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On the existence of positive solutions of p-Laplacian difference equations

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Abstract

In this paper, by means of fixed point theorem in a cone, the existence of positive solutions of p-Laplacian difference equations is considered.

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1. Introduction

For notation, given $a < b$ in Z, we employ intervals to denote discrete sets such as $[a,b]=\{a,a+1,\dots,b\}$, $[a,b)=\{a,a+1,\dots,b-1\}$, $[a,\infty)=\{a,a+1,\dots\}$, etc. Let $T \geqslant 1$ be fixed. In this paper, we are concerned with the following $p-$Laplacian difference equation:

$$\Delta[\phi_p(\Delta u(t-1))] + a(t)f(u(t)) = 0, \quad t \in [1, T+1], \tag{1}$$

satisfying the boundary conditions

$$\Delta u(0) = u(T+2) = 0, \tag{2}$$

where $\phi_p(s)$ is p-Laplacian operator, i.e., $\phi_p(s)=|s|^{p-2}s$, $p>1$, $(\phi_p)^{-1}=\phi_q$, $1/p+1/q=1$, and
(A) $f: R^+ \to R^+$ is continuous (R^+ denotes the nonnegative reals),
(B) $a(t)$ is a positive valued function defined on $[1, T+1]$.

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The motivation for the present work stems from many recent investigations in [1–4,6–8,10]. For the continuous case, boundary value problems analogous to (1) and (2) arise in various nonlinear phenomena for which only positive solutions are meaningful; see, for example [11,12].

2. Preliminaries

Let

$$f_0 = \lim_{u\to 0^+} \frac{f(u)}{u^{p-1}}, \qquad f_\infty = \lim_{u\to\infty} \frac{f(u)}{u^{p-1}}.$$

We note that $u(t)$ is a solution of (1) and (2), if, and only if

$$u(t) = \sum_{s=t}^{T+1} \phi_q\left(\sum_{i=1}^{s} a(i) f(u(i))\right), \quad t \in [0, T+2]. \tag{3}$$

Let

$$E = \{u : [0, T+2] \to R \,|\, \Delta u(0) = u(T+2) = 0\}$$

with norm, $\|u\| = \max_{t\in[0,T+2]} |u(t)|$, then $(E, \|\cdot\|)$ is a Banach space.

Define a cone, P, by

$$P = \{u \in E : u(t) \geqslant 0,\ t \in [0, T+2],\ \text{and}\ u(t) \geqslant \sigma(t)\|u\|\},$$

where $\sigma(t) = 1 - t/(T+2)$, $t \in [0, T+2]$.

The following two lemmas will play an important role in the proof of our results and can be found in the book in [9] as well as in the book in [5].

Lemma 2.1. *Let E be a Banach space, and let $P \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, and let*

$$T : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$$

be a completely continuous operator such that, either

(i) $\|Tu\| \leqslant \|u\|,\ u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \geqslant \|u\|,\ u \in P \cap \partial\Omega_2$, *or*
(ii) $\|Tu\| \geqslant \|u\|,\ u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \leqslant \|u\|,\ u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Lemma 2.2. *Let E be a Banach space, and let $P \subset E$ be a cone in E. For $\rho > 0$, define $P_\rho = \{u \in K : \|u\| \leqslant \rho\}$. Assume that*

$$T : P_\rho \to P$$

is a completely continuous operator such that, $Tu \neq u$ for $x \in \partial P_\rho = \{u \in P : \|u\| = \rho\}$.

(i) *If* $\|u\| \leqslant \|Tu\|,\ u \in \partial P_\rho$, *then* $i(T, P_\rho, P) = 0$;
(ii) *If* $\|u\| \geqslant \|Tu\|,\ u \in \partial P_\rho$, *then* $i(T, P_\rho, P) = 1$.

3. Solutions of (1) and (2) in a cone

Theorem 3.1. *Assume that conditions* (A) *and* (B) *are satisfied. If*

(i) $f_0 = 0,\ f_\infty = \infty$ *or*
(ii) $f_0 = \infty,\ f_\infty = 0.$

Then, there exists at least one solution of (1) *and* (2) *in* P.

Proof. Define a summation operator $T : P \to E$ by

$$(Tu)(t) = \sum_{s=t}^{T+1} \phi_q \left(\sum_{i=1}^{s} a(i) f(u(i)) \right), \quad t \in [0, T+2]. \tag{4}$$

We note that from (4), if $u \in P$, then $(Tu)(t) \geqslant 0,\ t \in [0, T+2]$.

Moreover, for $u \in P$, we have

$$\begin{aligned} \|Tu\| &= (Tu)(0) \\ &= \sum_{s=0}^{T+1} \phi_q \left(\sum_{i=1}^{s} a(i) f(u(i)) \right). \end{aligned}$$

Set

$$U(t) = (Tu)(t) - \left(1 - \frac{t}{T+2} \right) \|Tu\|, \quad t \in [0, T+2].$$

Then $U(0) = U(T+2) = 0,\ \Delta^2 U(t) \leqslant 0,\ t \in [0, T]$. By Lemma 2 in [6], we have $U(t) \geqslant 0,\ t \in [0, T+2]$, i.e. $(Tu)(t) \geqslant (1 - t/(T+2))\|Tu\|,\ t \in [0, T+2]$. Consequently, $T : P \to P$. It is also easy to check that $T : P \to P$ is completely continuous.

Case (i): Now, turning to f_0, there exits $H_1 > 0$ such that $f(u) \leqslant (\theta u)^{p-1}$ for $0 < u \leqslant H_1$, where $\theta > 0$ satisfies

$$\theta \sum_{s=0}^{T+1} \phi_q \left(\sum_{i=1}^{s} a(i) \right) \leqslant 1.$$

Thus, for $u \in P$ with $\|u\| = H_1$, implies that

$$\begin{aligned} \|Tu\| &= \sum_{s=0}^{T+1} \phi_q \left(\sum_{i=1}^{s} a(i) f(u(i)) \right) \\ &\leqslant \sum_{s=0}^{T+1} \phi_q \left(\sum_{i=1}^{s} a(i) (\theta u(i))^{p-1} \right) \\ &\leqslant \|u\| \theta \sum_{s=0}^{T+1} \phi_q \left(\sum\nolimits_{i=1}^{s} a(i) \right) \\ &\leqslant \|u\|. \end{aligned}$$

Therefore,

$$\|Tu\| \leqslant \|u\| \quad \text{for } u \in P \cap \partial\Omega_1, \tag{5}$$

where

$$\Omega_1 = \{u \in B : \|u\| < H_1\}.$$

If we next consider f_∞, there exists an $\bar{H}_2 > 0$ such that $f(u) \geqslant (\Theta u)^{p-1}$, for all $u \geqslant \bar{H}_2$, where $\Theta > 0$ satisfies

$$\frac{1}{2}\Theta \sum_{s\in Y} \phi_q\left(\sum_{i=1}^{s} a(i)\right) \geqslant 1,$$

where

$$Y = \left\{t \in Z : 0 \leqslant t \leqslant \frac{T+2}{2}\right\}.$$

Let $H_2 = \max\{2H_1, 2\bar{H}_2\}$, and define

$$\Omega_2 = \{u \in B : \|u\| < H_2\}.$$

Note that

$$\tfrac{1}{2}\|u\| \leqslant u(t) \leqslant \|u\| \quad \text{for } t \in Y,\ u \in P.$$

If $u \in P$ with $\|u\| = H_2$, then

$$\begin{aligned}
\|Tu\| &= \sum_{s=0}^{T+1} \phi_q\left(\sum_{i=1}^{s} a(i) f(u(i))\right) \\
&\geqslant \sum_{s\in Y} \phi_q\left(\sum_{i=1}^{s} a(i) f(u(i))\right) \\
&\geqslant \sum_{s\in Y} \phi_q\left(\sum_{i=1}^{s} a(i)(\Theta u(i))^{p-1}\right) \\
&\geqslant \frac{1}{2}\Theta\|u\| \sum_{s\in Y} \phi_q\left(\sum_{i=1}^{s} a(i)\right) \\
&\geqslant \|u\|.
\end{aligned}$$

Hence,

$$\|Tu\| \geqslant \|u\| \quad \text{for } u \in P \cap \partial\Omega_2. \tag{6}$$

From (5) and (6), Lemma 2.1(i) implies that T has a fixed point $u \in P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Case (ii): Beginning with f_0, there exits $H_1 > 0$ such that $f(u) \geqslant (vu)^{p-1}$ for $0 < u \leqslant H_1$, where $v > 0$ satisfies

$$\frac{1}{2} v \sum_{s\in Y} \phi_q\left(\sum_{i=1}^{s} a(i)\right) \geqslant 1.$$

So, for $u \in P$ with $\|u\| = H_1$, we have

$$\begin{aligned}
\|Tu\| &= \sum_{s=0}^{T+1} \phi_q \left(\sum_{i=1}^{s} a(i) f(u(i)) \right) \\
&\geqslant \sum_{s \in Y} \phi_q \left(\sum_{i=1}^{s} a(i) f(u(i)) \right) \\
&\geqslant \sum_{s \in Y} \phi_q \left(\sum\nolimits_{i=1}^{s} a(i)(vu(i))^{p-1} \right) \\
&\geqslant \frac{1}{2} v \|u\| \sum_{s \in Y} \phi_q \left(\sum_{i=1}^{s} a(i) \right) \\
&\geqslant \|u\|.
\end{aligned}$$

Therefore,

$$\|Tu\| \geqslant \|u\| \quad \text{for } u \in P \cap \partial\Omega_1, \tag{7}$$

where

$$\Omega_1 = \{u \in B : \|u\| < H_1\}.$$

Using the assumption concerning f_∞, there exists an $\bar{H}_2 > 0$ such that $f(u) \leqslant (\lambda u)^{p-1}$ for $u \geqslant H_2$, where $\lambda > 0$ satisfies

$$\lambda \sum_{s=0}^{T+1} \phi_q \left(\sum_{i=1}^{s} a(i) \right) \leqslant 1.$$

There are two cases: (a) f is bounded, and (b) f is unbounded.

For case (a), assume $M > 0$ is such that $f(u) \leqslant M^{p-1}$ for all $0 < u < \infty$. Let

$$H_2 = \max \left\{ 2H_1, M \sum_{s=0}^{T+1} \phi_q \left(\sum_{i=1}^{s} a(i) \right) \right\}.$$

Then, for $u \in P$ with $\|u\| = H_2$, we have

$$\begin{aligned}
\|Tu\| &= \sum_{s=0}^{T+1} \phi_q \left(\sum_{i=1}^{s} a(i) f(u(i)) \right) \\
&\leqslant M \sum_{s=0}^{T+1} \phi_q \left(\sum_{i=1}^{s} a(i) \right) \\
&\leqslant H_2 = \|u\|.
\end{aligned}$$

Hence

$$\|Tu\| \leqslant \|u\| \quad \text{for } u \in P \cap \partial\Omega_2, \tag{8}$$

where

$$\Omega_2 = \{u \in B : \|u\| < H_2\}.$$

For case (b), it can be shown without much difficulty that there is an $H_2 > \max\{2H_1, \bar{H}_2\}$ such that $f(u) \leqslant f(H_2)$, for $0 < u \leqslant H_2$. Choosing $u \in P$ with $\|u\| = H_2$,

$$\begin{aligned}
\|Tu\| &= \sum_{s=0}^{T+1} \phi_q\left(\sum_{i=1}^{s} a(i) f(u(i))\right) \\
&\leqslant \sum_{s=0}^{T+1} \phi_q\left(\sum_{i=1}^{s} a(i) f(H_2)\right) \\
&\leqslant (f(H_2))^{q-1} \sum_{s=0}^{T+1} \phi_q\left(\sum_{i=1}^{s} a(i)\right) \\
&\leqslant \lambda H_2 \sum_{s=0}^{T+1} \phi_q\left(\sum_{i=1}^{s} a(i)\right) \\
&\leqslant H_2 = \|u\|.
\end{aligned}$$

Therefore

$$\|Tu\| \leqslant \|u\| \quad \text{for } u \in P \cap \partial\Omega_2, \tag{9}$$

where

$$\Omega_2 = \{u \in B : \|u\| < H_2\}.$$

We apply Lemma 2.1 to conclude that T has a fixed point $u \in P \cap (\bar{\Omega}_2 \backslash \Omega_1)$. Thus, in either of the case, Lemma 2.1(ii) applied to (7) and (8) or (9) yields a fixed point of T which belongs to $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$. This fixed point u is a solution of (1) and (2). The proof of Theorem 3.1 is complete. □

Theorem 3.2. *Assume that conditions* (A) *and* (B) *are satisfied. If*

(i) $f_0 = f_\infty = \infty$,
(ii) *there exists* $\rho > 0$ *such that* $f(u) < (\eta\rho)^{p-1}$ *for* $0 < u \leqslant \rho$, *where*

$$\eta = \left[\sum_{s=0}^{T+1} \phi_q\left(\sum_{i=1}^{s} a(i)\right)\right]^{-1}.$$

Then, there exists at least two solutions u_1 *and* u_2 *of* (1) *and* (2) *in* P, *such that* $0 < \|u_1\| < \rho < \|u_2\|$.

Proof. Since $f_0=\infty$, there exits $d\in(0,\rho)$ such that $f(u)\geqslant(Mu)^{p-1}$ for $0\leqslant u\leqslant d$, where $M>0$ satisfies

$$M\sum_{s\in Y}\phi_q\left(\sum_{i=1}^{s}a(i)\left(1-\frac{i}{T+2}\right)^{p-1}\right)>1.$$

Thus, for $u\in P$ with $\|u\|=d$, we have

$$\begin{aligned}\|Tu\|&=\sum_{s=0}^{T+1}\phi_q\left(\sum_{i=1}^{s}a(i)f(u(i))\right)\\&\geqslant\sum_{s\in Y}\phi_q\left(\sum_{i=1}^{s}a(i)f(u(i))\right)\\&\geqslant\sum_{s\in Y}\phi_q\left(\sum_{i=1}^{s}a(i)(Mu(i))^{p-1}\right)\\&\geqslant M\|u\|\sum_{s\in Y}\phi_q\left(\sum_{i=1}^{s}a(i)\left(1-\frac{i}{T+2}\right)^{p-1}\right)\\&>\|u\|.\end{aligned}$$

Therefore,

$$i(T,P_d,P)=0.$$

Since $f_\infty=\infty$, there exists an $R>\rho$ such that $f(u)\geqslant(Nu)^{p-1}$, for all $u\geqslant R$, where $N>0$ satisfies

$$N\sum_{s\in Y}\phi_q\left(\sum_{i=1}^{s}a(i)\left(1-\frac{i}{T+2}\right)^{p-1}\right)>1.$$

If $u\in P$ with $\|u\|=R$, then

$$\begin{aligned}\|Tu\|&=\sum_{s=0}^{T+1}\phi_q\left(\sum_{i=1}^{s}a(i)f(u(i))\right)\\&\geqslant\sum_{s\in Y}\phi_q\left(\sum_{i=1}^{s}a(i)f(u(i))\right)\\&\geqslant N\|u\|\sum_{s\in Y}\phi_q\left(\sum_{i=1}^{s}a(i)\left(1-\frac{i}{T+2}\right)^{p-1}\right)\\&>\|u\|.\end{aligned}$$

Hence,

$$i(T, P_R, P) = 0.$$

If $u \in P$ with $\|u\| = \rho$, then

$$\|Tu\| = \sum_{s=0}^{T+1} \phi_q \left(\sum_{i=1}^{s} a(i) f(u(i)) \right)$$
$$< \eta\rho \sum_{s=0}^{T+1} \phi_q \left(\sum_{i=1}^{s} a(i)) \right)$$
$$= \rho = \|u\|.$$

Hence,

$$i(T, P_\rho, P) = 1.$$

Therefore,

$$i(T, P_R \backslash \bar{P}_\rho, P) = -1, \qquad i(T, P_\rho \backslash \bar{P}_d, P) = 1.$$

So, there exists at least two solutions u_1 and u_2 of (1), (2) in P, such that $0 < \|u_1\| < \rho < \|u_2\|$. □

Theorem 3.3. *Assume that conditions* (A) *and* (B) *are satisfied. If*

(i) $f_0 = f_\infty = 0$,
(ii) *there exists* $\rho > 0$ *such that* $f(u) > (\lambda\rho)^{p-1}$ *for* $\frac{1}{2}\rho \leqslant u \leqslant \rho$, *where*

$$\lambda = \left[\sum_{s \in Y} \phi_q \left(\sum_{i=1}^{s} a(i) \right) \right]^{-1}, \qquad Y = \left\{ t \in Z : 0 \leqslant t \leqslant \frac{T+2}{2} \right\}.$$

Then, there exists at least two solutions u_1 *and* u_2 *of* (1) *and* (2) *in* P, *such that* $0 < \|u_1\| < \rho < \|u_2\|$.

The proof of Theorem 3.3 is similar to that of Theorem 3.2. Here we omit it.

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