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On some operations and density of m-polar fuzzy graphs

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ABSTRACT

The theoretical concepts of graphs are highly utilized by computer science applications, social sciences, and medical sciences, especially in computer science for applications such as data mining, image segmentation, clustering, image capturing, and networking. Fuzzy graphs, bipolar fuzzy graphs and the recently developed m-polar fuzzy graphs are growing research topics because they are generalizations of graphs (crisp). In this paper, three new operations, i.e., direct product, semi-strong product and strong product, are defined on m-polar fuzzy graphs. It is proved that any of the products of m-polar fuzzy graphs are again an m-polar fuzzy graph. Sufficient conditions are established for each to be strong, and it is proved that the strong product of two complete m-polar fuzzy graphs is complete. If any of the products of two m-polar fuzzy graphs G_1 and G_2 are strong, then at least G_1 or G_2 must be strong. Moreover, the density of an m-polar fuzzy graph is defined, the notion of balanced m-polar fuzzy graph is studied, and necessary and sufficient conditions for the preceding products of two m-polar fuzzy balanced graphs to be balanced are established. Finally, the concept of product m-polar fuzzy graph is introduced, and it is shown that every product m-polar fuzzy graph is an m-polar fuzzy graph. Some operations, like union, direct product, and ring sum are defined to construct new product m-polar fuzzy graphs.

1. Introduction

Presently, science and technology is characterised by complex processes and phenomena for which complete information is not always available. For such cases, mathematical models are developed to handle various types of systems containing elements of uncertainty. A large number of these models is based on fuzzy sets. Graph theory has numerous applications to problems in computer science, electrical engineering, systems analysis, operations research, economics, networking routing, and transportation. Considering the fuzzy relations between fuzzy sets, Rosenfeld [18] introduced the concept of fuzzy graphs in 1975 and later developed the structure of fuzzy graphs. Mordeson and Nair [15] defined the complement of fuzzy graphs, which was further studied by Sunitha and Kumar [21]. The concepts of weak isomorphism, co-weak isomorphism and isomorphism between fuzzy graphs was introduced by Bhutani in Ref. [3]. Several researchers have worked on fuzzy graphs. Samanta and Pal introduced several types of fuzzy graphs, such as fuzzy planar graphs [29], fuzzy competition graphs [26,27], fuzzy tolerance graphs [22], and fuzzy threshold graphs [23]. Some more work on fuzzy graphs can be found on [4,12,16,17].

In 1994, Zhang [31–33] developed the concept of bipolar fuzzy sets as a generalization of fuzzy sets. The idea behind such description is connected with the existence of "bipolar information" (i.e., positive information and negative information) about the given set. Positive information represents what is granted to be possible, whereas negative information represents what is considered to be impossible. In 2011, using the concepts of bipolar fuzzy sets, Akram [1] introduced bipolar fuzzy graphs and defined different operations. Using this definition of bipolar fuzzy graphs, research is ongoing. Some work on bipolar fuzzy graphs may be found on [6,7,24,25,28,30]. Talebi and Rashmanlou [2] studied the complement and isomorphism of bipolar fuzzy graphs. Rashmanlou et al. [19,20] studied bipolar fuzzy graphs and bipolar fuzzy graphs with categorical properties.

In 2014, Juanjuan Chen et al. [5] introduced the notion of m-polar fuzzy sets as a generalization of bipolar fuzzy sets and showed that bipolar fuzzy sets and 2-polar fuzzy sets are cryptomorphic

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mathematical notions and that we can obtain one from the other. The idea behind this is that "multipolar information" (not just bipolar information, which corresponds to two-valued logic) exists because data of real world problems sometimes come from multiple agents. For example, the exact degree of telecommunication safety of mankind is a point in $[0,1]^n$ ($n \approx 7 \times 10^9$) because different persons have been monitored different times. There are many other examples, such as truth degrees of a logic formula that are based on n logic implication operators ($n \geq 2$), similarity degrees of two logic formulas that are based on n logic implication operators ($n \geq 2$), ordering results of a magazine, ordering results of a university, and inclusion degrees (accuracy measures, rough measures, approximation qualities, fuzziness measures, and decision preformation evaluations) of a rough set.

Ghorai and Pal [8] introduced the notion of generalized m-polar fuzzy graphs as a generalization of bipolar fuzzy graphs and defined different operations. In Ref. [9], they studied the complement and isomorphism of m-polar fuzzy graphs. Because of the importance of m-polar fuzzy graphs mentioned in Refs. [8,9], we investigated m-polar fuzzy graphs. In this paper, three new operations are defined on the m-polar fuzzy graph, including direct product, semi-strong product and strong product. Any of the products of m-polar fuzzy graphs are again an m-polar fuzzy graph. Sufficient conditions are established for each one to be strong, and it is proved that the strong product of two complete m-polar fuzzy graphs is complete. If any of the products of two m-polar fuzzy graphs G_1 and G_2 are strong, then at least G_1 or G_2 must be strong. Moreover, the density of an m-polar fuzzy graph is defined, the notion of balanced m-polar fuzzy graphs is studied, and the necessary and sufficient conditions for the preceding products of two m-polar fuzzy balanced graphs to be balanced are established. Finally, the concept of product m-polar fuzzy graphs is introduced, and it is proved that every product m-polar fuzzy graph is an m-polar fuzzy graph. Some operations, like union, direct product, and ring sum, are defined to construct new product m-polar fuzzy graphs.

2. Preliminaries

In this section, we recall some definitions of fuzzy graphs, m-polar fuzzy sets, and m-polar fuzzy relations, which are defined below. For further study, see Refs. [5,8,9,11,13–15].

Definition 2.1. [15] *A fuzzy graph with V as the underlying set is a triplet* $G = (V,\sigma,\mu)$, *where* $\sigma: V \to [0,1]$ *is a fuzzy subset of V and* $\mu: V \times V \to [0,1]$ *is a fuzzy relation on* σ, *such that* $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ *for all* $x,y \in V$, *where* $\wedge$ *stands for the minimum.*

The underlying crisp graph of G is denoted by $G^* = (\sigma^*,\mu^*)$, *where* $\sigma^* = \{x \in V: \sigma(x) > 0\}$ *and* $\mu^* = \{(x,y) \in V \times V: \mu(xy) > 0\}$.

Definition 2.2. [10] *A fuzzy graph* $G = (V,\sigma,\mu)$ *is complete if* $\mu(x,y) = \sigma(x) \wedge \sigma(y)$ *for all* $x,y \in V$.

The main purpose of this paper is to study m-polar fuzzy graphs based on m-polar fuzzy sets, which is defined below.

Throughout the paper, $[0,1]^m$ (*m-power of* $[0,1]$) *is considered to be a poset with point-wise order* $\leq$, *where m is a natural number.* $\leq$ *is defined by* $x \leq y \Leftrightarrow$ *for each* $i = 1,2,\ldots,m$; $p_i(x) \leq p_i(y)$ *where* $x,y \in [0,1]^m$ *and* $p_i: [0,1]^m \to [0,1]$ *is the i-th projection mapping.*

Definition 2.3. [5] *An m-polar fuzzy set (or a* $[0,1]^m$*-set) on X is a mapping* $A: X \to [0,1]^m$. *The set of all m-polar fuzzy sets on X is denoted by m(X).*

Definition 2.4. [8] *Let A and B be two m-polar fuzzy sets in X. Then* $A \cup B$ *and* $A \cap B$ *are also m-polar fuzzy sets in X defined by:*

$p_i \circ (A \cup B((x) = \{p_i \circ A(x) \vee p_i \circ B(x)\}$ *and*

$p_i \circ (A \cap B)(x) = \{p_i \circ A(x) \wedge p_i \circ B(x)\}$ *for* $i = 1,2,\ldots,m$ *and* $x \in X$ ($\vee$ *stands for maximum*).

$A \subseteq B$ *if and only if for each* $i = 1,2,\ldots,m$ *and* $x \in X$, $p_i \circ A(x) \leq p_i \circ B(x)$.

$A = B$ *if and only if for each* $i = 1,2,\ldots,m$ *and* $x \in X$, $p_i \circ A(x) = p_i \circ B(x)$.

Definition 2.5. [8] *Let A be an m-polar fuzzy set on a set X. An m-polar fuzzy relation on A is an m-polar fuzzy set B of* $X \times X$ *such that* $B(x,y) \leq min\{A(x),A(y)\}$ *for all* $x,y \in X$ *i.e., for each* $i = 1,2,\ldots,m$, *for all* $x,y \in X$, $p_i \circ B(x,y) \leq min\{p_i \circ A(x),p_i \circ A(y)\}$.

An m-polar fuzzy relation B on X is called symmetric if $B(x,y) = B(y,x)$ *for all* $x,y \in X$.

Definition 2.6. [10] *The semi-strong product of two fuzzy graphs* $G_1 = (\sigma_1,\mu_1)$ *and* $G_2 = (\sigma_2,\mu_2)$ *of the graphs* $G_1^* = (V_1,E_1)$ *and* $G_2^* = (V_2,E_2)$, *where it is assumed that* $V_1 \cap V_2 = \varnothing$, *is defined to be the fuzzy graph* $G_1 \bullet G_2 = (\sigma_1 \bullet \sigma_2, \mu_1 \bullet \mu_2)$ *of the graph* $G^* = (V_1 \times V_2, E)$ respectively, *such that* $E = \{(u,v_1)(u,v_2) | u \in V_1, v_1v_2 \in E_2\} \cup \{(u_1,v_1)(u_2,v_2) | u_1u_2 \in E_1, v_1v_2 \in E_2\}$,

$(\sigma_1 \bullet \sigma_2)(u,v) = \sigma_1(u) \wedge \sigma_2(v)$ *for all* $(u,v) \in V_1 \times V_2$,

$(\mu_1 \bullet \mu_2)((u,v_1)(u,v_2)) = \sigma_1(u) \wedge \mu_2(v_1v_2)$ *and*

$(\mu_1 \bullet \mu_2)((u_1,v_1)(u_2,v_2)) = \mu_1(u_1u_2) \wedge \mu_2(v_1v_2)$.

Definition 2.7. [10] *The strong product of two fuzzy graphs* $G_1 = (\sigma_1,\mu_1)$ *and* $G_2 = (\sigma_2,\mu_2)$ *of the graphs* $G_1^* = (V_1,E_1)$ *and* $G_2^* = (V_2,E_2)$ respectively, *where it is assumed that* $V_1 \cap V_2 = \varnothing$, *is defined to be the fuzzy graph* $G_1 \otimes G_2 = (\sigma_1 \otimes \sigma_2, \mu_1 \otimes \mu_2)$ *of the graph* $G^* = (V_1 \times V_2, E)$, *such that* $E = \{(u,v_1)(u,v_2) | u \in V_1, v_1v_2 \in E_2\} \cup \{(u_1,w)(u_2,w) | w \in V_2, u_1u_2 \in E_1\} \cup \{(u_1,v_1)(u_2,v_2) | u_1u_2 \in E_1, v_1v_2 \in E_2\}$,

$(\sigma_1 \otimes \sigma_2)(u,v) = \sigma_1(u) \wedge \sigma_2(v)$ *for all* $(u,v) \in V_1 \times V_2$,

$(\mu_1 \otimes \mu_2)((u,v_1)(u,v_2)) = \sigma_1(u) \wedge \mu_2(v_1v_2)$,

$(\mu_1 \otimes \mu_2)((u_1,w)(u_2,w)) = \sigma_2(w) \wedge \mu_1(u_1u_2)$ *and*

$(\mu_1 \otimes \mu_2)((u_1,v_1)(u_2,v_2)) = \mu_1(u_1u_2) \wedge \mu_2(v_1v_2)$.

Definition 2.8. [10] *The direct product of two fuzzy graphs* $G_1 = (\sigma_1,\mu_1)$ *and* $G_2 = (\sigma_2,\mu_2)$ *of the graphs* $G_1^* = (V_1,E_1)$ *and* $G_2^* = (V_2,E_2)$ respectively, *such that* $V_1 \cap V_2 = \varnothing$, *is defined to be the fuzzy graph* $G_1 \sqcap G_2 = (\sigma_1 \sqcap \sigma_2, \mu_1 \sqcap \mu_2)$ *of the graph* $G^* = (V_1 \times V_2, E)$, *such that* $E = \{(u_1,v_1)(u_2,v_2) | u_1u_2 \in E_1, v_1v_2 \in E_2\}$,

$(\sigma_1 \sqcap \sigma_2)(u,v) = \sigma_1(u) \wedge \sigma_2(v)$ *for all* $(u,v) \in V_1 \times V_2$, *and*

$(\mu_1 \sqcap \mu_2)((u_1,v_1)(u_2,v_2)) = \mu_1(u_1u_2) \wedge \mu_2(v_1v_2)$.

For a given set V, define an equivalence relation on $V \times V - \{(x,x) : x \in V\}$ *as follows:*

$(x_1,y_1) \sim (x_2,y_2) \Leftrightarrow$ *either* $(x_1,y_1) = (x_2,y_2)$ *or* $x_1 = y_2$ *and* $y_1 = x_2$.

The quotient set obtained in this way is denoted by $\widetilde{V^2}$, *and the equivalence class that contains the element (x,y) is denoted as xy or yx.*

Throughout this paper, G^* *represents a crisp graph, and G is an m-polar fuzzy graph of* G^*.

3. *m*-polar fuzzy graphs

In this section, we briefly recall some basic definitions related to m-polar fuzzy graphs.

Definition 3.1. [8] *An m-polar fuzzy graph of a graph* $G^* = (V,E)$ *is a pair* $G = (A,B)$ *where* $A: V \to [0,1]^m$ *is an m-polar fuzzy set in V and* $B: \widetilde{V^2} \to [0,1]^m$ *is an m-polar fuzzy set in* $\widetilde{V^2}$, *such that for each* $i = 1,2,\ldots,m$; $p_i \circ B(xy) \leq min\{p_i \circ A(x), p_i \circ A(y)\}$ *for all* $xy \in \widetilde{V^2}$ *and* $B(xy) = 0$ *for all* $xy \in \widetilde{V^2} - E$, ($0 = (0,0,\ldots,0)$ *is the smallest element in* $[0,1]^m$).

A is called the m-polar fuzzy vertex set of G, and B is called the m-polar fuzzy edge set of G.

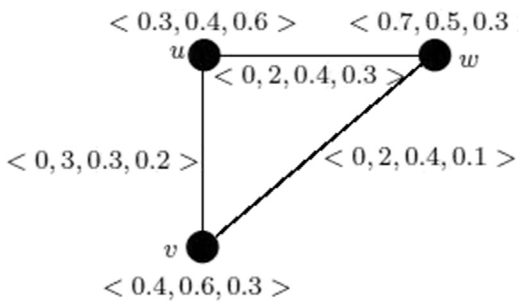

Fig. 1. Example of 3-polar fuzzy graph G

***Example* 3.2** *Let us consider the graph* $G^* = (V,E)$ *where* $V = \{u,v,w\}$ *and* $E = \{uv,vw,uw\}$. *A 3-polar fuzzy graph* G *of* G^* *is shown in* Fig. 1.

Definition 3.3. [8] *The m-polar fuzzy graph* $H = (P,C,D)$ *is called an m-polar fuzzy subgraph of* $G = (V,A,B)$ *induced by* P *if* $P \subseteq V$, $C(x) = A(x)$ *for all* $x \in P$ *and* $D(xy) = B(xy)$ *for all* $xy \in \widetilde{P^2}$.

***Example* 3.4** *H is a 3-polar fuzzy subgraph of G of Example 3.2* (see Fig. 2).

Definition 3.5. [8] *An m-polar fuzzy graph* $G = (A,B)$ *of the graph* $G^* = (V,E)$ *is called strong if* $B(xy) = \min\{A(x),A(y)\}$ *for all* $xy \in E$, *i.e., for each* $i = 1,2,\ldots,m$; $p_i \circ B(xy) = \min\{p_i \circ A(x), p_i \circ A(y)\}$ *for all* $xy \in E$.

Definition 3.6. [9] *An m-polar fuzzy graph* $G = (A,B)$ *of the graph* $G^* = (V,E)$ *is said to be complete if* $B(xy) = \min\{A(x),A(y)\}$ *for all* $x,y \in V$, *i.e., for each* $i = 1,2,\ldots,m$, $p_i \circ B(xy) = \min\{p_i \circ A(x), p_i \circ A(y)\}$ *for all* $x,y \in V$.

Definition 3.7. [9] *Let* $G = (A,B)$ *be an m-polar fuzzy graph of a graph* $G^* = (V,E)$. *The complement of G is an m-polar fuzzy graph* $\overline{G} = (\overline{A},\overline{B})$ *of* $\overline{G^*} = (V,\widetilde{V^2})$ *where* $\overline{A} = A$ *and* $\overline{B}$ *is defined by* $p_i \circ \overline{B}(xy) = \min\{p_i \circ A(x), p_i \circ A(y)\} - p_i \circ B(xy)$, *for each* $i = 1,2,\ldots,m$ *and for all* $xy \in \widetilde{V^2}$.

Definition 3.8. [8] *A strong m-polar fuzzy graph G is called self-complementary if* $G \cong \overline{G}$.

4. Products on *m*-polar fuzzy graphs

Here, the direct product of two m-polar fuzzy graphs is defined.

Definition 4.1. *Let* $G_1 = (A_1,B_1)$ *and* $G_2 = (A_2,B_2)$ *be two m-polar fuzzy graphs of the graphs* $G_1^* = (V_1,E_1)$ *and* $G_2^* = (V_2,E_2)$ respectively, *such that* $V_1 \cap V_2 = \emptyset$. *The direct product of* G_1 *and* G_2 *is defined to be the m-polar fuzzy graph* $G_1 \sqcap G_2 = (A_1 \sqcap A_2, B_1 \sqcap B_2)$ *of the graph*

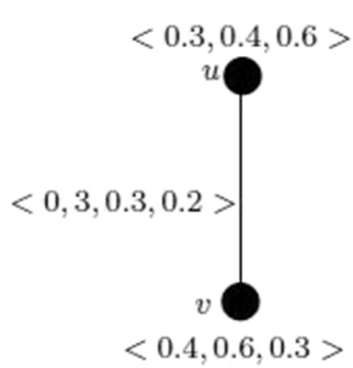

Fig. 2. 3-polar fuzzy subgraph H of G

$G^* = (V_1 \times V_2, E)$ *where,* $E = \{(u_1,v_1)(u_2,v_2) | u_1u_2 \in E_1,\ v_1v_2 \in E_2\} \subseteq \widetilde{V_1 \times V_2^2}$

and for each $i = 1,2,\ldots,m$

$p_i \circ (A_1 \sqcap A_2)(u,v) = p_i \circ A_1(u) \wedge p_i \circ A_2(v)$ *for all* $(u,v) \in V_1 \times V_2$,

$p_i \circ (B_1 \sqcap B_2)((u_1,v_1),(u_2,v_2)) = p_i \circ B_1(u_1u_2) \wedge p_i \circ B_2(v_1v_2)$ *for all* $u_1u_2 \in E_1$ *and* $v_1v_2 \in E_2$,

$p_i \circ (B_1 \sqcap B_2)((w,x),(y,z)) = 0$ *for all* $(w,x)(y,z) \in (\widetilde{V_1 \times V_2^2} - E)$.

Below, the direct product of m-polar fuzzy graphs is explained with an example.

***Example* 4.2** *Let* $G_1^* = (V_1,E_1)$ *and* $G_2^* = (V_2,E_2)$ *be two crisp graphs such that* $V_1 = \{u,v\}$, $V_2 = \{w,x\}$, $E_1 = \{uv\}$ *and* $E_2 = \{wx\}$. *Consider two 3-polar fuzzy graphs* $G_1 = (A_1,B_1)$ *and* $G_2 = (A_2,B_2)$ *of the graphs* $G_1^* = (V_1,E_1)$ *and* $G_2^* = (V_2,E_2)$. *Using the definition of direct product,* $G_1 \sqcap G_2$ *is calculated* (see Fig. 3).

It is easy to see that $G_1 \sqcap G_2$ *is a 3-polar fuzzy graph.*

Theorem 4.3. *The direct product* $G_1 \sqcap G_2$ *of two m-polar fuzzy graphs* G_1 *and* G_2 *is an m-polar fuzzy graph.*

Proof: *Let* $(u_1,v_1)(u_2,v_2) \in E$. *Then* $u_1u_2 \in E_1$ *and* $v_1v_2 \in E_2$. *Hence for each* $i = 1,2,\ldots,m$; *we have*

$p_i \circ (B_1 \sqcap B_2)((u_1,v_1),(u_2,v_2))$

$= p_i \circ B_1(u_1u_2) \wedge p_i \circ B_2(v_1v_2)$

$\leq p_i \circ A_1(u_1) \wedge p_i \circ A_1(u_2) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2)$ *(because* G_1 *and* G_2 *are m-polar fuzzy graphs)*

$= p_i \circ (A_1 \sqcap A_2)(u_1,v_1) \wedge p_i \circ (A_1 \sqcap A_2)(u_2,v_2)$

Also, for all $(w,x)(y,z) \in (\widetilde{V_1 \times V_2^2} - E)$, $i = 1,2,\ldots,m$;

$p_i \circ (B_1 \sqcap B_2)((w,x),(y,z)) = 0 \leq p_i \circ (A_1 \sqcap A_2)(w,y) \wedge p_i \circ (A_1 \sqcap A_2)(x,z)$.

This shows that $G_1 \sqcap G_2$ *is an m-polar fuzzy graph.*

Theorem 4.4. *If* $G_1 = (A_1,B_1)$ *and* $G_2 = (A_2,B_2)$ *are two strong m-polar fuzzy graphs, then* $G_1 \sqcap G_2$ *is also strong.*

Proof: *Let* $(u_1,v_1)(u_2,v_2) \in E$. *Because* G_1 *and* G_2 *are strong, we have for each* $i = 1,2,\ldots,m$

$p_i \circ (B_1 \sqcap B_2)((u_1,v_1),(u_2,v_2))$

$= p_i \circ B_1(u_1u_2) \wedge p_i \circ B_2(v_1v_2)$

$= p_i \circ A_1(u_1) \wedge p_i \circ A_1(u_2) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2)$

$= p_i \circ (A_1 \sqcap A_2)(u_1,v_1) \wedge p_i \circ (A_1 \sqcap A_2)(u_2,v_2)$.

Hence, $G_1 \sqcap G_2$ *is a strong m-polar fuzzy graph.*

Now, the semi-strong product is defined between two m-polar fuzzy graphs to construct a new m-polar fuzzy graph.

Definition 4.5. *The semi-strong product of two m-polar fuzzy graphs* $G_1 = (A_1,B_1)$ *of* $G_1^* = (V_1,E_1)$ *and* $G_2 = (A_2,B_2)$ *of* $G_2^* = (V_2,E_2)$, *where it is assumed that* $V_1 \cap V_2 = \emptyset$, *is defined to be the m-polar fuzzy graph* $G_1 \bullet G_2 = (A_1 \bullet A_2, B_1 \bullet B_2)$ *of* $G^* = (V_1 \times V_2, E)$, *where*

$E = \{(u,v_1)(u,v_2) | u \in V_1, v_1v_2 \in E_2\} \cup \{(u_1,v_1)(u_2,v_2) | u_1u_2 \in E_1, v_1v_2 \in E_2\} \subseteq \widetilde{V_1 \times V_2^2}$

satisfying the following: for each $i = 1,2,\ldots,m$

$p_i \circ (A_1 \bullet A_2)(u,v) = p_i \circ A_1(u) \wedge p_i \circ A_2(v)$ *for all* $(u,v) \in V_1 \times V_2$,

$p_i \circ (B_1 \bullet B_2)((u,v_1)(u,v_2)) = p_i \circ A_1(u) \wedge p_i \circ B_2(v_1v_2)$ *for all* $u \in V_1$ *and* $v_1v_2 \in E_2$,

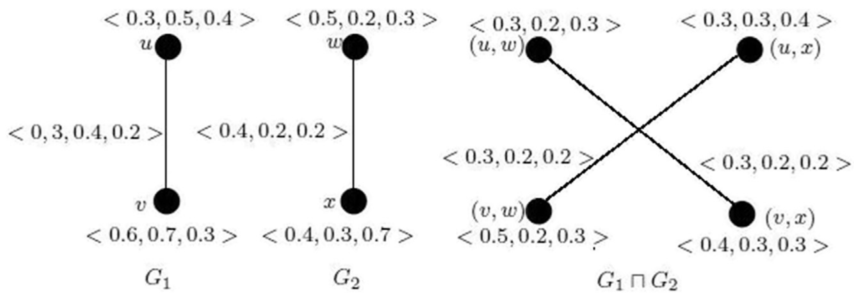

Fig. 3. Direct product of G_1 and G_2

$p_i \circ (B_1 \bullet B_2)((u_1,v_1)(u_2,v_2)) = p_i \circ B_1(u_1u_2) \wedge p_i \circ B_2(v_1v_2)$ for all $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$, and $p_i \circ (B_1 \bullet B_2)((w,x)(y,z)) = 0$ for all $(w,x)(y,z) \in (\widetilde{V_1 \times V_2^2} - E)$.

We demonstrate this product to construct a new m-polar fuzzy graphs, which is explained in the following example.

***Example* 4.6** *Consider the 3-polar fuzzy graphs $G_1 = (A_1,B_1)$ and $G_2 = (A_2,B_2)$ as in Example 4.2; then, $G_1 \bullet G_2$ is calculated using the above definition (see Fig. 4).*

It is easy to see that $G_1 \cdot G_2$ is a 3-polar fuzzy graph.

Theorem 4.7. *If $G_1 = (A_1,B_1)$ and $G_2 = (A_2,B_2)$ are m-polar fuzzy graphs, then $G_1 \cdot G_2$ is an m-polar fuzzy graph.*

Proof: *Let $(u,v_1)(u,v_2) \in E$. Then, $u \in V_1$ and $v_1v_2 \in E_2$. Because G_2 is an m-polar fuzzy graph, we have for each $i = 1,2,\ldots,m$*

$p_i \circ (B_1 \bullet B_2)((u,v_1)(u,v_2))$

$= p_i \circ A_1(u) \wedge p_i \circ B_2(v_1v_2)$

$\le p_i \circ A_1(u) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2)$

$= p_i \circ (A_1 \bullet A_2)(u,v_1) \wedge p_i \circ (A_1 \bullet A_2)(u,v_2)$.

Let $(u_1,v_1)(u_2,v_2) \in E$. Then $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$. G_1 and G_2 being m-polar fuzzy graphs, we have for each $i = 1,2,\ldots,m$

$p_i \circ (B_1 \bullet B_2)((u_1,v_1)(u_2,v_2))$

$= p_i \circ B_1(u_1u_2) \wedge p_i \circ B_2(v_1v_2)$

$\le p_i \circ A_1(u_1) \wedge p_i \circ A_1(u_2) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2)$

$= p_i \circ (A_1 \bullet A_2)(u_1,v_1) \wedge p_i \circ (A_1 \bullet A_2)(u_2,v_2)$.

Finally, for all $(w,x)(y,z) \in (\widetilde{V_1 \times V_2^2} - E)$, $i = 1,2,\ldots,m$;

$$p_i \circ (B_1 \bullet B_2)((w,x)(y,z)) = 0 \le p_i \circ (A_1 \bullet A_2)(w,y) \wedge p_i \circ (A_1 \bullet A_2)(x,z).$$

Theorem 4.8. *If $G_1 = (A_1,B_1)$ and $G_2 = (A_2,B_2)$ are strong m-polar fuzzy graphs, then $G_1 \bullet G_2$ is a strong m-polar fuzzy graph.*

Proof: *Let $(u,v_1)(u,v_2) \in E$. Then $u \in V_1$ and $v_1v_2 \in E_2 \bullet G_2$ being strong, we have for each $i = 1,2,\ldots,m$*

$p_i \circ (B_1 \bullet B_2)((u,v_1)(u,v_2))$

$= p_i \circ A_1(u) \wedge p_i \circ B_2(v_1v_2)$

$= p_i \circ A_1(u) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2)$

$= p_i \circ (A_1 \bullet A_2)(u,v_1) \wedge p_i \circ (A_1 \bullet A_2)(u,v_2)$.

If $(u_1,v_1)(u_2,v_2) \in E$. Then $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$.
Now, G_1 and G_2 being strong, we have for each $i = 1,2,\ldots,m$

$p_i \circ (B_1 \bullet B_2)((u_1,v_1)(u_2,v_2))$

$= p_i \circ B_1(u_1u_2) \wedge p_i \circ B_2(v_1v_2)$

$= p_i \circ A_1(u_1) \wedge p_i \circ A_1(u_2) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2)$

$= p_i \circ (A_1 \bullet A_2)(u_1,v_1) \wedge p_i \circ (A_1 \bullet A_2)(u_2,v_2)$.

Hence, $G_1 \bullet G_2$ is a strong m-polar fuzzy graph.

The strong product between m-polar fuzzy graphs is an important construction of m-polar fuzzy graphs, which is defined below.

Definition 4.9. *The strong product of two m-polar fuzzy graphs $G_1 = (A_1,B_1)$ of $G_1^* = (V_1,E_1)$ and $G_2 = (A_2,B_2)$ of $G_2^* = (V_2,E_2)$ such that $V_1 \cap V_2 = \varnothing$, is defined to be the m-polar fuzzy graph $G_1 \otimes G_2 = (A_1 \otimes A_2, B_1 \otimes B_2)$ of $G^* = (V_1 \times V_2,E)$, where*

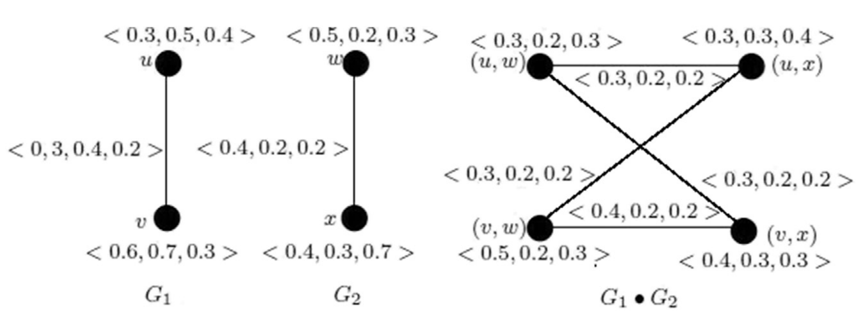

Fig. 4. *Semi-strong product of G_1 and G_2*

$E = \{(u,v_1)(u,v_2)|u \in V_1, v_1v_2 \in E_2\} \cup \{(u_1,w)(u_2,w)|w \in V_2,$
$u_1u_2 \in E_1\} \cup \{(u_1,v_1)(u_2,v_2)|u_1u_2 \in E_1, v_1v_2 \in E_2\} \subseteq \widetilde{V_1 \times V_2^2}$

such that the following condition holds: for each $i = 1,2,\ldots,m$

$p_i \circ (A_1 \otimes A_2)(u,v) = p_i \circ A_1(u) \wedge p_i \circ A_2(v)$ *for all* $(u,v) \in V_1 \times V_2$,

$p_i \circ (B_1 \otimes B_2)((u,v_1)(u,v_2)) = p_i \circ A_1(u) \wedge p_i \circ B_2(v_1v_2)$ *for all* $u \in V_1$ *and* $v_1v_2 \in E_2$,

$p_i \circ (B_1 \otimes B_2)((u_1,w)(u_2,w)) = p_i \circ B_1(u_1u_2) \wedge p_i \circ A_2(w)$ *for all* $w \in V_2$ *and* $u_1u_2 \in E_1$,

$p_i \circ (B_1 \otimes B_2)((u_1,v_1)(u_2,v_2)) = p_i \circ B_1(u_1u_2) \wedge p_i \circ B_2(v_1v_2)$ *for all* $u_1u_2 \in E_1$ *and* $v_1v_2 \in E_2$,

and $p_i \circ (B_1 \otimes B_2)((w,x)(y,z)) = 0$ *for all* $(w,x)(y,z) \in (\widetilde{V_1 \times V_2^2} - E)$.

We now give an example that shows that the strong product of m-polar fuzzy graphs is again an m-polar fuzzy graph.

***Example* 4.10** *Consider the* 3*-polar fuzzy graphs* $G_1 = (A_1,B_1)$ *and* $G_2 = (A_2,B_2)$ *as in Example* 4.2*. Also consider the strong product* $G_1 \otimes G_2$*, which is shown in* Fig. 5.

It is easily checked that $G_1 \otimes G_2$ is a 3-polar fuzzy graph.

Theorem 4.11. *The strong product* $G_1 \otimes G_2$ *of two m-polar fuzzy graphs is an m-polar fuzzy graph.*

Theorem 4.12. *If* $G_1 = (A_1,B_1)$ *and* $G_2 = (A_2,B_2)$ *are complete m-polar fuzzy graphs, then* $G_1 \otimes G_2$ *is complete.*

***Proof*:** *By Theorem* 4.11*, we have that the strong product of m-polar fuzzy graphs is an m-polar fuzzy graph. Because* G_1 *and* G_2 *are complete, every pair of vertices are adjacent in the graph* $G_1 \otimes G_2$ *and* $E = \widetilde{V_1 \times V_2^2}$.

Let $(u,v_1)(u,v_2) \in E$*. Because* G_2 *is complete, we have for each* $i = 1,2,\ldots,m$

$$p_i \circ (B_1 \otimes B_2)((u,v_1)(u,v_2))$$

$$= p_i \circ A_1(u) \wedge p_i \circ B_2(v_1v_2)$$

$$= p_i \circ A_1(u) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2)$$

$$= p_i \circ (A_1 \otimes A_2)(u,v_1) \wedge p_i \circ (A_1 \otimes A_2)(u,v_2).$$

Let $(u_1,w)(u_2,w) \in E$*. Because* G_1 *is complete, we have for each* $i = 1,2,\ldots,m$

$$p_i \circ (B_1 \otimes B_2)((u_1,w)(u_2,w))$$

$$= p_i \circ B_1(u_1u_2) \wedge p_i \circ A_2(w)$$

$$= p_i \circ A_1(u_1) \wedge p_i \circ A_1(u_2) \wedge p_i \circ A_2(w)$$

$$= p_i \circ (A_1 \otimes A_2)(u_1,w) \wedge p_i \circ (A_1 \otimes A_2)(u_2,w).$$

Finally, let $(u_1,v_1)(u_2,v_2) \in E$*. Then, because* G_1 *and* G_2 *are complete, we have for each* $i = 1,2,\ldots,m$

$$p_i \circ (B_1 \otimes B_2)((u_1,v_1)(u_2,v_2))$$

$$= p_i \circ B_1(u_1u_2) \wedge p_i \circ B_2(v_1v_2)$$

$$= p_i \circ A_1(u_1) \wedge p_i \circ A_1(u_2) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2)$$

$$= p_i \circ (A_1 \otimes A_2)(u_1,v_1) \wedge p_i \circ (A_1 \otimes A_2)(u_2,v_2).$$

Hence, $G_1 \otimes G_2$ *is complete.*

Theorem 4.13. *If* $G_1 = (A_1,B_1)$ *and* $G_2 = (A_2,B_2)$ *are m-polar fuzzy graphs such that* $G_1 \sqcap G_2$ *is strong, then at least* G_1 *or* G_2 *must be strong.*

***Proof*:** *Let us assume that both* G_1 *and* G_2 *are not strong m-polar fuzzy graphs. Then, there exists at least one* $u_1v_1 \in E_1$ *and* $u_2v_2 \in E_2$ *such that for each* $i = 1,2,\ldots,m$ $p_i \circ B_1(u_1v_1) < p_i \circ A_1(u_1) \wedge p_i \circ A_1(v_1)$ *and*

$$p_i \circ B_2(u_2v_2) < p_i \circ A_2(u_2) \wedge p_i \circ A_2(v_2).$$

Now, for each $i = 1,2,\ldots,m$*we have*

$$p_i \circ (B_1 \sqcap B_2)((u_1,v_1),(u_2,v_2))$$

$$= p_i \circ B_1(u_1u_2) \wedge p_i \circ B_2(v_1v_2)$$

$< p_i \circ A_1(u_1) \wedge p_i \circ A_1(u_2) \wedge p_i \circ A_2(v_1) \wedge p_i \circ A_2(v_2)$ *(from the above assumption)*

$$= p_i \circ (A_1 \sqcap A_2)(u_1,v_1) \wedge p_i \circ (A_1 \sqcap A_2)(u_2,v_2).$$

This shows that, $G_1 \sqcap G_2$ *is not strong, which is a contradiction. Therefore, our assumption is wrong. This means one of* G_1 *or* G_2 *is strong.*

The following result follows from the preceding theorem.

Theorem 4.14. *If* $G_1 = (A_1,B_1)$ *and* $G_2 = (A_2,B_2)$ *are m-polar fuzzy graphs such that* $G_1 \bullet G_2$ *or* $G_1 \otimes G_2$ *is strong, then at least* G_1 *or* G_2 *must be strong.*

5. Balanced *m*-polar fuzzy graphs

This section begins by defining the density of an *m*-polar fuzzy graph and balanced *m*-polar fuzzy graphs. Then it is proved that any complete *m*-polar fuzzy graph is balanced, but the converse is not always true.

Definition 5.1. *The density of an m-polar fuzzy graph* $G = (A,B)$ *of* $G^* = (V,E)$ *is* $D(G) = (p_1 \circ D(G), p_2 \circ D(G), \ldots, p_m \circ D(G))$*, where for each* $i = 1,2,\ldots,m$

$$p_i \circ D(G) = \frac{2(\sum_{u,v \in V} p_i \circ B(uv))}{\sum_{u,v \in V} (p_i \circ A(u) \wedge p_i \circ A(v))}.$$

G is said to be balanced if for each $i = 1,2,\ldots,m$

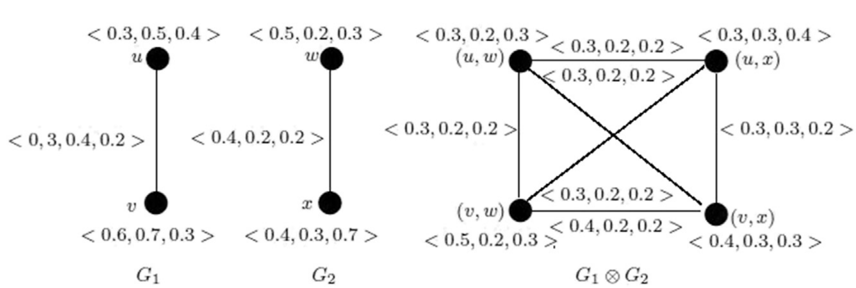

Fig. 5. Strong product of G_1 and G_2

$p_i\circ D(H)\le p_i\circ D(G)$ for all non-empty subgraphs H of G.

***Example* 5.2** Consider the 3-polar fuzzy graph $G=(A,B)$ of $G^*=(V,E)$ where $V=\{a,b,c\}$, $E=\{ab,bc,ca\}$,
$A=\left\{\frac{<0.3,0.4,0.5>}{a},\frac{<0.3,0.4,0.5>}{b},\frac{<0.3,0.4,0.5>}{c}\right\}$,
$B=\left\{\frac{<0.1,0.2,0.2>}{ab},\frac{<0.1,0.2,0.2>}{bc},\frac{<0.1,0.2,0.2>}{ca}\right\}$.

We have,

$$p_1\circ D(G)=\frac{2(p_1\circ B(ab)+p_1\circ B(bc)+p_1\circ B(ca))}{(p_1\circ A(a)\wedge p_1\circ A(b)+p_1\circ A(b)\wedge p_1\circ A(c)+p_1\circ A(c)\wedge p_1\circ A(a))}=\frac{2(0.1+0.1+0.1)}{0.3+0.3+0.3}=0.67.$$

Similarly, $p_2\circ D(G)=1$ and $p_3\circ D(G)=0.8$.

Hence, $D(G)=(0.67,1,0.8)$.

The non-empty subgraphs of G are $H_1=\{a,b\}$, $H_2=\{b,c\}$ and $H_3=\{c,a\}$.

Then,

$$D(H_1)=\left(\frac{2\times 0.1}{0.3},\frac{2\times 0.2}{0.4},\frac{2\times 0.2}{0.5}\right)=(0.67,1,0.8),$$

$$D(H_2)=\left(\frac{2\times 0.1}{0.3},\frac{2\times 0.2}{0.4},\frac{2\times 0.2}{0.5}\right)=(0.67,1,0.8)$$

and $D(H_3)=\left(\frac{2\times 0.1}{0.3},\frac{2\times 0.2}{0.4},\frac{2\times 0.2}{0.5}\right)=(0.67,1,0.8)$.

We see that, $D(H_1)=D(H_2)=D(H_3)=D(G)=(0.67,1,0.8)$.

Hence, G is a balanced 3-polar fuzzy graph (see Fig. 6).

Theorem 5.3. *Any complete m-polar fuzzy graph is balanced.*

***Proof*:** Let $G=(A,B)$ be a complete m-polar fuzzy graph and H be a non-empty subgraph of G. Then, for each $i=1,2,\ldots,m$

$$p_i\circ D(G)=\frac{2(\sum_{u,v\in V}p_i\circ B(uv))}{\sum_{u,v\in V}(p_i\circ A(u)\wedge p_i\circ A(v))}=\frac{2(\sum_{u,v\in V}p_i\circ A(u)\wedge p_i\circ A(v))}{\sum_{u,v\in V}(p_i\circ A(u)\wedge p_i\circ A(v))}=2$$

and

$$p_i\circ D(H)=\frac{2\left(\sum_{u,v\in V(H)}p_i\circ B(uv)\right)}{\sum_{u,v\in V(H)}(p_i\circ A(u)\wedge p_i\circ A(v))}\le\frac{2\left(\sum_{u,v\in V(H)}p_i\circ A(u)\wedge p_i\circ A(v)\right)}{\sum_{u,v\in V(H)}(p_i\circ A(u)\wedge p_i\circ A(v))}=2$$

(where $V(H)$ represents the vertex of H).

This shows that G is balanced.

The converse of the above theorem is not always true. For example, the 3-polar fuzzy graph in Fig. 6 is balanced but not complete.

Below, we discuss two types of m-polar fuzzy graphs, each with density equal to $1=(1,1,\ldots,1)$.

Theorem 5.4. *Every self-complementary strong m-polar fuzzy graph has density equal to $1=(1,1,\ldots,1)$.*

***Proof*:** Let $G=(A,B)$ be a self-complementary strong m-polar fuzzy graph of $G^*=(V,E)$. Then, by Proposition 6.12 of [8], we have for each $i=1,2,\ldots,m$ and $xy\in\widetilde{V^2}$,

$$\sum_{x\neq y}p_i\circ B(xy)=\frac{1}{2}\sum_{x\neq y}(p_i\circ A(x)\wedge p_i\circ A(y)).$$

Hence, $p_i\circ D(G)=\frac{2(\sum_{u,v\in V}p_i\circ B(uv))}{\sum_{u,v\in V}(p_i\circ A(u)\wedge p_i\circ A(v))}=1$ (by the above) for each $i=1,2,\ldots,m$. Thus, $D(G)=1$.

The converse of this theorem is not true in general. For example, the 3-polar fuzzy graph in Fig. 7 has density equal to $(1,1,1)$, but it is not self-complementary strong.

Here, we see that $D(G)=(1,1,1)$ but $G\ncong\overline{G}$.

Theorem 5.5. *Let $G=(A,B)$ be a strong m-polar fuzzy graph such that for each $i=1,2,\ldots,m$ and $uv\in\widetilde{V^2}$, $p_i\circ B(uv)=\frac{1}{2}(p_i\circ A(u)\wedge p_i\circ A(v))$. Then, $D(G)=1=(1,1,\ldots,1)$.*

***Proof*:** Because $G=(A,B)$ is a strong m-polar fuzzy graph such that for each $i=1,2,\ldots,m$ and $uv\in\widetilde{V^2}$, $p_i\circ B(uv)=\frac{1}{2}(p_i\circ A(u)\wedge p_i\circ A(v))$; therefore, by Proposition 6.13 of [8], we have G is self-complementary.

Hence, by Theorem 5.4, it follows that $D(G)=1$.

Next, necessary and sufficient conditions are established for the direct product, semi-strong product and strong product of two m-polar fuzzy graphs to be balanced.

***Theorem* 5.6** Let $G_1=(A_1,B_1)$ and $G_2=(A_2,B_2)$ be two m-polar fuzzy graphs of $G_1^*=(V_1,E_1)$ and $G_2^*=(V_2,E_2)$, respectively. Then, $D(G_k)\le D(G_1\sqcap G_2)$ for $k=1,2$ if and only if $D(G_1)=D(G_2)=D(G_1\sqcap G_2)$.

***Proof*:** Let $D(G_k)\le D(G_1\sqcap G_2)$ for $k=1,2$. Then for $i=1,2,\ldots,m$

$$p_i\circ D(G_1)=\frac{2(\sum_{u_1,u_2\in V_1}p_i\circ B_1(u_1u_2))}{\sum_{u_1,u_2\in V_1}(p_i\circ A_1(u_1)\wedge p_i\circ A_1(u_2))}$$

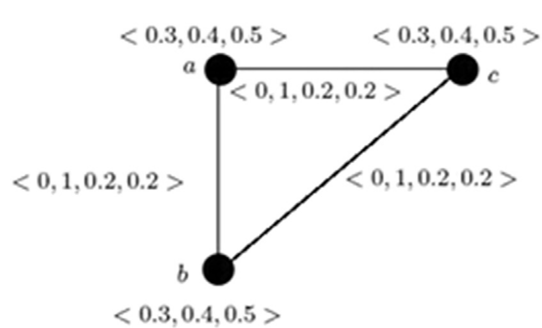

Fig. 6. 3-polar balanced fuzzy graph G

$$\geq \frac{2\left(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} p_i\circ B_1(u_1u_2)\wedge A_2(v_1)\wedge A_2(v_2)\right)}{\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} (p_i\circ A_1(u_1)\wedge p_i\circ A_1(u_2)\wedge A_2(v_1)\wedge A_2(v_2))}$$

$$= \frac{2\left(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} p_i\circ B_1(u_1u_2)\wedge p_i\circ B_2(v_1v_2)\right)}{\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} (p_i\circ A_1(u_1)\wedge p_i\circ A_1(u_2)\wedge A_2(v_1)\wedge A_2(v_2))}$$

$$= \frac{2\left(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} p_i\circ (B_1\sqcap B_2)((u_1,u_2)(v_1,v_2))\right)}{\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} p_i\circ (A_1\sqcap A_2)((u_1,u_2)(v_1,v_2))} = p_i\circ D(G_1\sqcap G_2).$$

Hence, $p_i\circ D(G_1)\geq p_i\circ D(G_1\sqcap G_2)$ *for each for* $i=1,2,\ldots,m$,
i.e., $D(G_1)\geq D(G_1\sqcap G_2)$.
Similarly, $D(G_2)\geq D(G_1\sqcap G_2)$.
Therefore, $D(G_1)=D(G_2)=D(G_1\sqcap G_2)$.

Theorem 5.7. *Let* $G_1=(A_1,B_1)$ *and* $G_2=(A_2,B_2)$ *be two balanced m-polar fuzzy graphs. Then,* $G_1\sqcap G_2$ *is balanced if and only if* $D(G_1)=D(G_2)=D(G_1\sqcap G_2)$.

Proof: *Suppose* $D(G_1\sqcap G_2)$ *is balanced. Then,* $D(G_k)\leq D(G_1\sqcap G_2)$ *for* $k=1,2$ *and by Theorem 5.6,* $D(G_1)=D(G_2)=D(G_1\sqcap G_2)$.

Conversely, let $D(G_1)=D(G_2)=D(G_1\sqcap G_2)$ *and H be a non-empty subgraph of* $G_1\sqcap G_2$*. Then there exist subgraphs* H_1 *of* G_1 *and* H_2 *of* G_2.

Let $p_i\circ D(G_1)=p_i\circ D(G_2)=\frac{q_i}{r_i}$, $p_i\circ D(H_1)=\frac{s_i}{t_i}$ *and* $p_i\circ D(H_2)=\frac{a_i}{b_i}$ *for* $i=1,2,\ldots,m$ *and* $a_i,b_i,q_i,r_i,s_i,t_i\in\mathbb{R}$.

Because G_1 *and* G_2 *are balanced, therefore for* $i=1,2,\ldots,m$
$p_i\circ D(H_1)=\frac{s_i}{t_i}\leq p_i\circ D(G_1)=\frac{q_i}{r_i}$ *and* $p_i\circ D(H_2)=\frac{a_i}{b_i}\leq p_i\circ D(G_2)=\frac{q_i}{r_i}$.
Thus, $s_ir_i+a_ir_i\leq t_iq_i+b_iq_i$ *i.e.,* $\frac{s_i+a_i}{t_i+b_i}\leq\frac{q_i}{r_i}$ *for* $i=1,2,\ldots,m$.
Hence, $p_i\circ D(H)\leq\frac{s_i+a_i}{t_i+b_i}\leq\frac{q_i}{r_i}=p_i\circ D(G_1\sqcap G_2)$ *for* $i=1,2,\ldots,m$.
Therefore, $G_1\sqcap G_2$ *is balanced.*
Similarly, we have the following results.

Theorem 5.8. *Let* $G_1=(A_1,B_1)$ *and* $G_2=(A_2,B_2)$ *be two balanced m-polar fuzzy graphs. Then (i)* $G_1\cdot G_2$ *is balanced if and only if* $D(G_1)=D(G_2)=D(G_1\cdot G_2)$.

(ii) $G_1\otimes G_2$ *is balanced if and only if* $D(G_1)=D(G_2)=D(G_1\otimes G_2)$.

6. Product *m*-polar fuzzy graphs

In this section, a new type of *m*-polar fuzzy graph, known as product *m*-polar fuzzy graphs, is defined.

Definition 6.1. *A product m-polar fuzzy graph of a graph* $G^*=(V,E)$ *is a pair* $G=(A,B)$ *where* $A:V\rightarrow[0,1]^m$ *is an m-polar fuzzy set in V and* $B:\widetilde{V^2}\rightarrow[0,1]^m$ *is an m-polar fuzzy set in* $\widetilde{V^2}$ *such that for each* $i=1,2,\ldots,m$; $p_i\circ B(xy)\leq p_i\circ A(x)\times p_i\circ A(y)$ *for all* $xy\in\widetilde{V^2}$.

***Remark* 6.2** *Because for each* $i=1,2,\ldots,m$; $p_i\circ A(x)$ *and* $p_i\circ A(y)$ *are less than or equal to 1, it follows that* $p_i\circ B(xy)\leq p_i\circ A(x)\times p_i\circ A(y)\leq p_i\circ A(x)\wedge p_i\circ A(y)$ *for all* $xy\in\widetilde{V^2}$*. Hence, every product m-polar fuzzy graph is an m-polar fuzzy graph.*

Definition 6.3. *A product m-polar fuzzy graph* $G=(A,B)$ *is said to be complete if for each* $i=1,2,\ldots,m$ *and* $x,y\in V$, $p_i\circ B(xy)=p_i\circ A(x)\times p_i\circ A(y)$.

Definition 6.4. *The complement of the product m-polar fuzzy graph* $G=(A,B)$ *is an m-polar fuzzy graph* $\overline{G}=(\overline{A},\overline{B})$ *where* $\overline{A}=A$ *and* $\overline{B}$ *is defined by*

$p_i\circ\overline{B}(xy)=p_i\circ A(x)\times p_i\circ A(y)-p_i\circ B(xy)$ *for each* $i=1,2,\ldots,m$ *and* $xy\in\widetilde{V^2}$.

***Remark* 6.5** *Because for all* $xy\in\widetilde{V^2}$, $i=1,2,\ldots,m$;

$$p_i\circ\overline{B}(xy)=p_i\circ A(x)\times p_i\circ A(y)-p_i\circ B(xy)\leq p_i\circ A(x)\times p_i\circ A(y),$$

Therefore, $\overline{G}$ *is a product m-polar fuzzy graph.*

Definition 6.6. *The union* $G_1\cup G_2=(A_1\cup A_2,B_1\cup B_2)$ *of two product m-polar fuzzy graphs* $G_1=(A_1,B_1)$ *of* $G_1^*=(V_1,E_1)$ *and* $G_2=(A_2,B_2)$ *of* $G_2^*=(V_2,E_2)$ *is defined as follows: for each* $i=1,2\ldots,m$

$$p_i\circ(A_1\cup A_2)(x)=\begin{cases} p_i\circ A_1(x) & \text{if } x\in V_1-V_2\\ p_i\circ A_2(x) & \text{if } x\in V_2-V_1\\ p_i\circ A_1(x)\vee p_i\circ A_2(x) & \text{if } x\in V_1\cap V_2.\end{cases}$$

and

$$p_i\circ(B_1\cup B_2)(xy)=\begin{cases} p_i\circ B_1(xy) & \text{if } xy\in E_1-E_2\\ p_i\circ B_2(xy) & \text{if } xy\in E_2-E_1\\ p_i\circ B_1(xy)\vee p_i\circ B_2(xy) & \text{if } xy\in E_1\cap E_2.\end{cases}$$

Proposition 6.7. *The direct product* $G_1\sqcap G_2$ *of two product m-polar fuzzy graphs* $G_1=(A_1,B_1)$ *and* $G_2=(A_2,B_2)$ *is a product m-polar fuzzy graph.*

Proof: *Let* $(u_1,v_1)(u_2,v_2)\in E$*. Then* $u_1u_2\in E_1$ *and* $v_1v_2\in E_2$.
Now, for each $i=1,2,\ldots,m$ *we have*

$$p_i\circ(B_1\sqcap B_2)((u_1,v_1),(u_2,v_2))$$

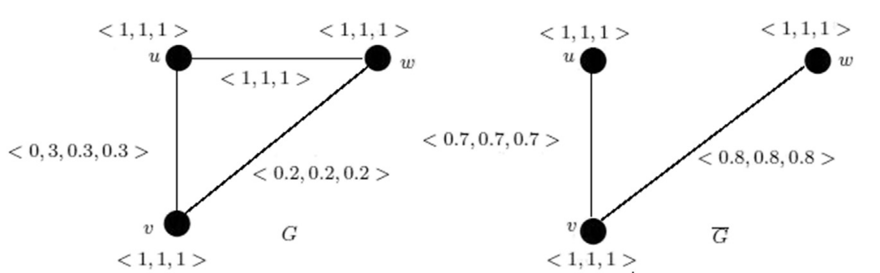

Fig. 7. *G is a 3-polar fuzzy graph with density* (1,1,1) *but is not self-complementary and strong.*

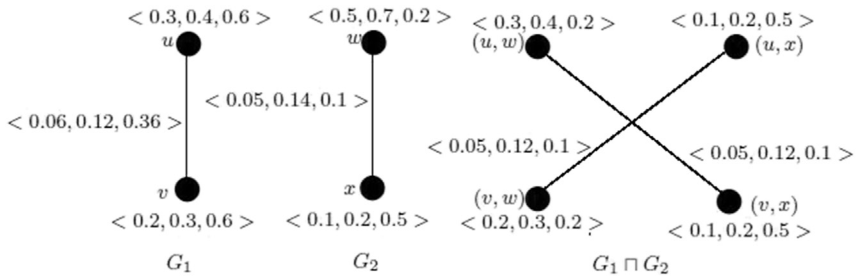

Fig. 8. *G_1 and G_2 are complete product 3-polar fuzzy graphs, but $G_1 \sqcap G_2$ is not complete.*

$$= min\{p_i \circ B_1(u_1u_2), p_i \circ B_2(v_1v_2)\}$$

$$\leq min\{p_i \circ A_1(u_1) \times p_i \circ A_1(u_2), p_i \circ A_2(v_1) \times p_i \circ A_2(v_2)\}$$

$$= min\{p_i \circ A_1(u_1), p_i \circ A_2(v_1)\} \times min\{p_i \circ A_1(u_2), p_i \circ A_2(v_2)\}$$

$$= p_i \circ (A_1 \sqcap A_2)(u_1, v_1) \times p_i \circ (A_1 \sqcap A_2)(u_2, v_2).$$

Hence, the result.

***Remark* 6.8** *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two complete product m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then, $G_1 \sqcap G_2$ may not be complete.*

For example, let us take product 3-polar fuzzy graphs G_1 and G_2, which are complete, but $G_1 \sqcap G_2$ is not complete (see Fig. 8)

Definition 6.9. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be the product m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then, the ring sum of G_1 and G_2 is denoted by $G = G_1 \oplus G_2 = (A_1 \oplus A_2, B_1 \oplus B_2)$ and defined as follows: for each $i = 1,2\ldots,m$*

$p_i \circ (A_1 \oplus A_2)(u) = p_i \circ (A_1 \cup A_2)(u)$ for all $u \in V_1 \cup V_2$ and

$$p_i \circ (B_1 \oplus B_2)(uv) = \begin{cases} p_i \circ B_1(uv) & \text{if} \quad uv \in E_1 - E_2 \\ p_i \circ B_2(uv) & \text{if} \quad uv \in E_2 - E_1 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6.10. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be the product m-polar fuzzy graphs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then, the ring sum $G = G_1 \oplus G_2 = (A_1 \oplus A_2, B_1 \oplus B_2)$ is a product m-polar fuzzy graph.*

***Proof*:** *We show the following: for each $i = 1,2,\ldots,m$*

$p_i \circ (B_1 \oplus B_2)(uv) \leq p_i \circ (A_1 \oplus A_2)(u) \times p_i \circ (A_1 \oplus A_2)(v)$ for all $uv \in E_1 \cup E_2$.

***Case (i)*:** *Let $uv \in E_1 - E_2$ and $u,v \in V_1 - V_2$. Then, for each $i = 1,2,\ldots,m$*

$$p_i \circ (B_1 \oplus B_2)(uv)$$

$$= p_i \circ B_1(uv) \leq p_i \circ A_1(u) \times p_i \circ A_1(v)$$
$$= p_i \circ (A_1 \oplus A_2)(u) \times p_i \circ (A_1 \oplus A_2)(v).$$

***Case (ii)*:** *Let $uv \in E_1 - E_2$ and $u \in V_1 - V_2$, $v \in V_1 \cap V_2$.*
Then, for each $i = 1,2,\ldots,m$

$$p_i \circ (B_1 \oplus B_2)(uv)$$

$$= p_i \circ B_1(uv)$$

$$\leq p_i \circ A_1(u) \times max\{p_i \circ A_1(v), p_i \circ A_2(v)\}$$

$$\leq p_i \circ (A_1 \cup A_2)(u) \times p_i \circ (A_1 \cup A_2)(v)$$

$$= p_i \circ (A_1 \oplus A_2)(u) \times p_i \circ (A_1 \oplus A_2)(v).$$

***Case (iii)*:** *Let $uv \in E_1 - E_2$ and $u,v \in V_1 \cap V_2$.*
Then, for each $i = 1,2,\ldots,m$

$$p_i \circ (B_1 \oplus B_2)(uv)$$

$$= p_i \circ B_1(uv)$$

$$\leq max\{p_i \circ A_1(u), p_i \circ A_2(u)\} \times max\{p_i \circ A_1(v), p_i \circ A_2(v)\}$$

$$\leq p_i \circ (A_1 \cup A_2)(u) \times p_i \circ (A_1 \cup A_2)(v)$$

$$= p_i \circ (A_1 \oplus A_2)(u) \times p_i \circ (A_1 \oplus A_2)(v).$$

Similarly, we can show that if $uv \in E_2 - E_1$, then for each $i = 1,2,\ldots,m$

$$p_i \circ (B_1 \oplus B_2)(uv) \leq p_i \circ (A_1 \oplus A_2)(u) \times p_i \circ (A_1 \oplus A_2)(v).$$

Hence, the result.

7. Conclusions

The theory of graphs is an extremely useful tool in solving combinatorial problems in different areas, including algebra, number theory, geometry, topology, operation research, optimization, and computer science. Because research and models of real world problems often involve multi-agent, multi-attribute, multi-object, multi-index, and multi-polar information and uncertainty, the study of *m*-polar fuzzy graphs is significant. The *m*-polar fuzzy models gives increasing precision, flexibility, and comparability compared to the classical, fuzzy and bipolar fuzzy models. Therefore, we studied some properties of *m*-polar fuzzy graphs. In this study, we defined three new operations, density and balanced *m*-polar fuzzy graphs, but the degree of the resultant graphs was not determined. We are currently working on this topic. The results of this paper will help to find new algorithms and more important results. Our next plan is to extend our research work to regular and irregular *m*-polar fuzzy graphs, *m*-polar fuzzy intersection graphs, *m*-polar fuzzy interval graphs, and *m*-polar fuzzy hypergraphs.

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