A construction for 2-chromatic Steiner quadruple systems

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Abstract

In 1971, Doyen and Vandensavel gave a special doubling construction that gives a direct construction of 2-chromatic SQS\((v)\) for all \(v \equiv 4 \text{ or } 8 \pmod{12}\). In this paper, we introduce the concept of a 2-chromatic candelabra quadruple system, and use it to provide a construction for 2-chromatic SQS. It is proved that a 2-chromatic SQS\((v)\) exists if \(v \equiv 10 \text{ or } 26 \pmod{48}\), or if \(v \equiv 2 \text{ or } 34 \pmod{96}\) with the possible exception \(v = 98\).

1. Introduction

A \(t\)-wise balanced design (\(t\)BD) is a pair \((X, \mathcal{B})\), where \(X\) is a finite set of points and \(\mathcal{B}\) is a set of subsets of \(X\), called blocks, with the property that every \(t\)-element subset of \(X\) is contained in a unique block. If \(|X| = v\) and the block sizes of \(\mathcal{B}\) are all from \(K\), we denote the \(t\)BD by \(S(t, K, v)\). When \(K = \{k\}\), we simply write \(k\) for \(K\). An \(S(t, k, v)\) is called a Steiner system.

An \(S(3, 4, v)\) is called a Steiner quadruple system of order \(v\) (briefly an SQS\((v)\)). It is well known (see [2]) that an SQS\((v)\) exists if and only if \(v \equiv 2 \text{ or } 4 \pmod{6}\).

As with a hypergraph, a \(k\)-coloring of an SQS\((v)\) is a partition of the set \(X\) into \(k\) parts or color classes such that no block of \(\mathcal{B}\) is contained in any color class. An SQS\((v)\) is \(k\)-chromatic if it is \(k\)-colorable but not \((k - 1)\)-colorable.

Steiner quadruple systems can be 2-colored as was noted by Doyen and Vandensavel [1]. In that paper, they pointed out that if an SQS\((v)\) can be 2-colorable, then the color classes each
Theorem 1.1. A 2-chromatic SQS(v) exists if \( v \equiv 4 \) or \( 8 \) (mod \( 12 \)), or if \( v = 2 \cdot 5^a 13^b 17^c \), \( a, b, c \geq 0 \), and proved that there does not exist a 2-chromatic SQS(14). Recently, Phelps in [8] enumerated 2-chromatic SQS(22) having a cyclic automorphism of order 11. We state these known results on 2-chromatic SQS in the following theorem.

Theorem 1.2. A 2-chromatic SQS(v) exists if \( v \equiv 4 \) or \( 8 \) (mod \( 12 \)), or if \( v \equiv 10 \) or \( 26 \) (mod \( 48 \)), or if \( v \equiv 2 \) or \( 34 \) (mod \( 96 \)) with the possible exception of \( v = 98 \), or if \( v = 22 \). A 2-chromatic SQS(14) does not exist.

2. A recursive construction for 2-chromatic SQS

In Lemma 2.2 of this section we describe a recursive construction for 2-chromatic SQS from 2-chromatic candelabra quadruple systems.

Let \( v \) be a non-negative integer, let \( t \) be a positive integer and let \( K \) be a set of positive integers. A candelabra \( t \)-system (or \( t-CS \) as in [7]) of order \( v \), and block sizes from \( K \) denoted by \( CS(t, K, v) \), is a quadruple \((X, S, G, A)\) that satisfies the following properties:

1. \( X \) is a set of \( v \) elements (called points);
2. \( S \) is an \( s \)-subset (called the stem of the candelabra) of \( X \);
3. \( G = \{G_1, G_2, \ldots\} \) is a set of non-empty subsets (called groups or branches) of \( X \setminus S \), which partition \( X \setminus S \);
4. \( A \) is a family of subsets (called blocks) of \( X \), each of cardinality from \( K \);
5. every \( t \)-subset \( T \) of \( X \) with \( |T \cap (S \cup G_i)| < t \), for all \( i \), is contained in a unique block and no \( t \)-subsets of \( S \cup G_i \), for all \( i \), are contained in any block.

By the group type (or type) of a \( t-CS \) \((X, S, \Gamma, A)\) we mean the list \(((|G|, G \in \Gamma) : |S|)\) of group sizes and stem size. The stem size is separated from the group sizes by a colon. If a \( t-CS \) has \( n_i \) groups of size \( g_i \), \( 1 \leq i \leq r \), and stem size \( s \), then we use the notation \((g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r} : s)\) to denote the group type. When \( t = 3 \) and \( K = \{4\} \), such a system is called a candelabra quadruple system (as in [4]) and denoted for short by \( \text{CQS}(g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r} : s) \).

Let \( g_1, \ldots, g_r \) and \( s \) be all even. A \( \text{CQS}(g_1^{a_1} g_2^{a_2} \cdots g_r^{a_r} : s) \) \((X, S, \mathcal{G}, \mathcal{B})\) is called 2-chromatic if the point set \( X \) can be partitioned into two color classes \( X_1 \) and \( X_2 \) such that \(|G \cap X_1| = |G \cap X_2|\) for \( G \in \mathcal{G} \cup \{S\} \), and such that no block of \( \mathcal{B} \) is contained in any color class.

We state one example which will play an important role in the following.

Lemma 2.1. There exists a 2-chromatic CQS(\(8^4 : 2)\).
Proof. The desired design is constructed on \((Z_{16} \cup \{\infty\}) \times Z_2\) with groups \([i, i + 4, i + 8, i + 12] \times Z_2\) for \(0 \leq i \leq 3\) and a stem \([\infty] \times Z_2\). Its two color classes are \((Z_{16} \cup \{\infty\}) \times \{j\}, j = 0, 1\). Its blocks are generated by 86 base blocks under \((\text{mod} 16, -)\), of which we list 45 base blocks. The other 41 base blocks can be obtained from the last 41 base blocks under the mapping 
\[f : (a, b) \mapsto (-a, b + 1).\]  
Note that \(\infty + i = \infty\).

A holey quadruple system of order \(v\) with a hole of order \(s\), denoted by HQS\((v : s)\), is a triple \((X, S, \mathcal{A})\) where \(X\) is a set of size \(v\), \(S\) is an \(s\)-subset of \(X\), and \(\mathcal{A}\) is a set of \(4\)-subsets (called blocks) of \(X\) such that every 3-subset \(T \subseteq X\) with \(T \not\subseteq S\) is contained in a unique block and no 3-subset of \(S\) is contained in any block. Let \(v, s\) be even. An HQS\((v : s)\) is called 2-chromatic if the point set \(X\) can be partitioned into two color classes \(X_1\) and \(X_2\) of size \(v/2\) such that \(|S \cap X_1| = |S \cap X_2|\), and such that no block of \(\mathcal{A}\) is contained in any color class.

Using 2-chromatic CQS and HQS, we have the following.

Lemma 2.2. Suppose that there is a 2-chromatic CQS\((g_0^1, g_2^1, \ldots, g_r^r : s)\). If there is a 2-chromatic HQS\((g_i + s : s)\) for \(1 \leq i \leq r\), then there is a 2-chromatic HQS\((g_0 + s + \sum_{1 \leq i \leq r} a_i g_i : g_0 + s)\). Further, if there is a 2-chromatic SQS\((g_0 + s)\), then there is a 2-chromatic SQS\((g_0 + s + \sum_{1 \leq i \leq r} a_i g_i)\).

Proof. Let \((X, S, G, \mathcal{A})\) be the given 2-chromatic CQS\((g_0^1, g_2^1, \ldots, g_r^r : s)\) with two color classes \(X_1\) and \(X_2\). Let \(G\) be a group of size \(g_0\).

For any \(G' \in G\) with \(G' \neq G\), construct a 2-chromatic HQS\((|G'| + s : s)\) on \(G' \cup S\) with \(S\) as a hole and \((G' \cup S) \cap X_i\) \((i = 0, 1)\) as two color classes. Such a design exists by assumption. Denote its block set by \(B_{G'}\). Then \((X, \mathcal{A} \cup (\cup_{G' \in G, G' \neq G} B_{G'}))\) is a 2-chromatic HQS\((g_0 + s + \sum_{1 \leq i \leq r} a_i g_i : g_0 + s)\) with \(G \cup S\) as a hole, \(X_1\) and \(X_2\) as two color classes.

Further, suppose the given 2-chromatic SQS\((g_0 + s)\) on \(G \cup S\) with \((G \cup S) \cap X_i\) \((i = 0, 1)\) as two color classes has block set \(B_G\). Then \((X, \mathcal{A} \cup B_G \cup (\cup_{G' \in G, G' \neq G} B_{G'}))\) is a 2-chromatic SQS\((g_0 + s + \sum_{1 \leq i \leq r} a_i g_i)\) with two color classes \(X_1\) and \(X_2\). 

The above lemma demonstrates that the 2-chromatic CQS is useful in the construction of 2-chromatic SQS. To obtain such CQS, we first state a fundamental construction for 3-CS which is a special case of the fundamental construction of Hartman [3], then give a recursive construction for 2-chromatic CQS.

Let \((X, S, G, \mathcal{A})\) be a CS\((3, K, v)\) of type \((s_1^{a_1} \cdots g_r^{a_r} : s)\) and \(S = \{\infty_1, \infty_2, \ldots, \infty_3\}\) with \(s \geq 1\). For \(1 \leq i \leq s\), let \(B_i = \{A \setminus \{\infty_i\} : B \in \mathcal{A}, \infty_i \in A\}\) and \(T = \{A \in \mathcal{A} : A \cap S = \emptyset\}\). The \((s + 3)\)-tuple \((X, S, G, B_1, B_2, \ldots, B_s, T)\) is called an s-fan design (as in [3]). Its type is the list \(|G||G| \in G\). If block sizes of \(B_i\) and \(T\) are from \(K_i\) \((1 \leq i \leq s)\) and \(K_T\), respectively, then the s-fan design is denoted by s-FG\((3, (K_1, K_2, \ldots, K_s, K_T), \sum_{i=1}^r a_i g_i)\) of type \(g_1^{a_1} g_2^{a_2} \cdots g_r^{a_r}\).
Let $v$ be a non-negative integer, let $t$ be a positive integer and $K$ be a set of positive integers. A group divisible $t$-design of order $v$ and with block sizes from $K$ denoted by GDD($t$, $K$, $v$) is a triple $(X, G, B)$ such that

(1) $X$ is a set of $v$ elements (called points),

(2) $G = \{G_1, G_2, \ldots\}$ is a set of non-empty subsets of $X$ (called groups) such that $(X, G)$ is a 1-wise balanced design,

(3) $B$ is a family of subsets of $X$ (called blocks) each of cardinality from $K$ such that each block intersects any given group in at most one point,

(4) each $t$-set of points from $t$ distinct groups is contained in exactly one block.

The type of the GDD($t$, $K$, $v$) is defined as the list $|G||G \in G$.

**Theorem 2.3** ([3]). Suppose that there exists an e-FG(3, ($K_1$, $\ldots$, $K_e$, $K_T$), $v$) of type $s_1 \leq s_2 \leq \ldots \leq s_e$. Suppose that there exist a GDD(3, $L$, $b(k_1 + s_1)$) of type $b(k_i)$ for any $k_i \in K_i$ with $2 \leq i \leq e$, a CS(3, $L$, $b(k_1 + s_1)$) of type $(b(k_i))$ for any $k_i \in K_1$ and a GDD(3, $L$, $b(k_j)$) of type $b(k_j)$ for any $k_j \in K_T$. Then there exists a CS(3, $L$, $vb + \sum_{1 \leq i \leq e} s_i$) of type $(bgr_1)^a_1(bgr_2)^a_2(\ldots(bgr_e)^a_e) : \sum_{1 \leq i \leq e} s_i$.

Let $g_1, \ldots, g_r$ be all even. A GDD(3, $4$, $v$) of type $g_1 g_2 g_3 \ldots g_r$ $(X, G, A)$ is said to be 2-chromatic if the vertex set of $X$ can be partitioned into two color classes $X_1$ and $X_2$ such that $|G \cap X_1| = |G \cap X_2|$ for any $G \in G$ and no block of $A$ is contained in any color class.

Taking a 2-chromatic CQS and a 2-chromatic GDD(3, $4$, $v$) as input designs in Theorem 2.3, we have the following construction for 2-chromatic CQS.

**Lemma 2.4.** Suppose that there exists an e-FG(3, ($K_1$, $\ldots$, $K_e$, $K_T$), $v$) of type $s_1 \leq s_2 \leq \ldots \leq s_e$. Suppose that there exist a 2-chromatic GDD(3, $4$, $b(k_j + s_j)$) of type $b(k_j s_j)$ for any $k_j \in K_j$ with $2 \leq j \leq e$, a 2-chromatic CQS($b(k_i)$) for any $k_i \in K_1$ and a 2-chromatic GDD(3, $4$, $b(k_j)$) of type $b(k_j)$ for any $k_j \in K_T$. Then there exists a 2-chromatic CQS($s_1 \leq s_2 \leq \ldots \leq s_e$).

**Proof.** Let $(X, G, B, T)$ be the given e-FG. By the definition, $b$ and $s_j$ are all even. Let

$s = \sum_{1 \leq j \leq e} s_j$ and $S = \{\infty\} \times Z_2^s \times Z_2^t$. Suppose that $\infty \notin X$. We shall construct a 2-chromatic CQS($s_1 \leq s_2 \leq \ldots \leq s_e$) on $X' = (X \times Z_2^t) \cup S$ having a group set $G' = \{G \times Z_2^t : G \in G\}$ and a stem $S$. Let $X_i = (X \times Z_2^t \times \{i\}) \cup (\infty \times Z_2^t \times \{i\}) (i = 0, 1)$ be two color classes.

For convenience, define $G_x = \{x\} \times Z_2^t \times Z_2$ for $x \in X$ and $S_i \cup S_2 \cup \ldots \cup S_e$, where $S_1 = \{\infty\} \times Z_2^s / 2 \times Z_2$, $S_j = S'_j \times Z_2$ and $S'_j = (\infty, \sum_{1 \leq j \leq s_i / 2 - 1})$ for $2 \leq j \leq e$.

For each block $B \in T$, construct a 2-chromatic GDD(3, $4$, $b|B|$) on $B \times Z_2^t \times Z_2$ with $G_x$ ($x \in B$) as its groups and $B \times Z_2^t \times \{i\}$ ($i = 0, 1$) as two color classes. Such a design exists by assumption. Denote its block set by $A_B$.

For any $2 \leq j \leq e$ and any block $B_j \in B_j$, construct a 2-chromatic GDD(3, $4$, $b|B_j| + s_j$) on $(B_j \times Z_2^t \times Z_2) \cup S_j$ with $G_x$ ($x \in B_j$) and $S_j$ as its groups, $((B_j \times Z_2^t) \cup S'_j) \times \{i\}$ ($i = 0, 1$) as two color classes. Such a design exists by assumption. Denote its block set by $A_B$.

For each block $B_j \in B_j$, construct a 2-chromatic CQS($s_1 \leq s_2 \leq \ldots \leq s_e$) on $(B_j \times Z_2^t \times Z_2) \cup S_1$ with $G_x$ ($x \in B_1$) as its branches and $S_1$ as its stem, such that $(B_1 \times Z_2^t \times \{i\}) \cup (\infty \times Z_2^s) \times \{i\}$ ($i = 0, 1$) as two color classes. Such a design exists by assumption. Denote its block set by $A_B$.

Let

$F = (\cup_{B \in T} A_B) \cup (\cup_{1 \leq j \leq e, B_j \in B_j} A_B)$. 

By Theorem 2.3 $(X', S, G', F)$ is a CQS $((bg_1)^{a_1}(bg_2)^{a_2}\cdots (bg_r)^{a_r} : \sum_{1 \leq i \leq e} s_i)$. Further, each block in any input design is not contained in any color class. It follows that each block of $F$ is not contained in any color class and such a CQS is also 2-chromatic. \square

In the next section, we shall use Lemma 2.4 to obtain some 2-chromatic CQS which will produce some new infinite classes of 2-chromatic SQS.

3. Main result

We shall use the result on $S(3, \{4, 5\}, v)$ and apply Lemma 2.4 to prove our main result. We first construct the required input design.

A recursive construction for 2-chromatic GDD$(3, 4, v)$ is given below.

**Lemma 3.1.** Suppose that there is a GDD$(3, K, v)$ of type $g_1^{a_1}\cdots g_r^{a_r}$. If there is a 2-chromatic GDD$(3, 4, bk)$ of type $b^k$ for any $k \in K$, then there is a 2-chromatic GDD$(3, 4, bv)$ of type $(bg_1)^{a_1}\cdots (bg_r)^{a_r}$.

**Proof.** Let $(X, G, A)$ be the given GDD$(3, K, v)$. By the definition of the 2-chromatic GDD, $b$ is even. We shall construct the desired design on $X' = X \times Z_{b/2} \times Z_2$ having a group set $G' = \{G \times Z_{b/2} \times Z_2 : G \in G\}$ and two color classes $X \times Z_{b/2} \times \{i\}, i = 0, 1$.

For each block $A \in A$, construct a 2-chromatic GDD$(3, 4, b|A|)$ on $A \times Z_{b/2} \times Z_2$ with groups $\{x\} \times Z_{b/2} \times Z_2$, such that $A \times Z_{b/2} \times \{i\}$ is a two color classes. Such a design exists by assumption. Denote its block set by $B_A$.

Let $B = \cup_{A \in A} B_A$. Then, it is easy to check that $(X', G', B)$ is a GDD$(3, 4, bv)$ of type $(bg_1)^{a_1}\cdots (bg_r)^{a_r}$. Further, each block in any input design is not contained in any color class. It follows that each block in the GDD$(3, 4, bv)$ is not contained in any color class and such a design is also 2-chromatic. \square

A GDD$(3, 4, ur)$ of type $r^u$ is called an $H$-design [6]. Mills proved the following.

**Lemma 3.2** (Mills [6]). For $u > 3$ and $u \neq 5$, a GDD$(3, 4, ur)$ of type $r^u$ exists if and only if $ru$ is even and $r(u - 1)(u - 2)$ is divisible by 3. For $u = 5$, a GDD$(3, 4, 5r)$ of type $r^5$ exists if $r$ is divisible by 4 or 6.

From the known GDD in Lemma 3.2, we have the following.

**Lemma 3.3.** For $u > 3$ and $u \neq 5$, if $ru$ is even and $r(u - 1)(u - 2)$ is divisible by 3, then there is a 2-chromatic GDD$(3, 4, urb)$ of type $(br)^u$ for any even $b$. For $u = 5$, if $r$ is divisible by 4 or 6, then there is a 2-chromatic GDD$(3, 4, 5br)$ of type $(br)^5$ for any even $b$.

**Proof.** First we construct a 2-chromatic GDD$(3, 4, 8)$ of type $2^4$ on $Z_8$ with groups $G_i = \{i, i + 4\} (0 \leq i \leq 3)$ and two color classes $X_1 = \{0, 1, 2, 3\}, X_2 = \{4, 5, 6, 7\}$. Its block set $A$ contains the following blocks.

$0 1 2 7$  $0 1 3 6$  $0 2 3 5$  $0 5 6 7$  $1 2 3 4$  $1 4 6 7$  $2 4 5 7$  $3 4 5 6$

From such a GDD, we shall construct a 2-chromatic GDD$(3, 4, 4b)$ on $Z_8 \times Z_{b/2}$ with groups $\{G_i\} \times Z_{b/2}, 0 \leq i \leq 3$, and two color classes $X_j \times Z_{b/2}, j = 0, 1$. 


For each block $A = \{x_1, x_2, x_3, x_4\} \in \mathcal{A}$, construct a GDD($3, 4, 2b$) of type $(b/2)^4$ on $A \times Z_{b/2}$ with groups $\{x\} \times Z_{b/2}$. It has the following blocks.

$\{(x_1, i), (x_2, j), (x_3, k), (x_4, l)\} : i, j, k, l \in Z_{b/2}, \quad i + j + k + l \equiv 0 \pmod{b/2}.$

Denote its block set by $\mathcal{B}_A$.

Let $\mathcal{B} = \bigcup_{A \in \mathcal{A}} \mathcal{B}_A$. It is easy to check that $\mathcal{B}$ is the block set of a GDD($3, 4, 4b$) of type $b^4$. Since each block in $\mathcal{B}_A$ is not contained in any color class, such a GDD is also 2-chromatic.

For the given $r, u$, by Lemma 3.2 there is a GDD($3, 4, ur$) of type $r^u$. Using the above 2-chromatic GDD($3, 4, 4b$) of type $b^4$ as the input design, by Lemma 3.1 there exists a 2-chromatic GDD($3, 4, bur$) of type $(br)^u$. □

A small 2-chromatic CQS is constructed directly.

**Lemma 3.4.** There exists a 2-chromatic CQS($8^3 : 2$).

**Proof.** The desired design is constructed on $(Z_{12} \cup \{\infty\}) \times Z_2$ with groups $\{i, i + 3, i + 6, i + 9\} \times Z_2 (0 \leq i \leq 2)$ and a stem $\{\infty\} \times Z_2$. Its two color classes are $(Z_{12} \cup \{\infty\}) \times \{j\}, j = 0, 1$. Its blocks are generated by $96$ base blocks under $\langle +2 \pmod{12}, -\rangle$, of which we list $48$ base blocks as follows. The other $48$ base blocks can be obtained from them under the mapping $f : (a, b) \mapsto (-a, b + 1)$. Note that $\infty + i = \infty$ and that the first two base blocks $A, B$, together with $f(A), f(B)$, each generate exactly two distinct blocks.

\[
\begin{array}{cccccccc}
0_0 40 80 \infty & 0_0 50 90 \infty & 0_0 41 81 \infty & 0_0 51 91 \infty & 0_0 10 20 \infty & 0_0 60 110 \infty \\
0_1 11 \infty & 0_1 61 11 \infty & 0_1 10 21 \infty & 0_1 60 11 \infty & 0_1 11 20 \infty & 0_1 61 10 \infty \\
0_0 30 60 1 & 0_0 40 50 1 & 0_0 50 40 1 & 0_0 10 60 51 & 0_0 10 70 81 & 0_0 10 80 11 \\
0_0 10 90 71 & 0_0 10 100 31 & 0_0 10 110 91 & 0_0 20 40 111 & 0_0 20 50 01 & 0_0 20 60 71 \\
0_0 20 70 41 & 0_0 20 80 101 & 0_0 20 90 21 & 0_0 20 110 61 & 0_0 30 40 41 & 0_0 30 50 11 \\
0_0 30 70 71 & 0_0 30 80 21 & 0_0 30 110 81 & 0_0 40 90 11 & 0_0 40 110 71 & 0_0 50 60 10 \\
0_0 50 90 51 & 0_0 50 110 11 & 0_0 70 90 101 & 0_0 70 110 21 & 0_0 90 110 11 & 0_1 30 50 01 \\
0_1 30 70 51 & 0_1 30 90 21 & 0_1 10 60 101 & 0_1 20 31 81 & 0_1 70 11 91 & 0_0 110 31 51 \\
\end{array}
\]

□

A result on 3BD is given in [5].

**Lemma 3.5 ([5]).** There is an S($3, \{4, 5\}, v$) for $v \equiv 1, 2, 4, 5, 8, 10 \pmod{12}$ with $v \neq 13$.

We are in a position to prove our main result.

**Proof of Theorem 1.2.** By Theorem 1.1, we need only to consider the cases $v \equiv 10, 26 \pmod{48}$ and $v \equiv 2, 34 \pmod{96}$ with $v \neq 98$. Each such $v$ can be written as $v = 8k + 2$ where $k \equiv 0, 1, 3, 4, 7, 9 \pmod{12}$ and $k \neq 12$.

For $k = 0$, it exists trivially. For $k = 1$, a 2-chromatic SQS($8k + 2$) exists from Theorem 1.1. For each $k \equiv 0, 1, 3, 4, 7, 9 \pmod{12}$, $k \geq 3$ and $k \neq 12$, by Lemma 3.5 there is an S($3, \{4, 5\}, k + 1$). Deleting a point from this 3BD yields a 1-FG($3,$ $\{(3, 4), \{4, 5\}\}$, $k$) of type $1^k$. Apply Lemma 2.4 with $b = 8$. By Lemmas 2.1 and 3.4 there is a 2-chromatic CQS($8^i : 2$) for $i = 3, 4$. By Lemma 3.3 there is a 2-chromatic GDD($3, 4, 8j$) of type $8^j$ for $j = 4, 5$. Then we obtain a 2-chromatic CQS($8^k : 2$). Since a 2-chromatic SQS($10$) is also a 2-chromatic HQS($10 : 2$), by applying Lemma 2.2 we obtain a 2-chromatic SQS($8k + 2$). This completes the proof. □
This paper presents a construction for 2-chromatic SQS, which is used to study the difficult class of orders \( v \equiv 2 \) or \( 10 \) (mod 12). Three eighths of this problem is settled. The orders \( v \equiv 14 \) or \( 22 \) (mod 24), and \( v \equiv 50 \) or \( 82 \) (mod 96), and \( v = 98 \) are what remains be done to settle the existence question for 2-chromatic SQS completely.

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