = 0$ and $\delta(\gamma, by) = 0$ so that $ax \leq \gamma \leq \gamma \leq (by)'$.

We have already seen that, given a fuzzy proximity $\delta$ on $X$, we have $\delta(\mu, \rho) = \delta(\bar{\mu}, \rho)$: hence $\delta(\mu, \rho) = 0$ implies $\bar{\mu} \leq (\rho)'$; therefore one may wander if, given a fuzzy topological space, the position

$$\delta(\mu, \rho) = 0 \quad \text{iff} \quad \bar{\mu} \leq (\rho)'$$

defines a fuzzy proximity. The next theorem gives a solution which is analogous to the classical one.

3.5 **Theorem.** Let $(X, \mathcal{F})$ be a fuzzy topological space. Define

$$\delta(\mu, \rho) = 0 \quad \text{iff} \quad \bar{\mu} \leq (\rho)'$$

Then:

(i) $\delta$ is a proximity if and only if $(X, \mathcal{F})$ is normal,

(ii) if $\delta$ is a proximity, then $\mathcal{F}_\delta$ is coarser than $\mathcal{F}$.

(iii) if $(X, \mathcal{F})$ is normal and $T_1$, then $\mathcal{F}_\delta$ and $\mathcal{F}$ coincide, and $\delta$ is the finest proximity which induces $\mathcal{F}$.

**Proof.** (i) $\Rightarrow$ Take $\mu, \rho \in$ closed sets, $\bar{\mu} \leq \rho'$; by the definition of $\delta$, it is $\delta(\mu, \rho) = 0$, hence there exists $\gamma$ such that $\delta(\mu, \gamma') = 0$, $\delta(\gamma, \rho) = 0$, which imply $\mu \leq \gamma \leq \gamma \leq \rho'$. 

\( \leq (P1), (P2), (P3) \) and \( (P5) \) are trivial. To check \( (P4) \), take \( \mu, \rho \) such that \( \delta(\mu, \rho) = 0 \), which means \( \bar{\mu} \leq (\bar{\rho})' \); by normality there exists an open set \( \gamma \) such that \( \bar{\mu} \leq \gamma \leq \bar{\gamma} \leq (\bar{\rho})' \) and conclude that \( \delta(\mu, \gamma) = 0 \) and \( \delta(\gamma, \rho) = 0 \).

(ii) Let \( \mu \) be a \( \mathcal{F} \)-open set: it means that \( \mu = \text{sup}\{\rho_i : i \in I\} \), where \( \delta(\rho_i, \mu') = 0 \) for every \( i \in I \), that is, \( \text{cl}_\mathcal{F}(\rho_i) \leq (\text{cl}_\mathcal{F}(\mu'))' \); hence \( \mu \leq \bigvee_{i \in I} \text{cl}_\mathcal{F}(\rho_i) \leq (\text{cl}_\mathcal{F}(\mu'))' \leq \mu \) and \( \mu = (\text{cl}_\mathcal{F}(\mu'))' \) is a \( \mathcal{F} \)-open set.

(iii) If \( \mu \) is a \( \mathcal{K} \)-open set and \( ax \leq \mu \), then, by the definitions, \( \delta(ax, \mu') = 0 \), which implies that \( ax \leq \text{int}_\mathcal{F}(\mu) \).

The last remark is an obvious consequence of 2.8(i) and \( (P5) \). 1

We conclude this section providing another approach to fuzzy proximity spaces, which clearly resembles what happens in the usual case.

Take a fuzzy proximity \( \delta \) and consider the binary relation \( \leq \) on \( L^X \) given by \( \mu \leq \rho \) if \( \delta(\mu, \rho') = 0 \). It is easy to show that the relation \( \leq \) verifies the following conditions:

\[
1 \leq 1, \\
\mu \leq \rho \leq \sigma \leq \tau \text{ implies } \mu \leq \tau. \\
\mu \leq \rho_i, i = 1, ..., n, \text{ implies } \mu \leq \bigwedge_{i=1}^n \rho_i, \\
\mu \leq \rho \text{ implies } \rho' \leq \mu', \\
\mu \leq \rho \text{ implies that there exists } \gamma \text{ such that } \mu \leq \gamma \leq \rho, \\
\mu \leq \rho \text{ implies } \mu \leq \rho;
\]

if \( \delta \) is separated, then we also have:

for every pair of fuzzy points \( ax, by \), we have

\[
ax \leq (by)' \iff ax \leq (by)'.
\]

One sees immediately that \( (Q2) \) and \( (Q6') \) imply \( (Q6) \).

Vice versa, given a relation \( \leq \) on \( L^X \), which satisfies the properties \( (Q1)-(Q6) \), one obtains a fuzzy proximity putting

\[
\delta(\mu, \rho) = 0 \iff \mu \leq \rho'.
\]

If the relation satisfies \( (Q1)-(Q5), (Q6') \), such a proximity is separated.

4. CONNEXIONS BETWEEN FUZZY PROXIMITIES AND FUZZY UNIFORMITIES

In this section we shall study some connexions between fuzzy uniformities and fuzzy proximities: namely, we shall show that any \( \mathcal{F} \)-uniformity induces
a f. proximity in a canonical way, and vice versa: this correspondence works nicely.

Let \( \mathcal{U} \) be a f. uniformity and, for \( \mu, \rho \in L^X \), define

\[
\delta_{\mathcal{U}}(\mu, \rho) = 0 \quad \text{if and only if there exists } U \in \mathcal{U} \text{ such that } U(\mu) \leq \rho'.
\]

**4.1 Theorem.** \( \delta_{\mathcal{U}} \), as defined above, is a f. proximity.

**Proof.** We verify properties (P1)–(P5).

(P1) Trivial.

(P2) \( \delta_{\mathcal{U}}(\mu, \rho) = \delta_{\mathcal{U}}(\rho, \mu) \) since, for \( U \in \mathcal{U} \), \( U(\mu) \leq \rho' \) iff \( U^{-1}(\rho) \leq \mu' \) [5, Proposition 10].

(P3) It is enough to prove that \( \delta_{\mathcal{U}}(\mu, \rho) = 0 = \delta_{\mathcal{U}}(v, \rho) \) implies \( \delta_{\mathcal{U}}(\mu \vee v, \rho) = 0 \) since the converse is trivial. If \( U(\mu) \leq \rho' \), \( V(v) \leq \rho' \), we have \( (U \wedge V)(\mu \vee v) \leq \rho' \); then \( \delta_{\mathcal{U}}(\mu \vee v, \rho) = 0 \).

(P4) Let \( \delta_{\mathcal{U}}(\mu, \rho) = 0 \); there exists \( U \in \mathcal{U} \) such that \( U(\mu) \leq \rho' \). Take \( V \in \mathcal{U} \) which verifies \( V = V^{-1} \). \( V \circ V \leq U \); then \( V(V(\mu)) \leq \rho' \) implies \( V(\rho) \leq (V(\mu))' \), hence, for \( \gamma = V(\rho) \), we have \( \delta_{\mathcal{U}}(\mu, \gamma) = 0 = \delta_{\mathcal{U}}(\rho, \gamma') \).

(P5) Trivial.

**4.2 Remark.** We say that a f. uniformity \( \mathcal{U} \) is separated if, given f. points \( ax, by \) such that \( ax \leq (by)' \), there exists \( U \in \mathcal{U} \) such that \( U(ax) \leq (by)' \).

It is easy to show that a separated uniformity induces a Hausdorff topology and that \( \delta_{\mathcal{U}} \) is a separated proximity.

**4.3 Theorem.** Let \( \mathcal{U} \) be a f. uniformity. \( \mathcal{U} \) and \( \delta_{\mathcal{U}} \) induce the same topology.

**Proof.** Given a fuzzy set \( \mu \), observe that

\[
\{ v : \text{there exists } U \in \mathcal{U} \text{ such that } U(v) \leq \mu \} = \{ v : \delta_{\mathcal{U}}(v, \mu') = 0 \}
\]

and the supremum of the first member of the equality is the interior of \( \mu \) in the topology induced by \( \mathcal{U} \), while the supremum of the second one is the interior of \( \mu \) in the topology induced by \( \delta_{\mathcal{U}} \).

Now we tackle this problem: how can one construct a fuzzy uniformity when a fuzzy proximity is given? As in the classical case, the solution presents some difficulties.

Let \( (X, \delta) \) be a f. proximity space and put

\[
\mathcal{A} = \{ (\mu, \rho) \in L^X \times L^Y : \delta(\mu, \rho) = 0 \}.
\]
For every \((\mu, \rho) \in L^X \times L^X\), define \(U_{\mu \rho}: L^X \to L^X\) as follows: \(U_{\mu \rho}(Q) = 0\), \(U_{\mu \rho}(v) = \rho'\) if \(v \leq \mu\), \(U_{\mu \rho}(v) = 1\) otherwise; denote by \(\mathcal{F}\) the set \(\{U_{\mu \rho}: (\mu, \rho) \in \mathcal{F}\}\).

4.4 Theorem. \(\mathcal{F}\) is a subbasis for a uniformity.

Proof. Trivially every member of \(\mathcal{F}\) satisfies (A1)-(A3). Now we are going to show that \((U_{\mu \rho})^{-1} = U_{\mu \rho}\): in fact, \((U_{\mu \rho})^{-1}(v) = \inf\{\sigma: U_{\mu \rho}(\sigma') \leq v'\}\); this implies that

1. \((U_{\mu \rho})^{-1}(0) = 0\),
2. if \(0 \neq v \leq \rho\), i.e., \(1 \neq v' \geq \rho'\), then \(U_{\mu \rho}(\sigma') \leq v'\) iff \(U_{\mu \rho}(\sigma') \leq \rho'\) iff \(\sigma' \leq \mu\), i.e., \(\sigma \geq \mu'\); hence \((U_{\mu \rho})^{-1}(v) = \inf\{\sigma: \sigma \geq \mu'\} = \mu'\);
3. if \(v \leq \rho\), i.e., \(v' \geq \rho'\), then \(U_{\mu \rho}(\sigma') \leq v'\) iff \(U_{\mu \rho}(\sigma') = 0\) iff \(\sigma = 1\), that is, \((U_{\mu \rho})^{-1}(v) = 1\).

At this point observe that if \((\mu, \gamma), (\gamma', \rho)\) belong to \(L^X \times L^X\), we have

\[U_{\gamma' \rho} \circ U_{\mu \gamma} = U_{\mu \rho}.\]

Fix \((\mu, \rho) \in \mathcal{F}\): by (P4) there exists \(\gamma \in L^X\) such that \((\mu, \gamma), (\gamma', \rho)\) belong to \(\mathcal{F}\), hence the element \(V = U_{\gamma' \rho} \land U_{\mu \gamma}\) satisfies \(V \circ V \leq U_{\mu \rho}\), which completes the proof.  

Given a fuzzy proximity \(\delta\), we shall denote by \(\mathcal{H}_\delta\) the fuzzy uniformity which has the collection \(\mathcal{F}\) introduced above as a subbasis.

With a technique quite analogous to the one used by Hutton in the proof of Lemma 3 in [5] (which requires that \(L^X\) be completely distributive), one can show that for any finite family \(U_1, ..., U_n\) of elements of \(\mathcal{F}\), the following equality holds, for any \(v \in L^X\):

\[(U_1 \wedge \cdots \wedge U_n)(v) = \inf\{U_1(v_1) \cup \cdots \cup U_n(v_n): v_1 \cup \cdots \cup v_n = v\}.\quad (*)\]

This formula enables us to characterize the finite infima of elements of \(\mathcal{F}\).

4.5 Lemma. Let \(\delta\) be a given f. proximity, \(v \in L^X\), \(\mathcal{F}\) as above, \(U_{\mu_i, \rho_i}\) elements of \(\mathcal{F}\) for \(i = 1, ..., n\). Denote by \(\mathcal{F}\) the set

\[\mathcal{J} = \left\{ J: J \subseteq \{1, ..., n\}, \bigvee_{j \in J} \mu_j \geq \rho \right\},\]

and, in order to simplify the notations, put

\[V = \bigwedge_{i=1}^n U_{\mu_i, \rho_i}; \tau_r = \left(\bigwedge_{j \in J} \rho_j\right)'\quad \text{for any nonempty subset } J \text{ of } \{1, ..., n\}.\]
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Then

\[ V(v) - \inf \{ \tau_j : J \subseteq \mathcal{F} \} \quad (**) \]

(we agree that the second member is 1 when \( \mathcal{F} = \emptyset \)).

**Proof.** First, notice that if \( J, K \) are nonempty subsets of \{1,..., n\} and \( J \subseteq K \), then \( \tau_j \leq \tau_K \). We want to show that the left member of equality (**) is less than or equal to the right one; this is trivial if \( \mathcal{F} \) is empty. For any nonempty \( J \subseteq \mathcal{F} \) such that

\[ H \subseteq J \text{ implies } \bigvee_{h \in H} \mu_h \geq v. \]

and for \( i = \{1,..., n\} \), put

\[ v_i = v \land \mu_i \quad \text{for } i \in J \]
\[ = 0 \quad \text{otherwise}. \]

We obtain \( \bigvee_{i=1}^{n} U_{\mu_i}(v_i) = \tau_j \); therefore by (*), \( V(v) \leq \inf \{ \tau_j : J \subseteq \mathcal{F} \} \).

The converse inequality is now trivial if \( V(v) = 1 \); otherwise, if \( \tau = \bigvee_{i=1}^{n} U_{\mu_i}(v_i) \) is an element different from 1 which occurs in the family at the second member of (*), consider \( J = \{ j : j \in \{1,..., n\}, (\mu_j)' \leq \tau \} \); clearly, \( \tau = \tau_j \). Now if \( i \notin J \), then \( (\mu_i)' \leq \tau \); hence \( U_{\mu_i}(v_i) = 0 \), which implies \( v_i = 0 \); therefore \( v = \bigvee_{i=1}^{n} v_i = \bigvee_{j \in J} v_j \leq \bigvee_{j \in J} \mu_j \) (indeed, for \( j \in J \), it is \( v_j \leq \mu_j \) since \( U_{\mu_i}(v_i) \neq 1 \)), that is, \( J \subseteq \mathcal{F} \) and \( \tau \geq \inf \{ \tau_j : J \subseteq \mathcal{F} \} \).

The proof is complete. \[
\]

4.6 **Theorem.** Let \((X, \delta)\) be a fuzzy proximity space, and denote by \( \tilde{\delta} \) the fuzzy proximity \( \delta_{\mathcal{H}_{\delta}} \); then we have \( \delta = \tilde{\delta} \).

**Proof.** Let us suppose \( \delta(\mu, \rho) = 0 \); then \( U_{\mu \rho} \in \mathcal{H}_{\delta} \) and \( U_{\mu \rho}(\mu) = \rho' \), which gives \( \delta(\mu, \rho) = 0 \).

Conversely, let \( \tilde{\delta}(\mu, \rho) = 0 \); if \( \rho = 0 \), trivially \( \delta(\mu, \rho) = 0 \), then assume \( \rho \neq 0 \) and observe that \( \delta(\mu, \rho) = 0 \) means that there exists a basic element of \( \mathcal{H}_{\delta} \), say \( V = U_{\mu_1 \mu_1} \land \cdots \land U_{\mu_n \mu_n} \) such that \( V(\mu) \leq \rho' \). Using the same notations introduced above, \( \mathcal{F} \) is nonempty since \( \rho \neq 0 \), and by the previous lemma \( V(\mu) = \bigwedge_{j \in \mathcal{F}} \tau_j \), which implies \( \rho' \geq \bigwedge_{j \in \mathcal{F}} \tau_j = (\bigvee_{j \in \mathcal{F}} (\bigwedge_{j \in \mathcal{F}} \mu_j))' \). That is, \( \rho \leq \bigvee_{j \in \mathcal{F}} (\bigwedge_{j \in \mathcal{F}} \mu_j) \); on the other hand observe that \( \mu \leq \bigvee_{j \in \mathcal{F}} (\bigwedge_{j \in \mathcal{F}} \mu_j) \). Since \( \delta(\mu_i, \rho_i) = 0 \) for \( i = 1,..., n \), a double application of Lemma 2.3(ii) enables us to say that

\[ \delta \left( \bigwedge_{j \in \mathcal{F}} \left( \bigvee_{j \in \mathcal{F}} \mu_j \right), \bigvee_{j \in \mathcal{F}} \left( \bigwedge_{j \in \mathcal{F}} \rho_j \right) \right) = 0, \]

so that \( \delta(\mu, \rho) = 0 \) too. \[
\]
4.7 **Corollary.** Let $\delta$ be a fuzzy proximity on $X$. Then $\delta$ and $\mathcal{V}_\delta$ induce the same fuzzy topology.

*Proof.* Use Theorems 4.3 and 4.6. ■

4.8 **Corollary.** A fuzzy topological space $(X, \mathcal{V})$ is completely regular if and only if $\mathcal{V}$ can be induced by a fuzzy proximity.

*Proof.* It follows from 4.3, 4.7 and [5, Theorem 17]. ■

Given a fuzzy uniform space $(X, \mathcal{U})$, we shall denote by $p\mathcal{U}$ the uniformity $\mathcal{U}_{p\mathcal{U}}$: a description of $p\mathcal{U}$ and a study of its properties are the object of the next section. For now we can easily prove the following proposition.

4.9 **Proposition.** The fuzzy uniformity $p\mathcal{U}$ introduced above is coarser than $\mathcal{U}$.

*Proof.* It is enough to show that $U_{\mu, \rho} \in \mathcal{U}$ whenever $\delta_{\mathcal{U}}(\mu, \rho) = 0$, that is, whenever there exists $U \in \mathcal{U}$ such that $U(\mu) \leq \rho'$; then, owing to the definition of $U_{\mu, \rho}$, we have $U \subseteq U_{\mu, \rho}$, hence $U_{\mu, \rho} \in \mathcal{U}$. ■

The next example shows that indeed $p\mathcal{U}$ can be properly coarser than $\mathcal{U}$.

4.10 **Example.** Let $\mathcal{U}$ be the fuzzy uniformity on $I^1$ which has as a basis the element $U: I^1 \rightarrow I^1$ so defined:

$$U(\mu) = \sup \{\mu(x) : x \in I\}.$$ 

We show that $\mathcal{U} \neq p\mathcal{U}$. In fact the inequality

$$U \supseteq U_{\mu, \rho_1} \wedge \ldots \wedge U_{\mu, \rho_n},$$

where $\delta_{\mathcal{U}}(\mu_i, \rho_i) = 0$ for $i = 1, \ldots, n$, cannot be satisfied by any choice of the pairs $(\mu_i, \rho_i)$; first it is not restrictive to suppose $\mu_i \neq 0$ since $U_{\rho, \rho}(v) = 1$ for any $\rho \in I^1$ and any $v \in I^1$, $v \neq 0$; then observe that, putting $m_i = U(\mu_i)$, $r_i = U(\rho_i)$, trivially it is $U_{\mu, \rho_i} \subseteq U_{\mu, \rho_i}$; then take $0 \leq m \leq \wedge_{i=1}^{n} m_i$ and, using the equality (*), we get

$$m = U(m) < \bigwedge_{i=1}^{n} m_i \leq \bigwedge_{i=1}^{n} (1 - r_i) = (U_{m, r_1} \wedge \ldots \wedge U_{m, \rho_n})(m);$$

clearly, in this example, we use the same symbols to denote both real numbers and their corresponding constant functions.
5. Totally Bounded Fuzzy Uniformities

In this section we shall introduce a reflective full subcategory of the category of f. uniform spaces and f. uniform maps, by which we provide a characterization of those uniformities that are induced by a f. proximity.

5.1 Definition. We say that a fuzzy uniformity \( \mathcal{U} \) is totally bounded if there exists a basis \( \mathcal{B} \) of \( \mathcal{U} \) such that for any \( U \in \mathcal{B} \), the set \( \{ U(\mu) : \mu \in L^X \} \) is finite.

The definition introduced above is coherent with the well-known definition of totally bounded uniformity: that is, in the usual case, i.e., if \( L = \{0, 1\} \), one can show that the two definitions are equivalent.

Remark that, in view of (*), we can replace the word "basis" by "subbasis" in Definition 5.1.

5.2 Proposition. For any fuzzy proximity \( \delta \), the fuzzy uniformity \( \mathcal{U}_\delta \) is totally bounded.

Proof. Let \( \mathcal{S} \) be the subbasis of \( \mathcal{U}_\delta \) which we have described in Section 4: every element of \( \mathcal{S} \) assumes at most three values. 

We remark that parts (ii) and (iii) of the next lemma are extensions of Lemmas 36 and 30 of [6, Chap. II], respectively.

5.3 Lemma. Let \((X, \mathcal{U})\) be a fuzzy uniform space, \( U \) an element of \( \mathcal{U} \) such that the set \( \{ U(\mu) : \mu \in L^X \} = \{ \rho_1, \ldots, \rho_n \} \) is finite. Then

(i) \( \{ U^{-1}(\mu) : \mu \in L^X \} \) is finite;

(ii) there exist elements \( U_1, \ldots, U_n \in \mathcal{U} \) such that every \( U_i \) assumes only one nontrivial value and \( U = U_1 \wedge \cdots \wedge U_n \);

(iii) there exists \( V \in \mathcal{U} \) such that the set \( \{ V(\mu) : \mu \in L^X \} \) is finite and \( V \circ V \leq U \).

Proof: (i) \( U^{-1}(\mu) \neq U^{-1}(\rho) \) implies that \( \mu' \) and \( \rho' \) do not exceed the same elements \( \rho_i \): hence \( U^{-1} \) can assume at most \( 2^n \) different values.

(ii) Put \( \mu_i = \sup \{ \mu : U(\mu) \leq \rho_i \} \). Since \( U \) preserves suprema, we have \( U(\mu_i) \leq \rho_i \), hence \( \delta_{\mu'}(\mu_i, \rho_i) = 0 \); therefore \( U_{\mu, \rho_i} \) belongs to \( \mathcal{U} \) for every \( i = 1, \ldots, n \) by 4.9. It is obvious, by the definitions, that \( U_{\mu, \rho_i} \leq U \) for every \( i \).

On the other hand \( U \geq U_{\mu, \rho_1} \wedge \cdots \wedge U_{\mu, \rho_n} \) because \( U(\mu) = \rho_j, j \in \{1, \ldots, n\} \), implies \( \mu \leq \mu_j \), hence \( U_{\mu, \rho}(\mu) = \rho_j \). The equality is now evident.

(iii) By (ii), \( U \) is a finite infimum of elements of the canonical subbasis of \( \rho \mathcal{U} \); hence the thesis follows from Proposition 5.2. 

5.4 Theorem. Let \( \mathcal{U} \) be a f. uniformity on \( X \). The collection \( \{U \in \mathcal{U} : \mu \in L^X \text{ is finite}\} \) is a basis for a (totally bounded) uniformity, which is nothing but \( p\mathcal{U} \); therefore \( p\mathcal{U} \) is the finest totally bounded uniformity coarser than \( \mathcal{U} \).

Proof. The collection in the statement is a basis for a uniformity by Lemma 5.3 and the equality \( (\ast) \); such a uniformity is finer than \( p\mathcal{U} \) by Propositions 4.9 and 5.2, and coarser by 5.3(ii).

At this point it is clear that the results obtained in this section imply the following important theorem:

5.5 Theorem. A fuzzy uniformity \( \mathcal{U} \) is totally bounded if and only if \( \mathcal{U} = p\mathcal{U} \).

The next theorem shows that the behaviour of the correspondence \( (X, \mathcal{U}) \to (X, p\mathcal{U}) \) is the best one which can be expected.

5.6 Definition. Given f. uniform spaces \( (X, \mathcal{U}), (Y, \mathcal{V}) \), we say that a function \( f : (X, \mathcal{U}) \to (Y, \mathcal{V}) \) is a proximity map if it is a proximity map between the induced fuzzy proximity spaces; i.e., if for every \( v, \sigma \in L^X \), for which there exists \( V \in \mathcal{V} \) such that \( V(v) \leqslant \sigma \), there exists \( U \in \mathcal{U} \) such that \( U(f^{-1}(v)) \leqslant (f^{-1}(\sigma))^\prime \).

5.7 Lemma. Let \( f : X \to Y \) be an arbitrary function, and \( v, \sigma \) belong to \( L^X \), then \( f^{-1}(U_v \sigma) = U_{v \circ f \sigma} \).

Proof. We have, by definition, for \( \mu \in L^X \), \( f^{-1}(U_v \sigma)(\mu) = (U_v(f(\mu))) \circ f \). Now observe that \( f(0) = 0 \), hence \( f^{-1}(U_v \sigma)(0) = 0 \); afterwards \( 0 < \mu \leqslant v \circ f \) implies \( f(\mu) \leqslant f(v \circ f) \leqslant v \), hence \( f^{-1}(U_v \sigma)(\mu) = \sigma \circ f = (\sigma \circ f)'^\prime \); finally, \( \mu \leqslant v \circ f \) implies \( f(\mu) \leqslant v \), otherwise \( f^{-1}(v) \geq f^{-1}(f(\mu)) \geq \mu \). Hence \( f^{-1}(U_v \sigma)(\mu) = 1 \).

In conclusion, we have just proved that \( f^{-1}(U_v \sigma) = U_{v \circ f \sigma} \).

5.8 Theorem. Let \( (X, \mathcal{U}), (Y, \mathcal{V}) \) be f. uniform spaces. Then:

(i) the correspondence \( (X, \mathcal{U}) \to (X, p\mathcal{U}) \) is a reflection from the category of f. uniform spaces and f. uniform maps onto the full subcategory of totally bounded f. uniform spaces;

(ii) a function \( f : (X, \mathcal{U}) \to (Y, \mathcal{V}) \) is a f. proximity map if and only if \( f : (X, \mathcal{U}) \to (Y, p\mathcal{V}) \) is a f. uniform map; therefore \( (X, \mathcal{U}) \) and \( (Y, \mathcal{V}) \) are "proximally isomorphic" if and only if \( (X, p\mathcal{U}) \) and \( (Y, p\mathcal{V}) \) are uniformly isomorphic.

Proof. (i) By 4.9 the identity \( i : (X, \mathcal{U}) \to (X, p\mathcal{U}) \) is a uniform map.
Moreover if \((Z, \mathcal{H})\) is a totally bounded \(f\)-uniform space and \(f: (X, \mathcal{H}) \rightarrow (Z, \mathcal{H})\) a \(f\)-uniform map, then \(f^* : (X, p\mathcal{H}) \rightarrow (Z, \mathcal{H})\) is still uniform. Indeed if \(W \in \mathcal{H}\) is such that \(\{W(\tau): \tau \in L^X\}\) is finite, then \(\{f^*(W)(\mu): \mu \in L^X\}\) is finite too, and so \(f^*(W) \in p\mathcal{H}\) by 5.4.

(ii) \(\Rightarrow\) By Proposition 1.1 it is enough to show that \(f^*(U_{\nu\sigma})\) belongs to \(\mathcal{H}\) whenever \(\nu, \sigma\) are elements of \(L^Y\) for which there exists \(V \in \mathcal{T}\) such that \(V(\nu) \leq \sigma'\). Indeed, by the hypothesis, there exists \(U \in \mathcal{H}\) such that \(U(f^*(\nu)) \leq (f^*(\sigma))'\); hence the element \(U_{f^*(\nu)f^*(\sigma)} = f^*(U_{\nu\sigma})\) belongs to \(\mathcal{H}\).

\(<\) Let \(\nu, \sigma \in L^Y, V \in \mathcal{T}\) such that \(V(\nu) \leq \sigma'\); then \(U_{\nu\sigma}\) belongs to \(p\mathcal{T}\) and \(f^*(U_{\nu\sigma})\) belongs to \(\mathcal{H}\); moreover, by the lemma above, \(f^*(U_{\nu\sigma})(f^*(\nu)) = (f^*(\sigma))'\), which implies that \(f\) is a proximity map.

The last assertion is clear.

5.9 Remark. If \((X, \mathcal{H})\) is a totally bounded \(f\)-uniform space, then for every \(U \in \mathcal{H}\) there exists a finite subset \(F\) of \(X\) such that \(U(F) = 1\). In fact, take \(V \leq U\) such that \(\{V(\mu): \mu \in L^X\}\) has a finite number of elements, say: \(0, 1, \mu_1, \ldots, \mu_n\). For every \(i \in \{1, \ldots, n\}\), take \(x_i \in X\) such that \(\mu_i(x_i) < 1\) and put \(F = \{x_1, \ldots, x_n\}\); then observe that \(U(F) \geq V(F) \geq V(x_i) \leq \mu_i\) for every \(i\). Hence \(V(F) \neq \mu_i\) for every \(i\), which implies \(U(F) = V(F) = 1\).

However this condition is properly weaker than the total boundedness: one can easily check that the space of Example 4.10 verifies this condition, while we have already proved that it is not totally bounded.

We conclude the paper providing at the same time, an example of a nontrivial totally bounded \(f\)-uniform space and an extension of a classical result.

Let \(I(L)\) be the fuzzy unit interval, equipped with the usual \(f\)-topology and usual \(f\)-uniformity introduced in [4, 5]. We recall that a subbasis for the topology is given by \(\{L_t, R_t: t \in \mathbb{R}\}\), where

\[
L_t: I(L) \rightarrow L \quad \text{and} \quad R_t: I(L) \rightarrow L
\]

are defined by \(L_t(\mu) = (\mu(t-))', R_t(\mu) = \mu(t+)\); a subbasis for the uniformity is the set \(\{B_\varepsilon, B_\varepsilon^{-1}: \varepsilon \in \mathbb{R}, \varepsilon > 0\}\), where

\[
B_\varepsilon: L^{I(L)} \rightarrow L^{I(L)}
\]

is defined by

\[
B_\varepsilon(0) = 0 \quad \text{and} \quad B_\varepsilon(\mu) = R_{t-\varepsilon},
\]

where \(t\) is the greatest \(s \in \mathbb{R}\) such that \(\mu \leq L_s' = \inf \{R_{s-\varepsilon}: \mu \leq L_s'\}\).

5.10 Theorem. The usual fuzzy uniformity on \(I(L)\) is totally bounded.
Proof. Let \( n \) be a fixed natural number. For \( \mu \in L^{(1)} \) define
\[
\tilde{B}_{1/n}(\mu) = R_{t-1/2n}
\]
where \( \tilde{t} \) is the greatest element of the form \( m/2n, m \in \{0, 1, ..., 2n + 1\} \), such that \( \mu \leq L_{m/2n}' \)
\[
= \inf\{R_{(m-1)/2n} : m \in \{0, 1, ..., 2n + 1\}, \mu \leq L_{m/2n}' \}.
\]
Clearly \( \tilde{B}_{1/n} \) assumes a finite number of values (to be precise, the values that it assumes are \( 2n + 2 \)); moreover, if \( \tilde{B}_{1/n}(\mu) = R_{\tilde{t}-1/2n} \) and \( B_{1/n}(\mu) = R_{t-1/n} \), it is easily seen that \( t - 1/2n < \tilde{t} \leq t \), hence that \( t - 1/n < \tilde{t} - 1/2n \leq t - 1/2n \) and that \( R_{t-1/n} > R_{\tilde{t}-1/2n} \geq R_{t-1/2n} \); finally, by the last inequality, we can say that \( B_{1/n}(\mu) > \tilde{B}_{1/n}(\mu) \geq B_{1/2n}(\mu) \). This clearly implies that the set \( \{ B_{1/n}, \tilde{B}_{1/n} : n \in \mathbb{N} \} \) is a subbasis for the usual \( f \) uniformity on \( I(L) \); now we have only to recall Lemma 5.3(i), in order to conclude that the usual \( f \) uniformity is totally bounded. 

REFERENCES