Factor-complement partitions of ascending $k$-parameter words

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Abstract

In this paper variations of the $\star$-version of the Hales--Jewett theorem by Voigt for arbitrary ascending parameter words are used to prove versions of the Hales--Jewett and Graham--Rothschild theorems where every factor of every word is coloured. To facilitate the proof three-alphabet ascending parameter words are introduced. Primitive recursive bounds are given for all new results.

0. Introduction

The idea of parameter words and sets originated with Hales and Jewett in 1963 [3], and was formalized and extended by Graham and Rothschild in 1971 [2].

The fundamental result on the partitioning of 0-parameter words by Hales and Jewett [3] was proved in primitive recursive arithmetic by Shelah in [5]. Recently, Fouché gave a primitive recursive proof of a version of the Graham--Rothschild theorem [1].

Based on these results, we prove analogues and variations of the $\star$-version of the Hales--Jewett theorem [6] for arbitrary ascending parameter words. One can think of $\star$-words of length $n$ as equivalence classes of words of length $n$ determined by all the different left factors of the words. (Also see the remark following Theorem 1.2 in Section 1.) We generalize this to the case where the equivalence classes are determined by all the factors of the words.

The main results of this paper are generalizations of the Hales--Jewett and Graham--Rothschild theorems to the case of three-alphabet ascending parameter words. A
colouring of these words can be interpreted as a colouring of every factor of every word. In order to do this, three-alphabet ascending parameter words are introduced. One can show that in these cases a monochromatic parameter word cannot be found. However, one can find a parameter word such that the colour of a word in the corresponding parameter set depends only on the transition from one alphabet to the other, a result which is the best that can be expected.

Primitive recursive bounds are given for all the new results.

1. A Hales–Jewett theorem for ascending parameter words

We recall that if \( A \) is a finite set, then a \( k \)-parameter word \( f \) of length \( n \) over \( A \) is a word of length \( n \) over \( A \cup \{\lambda_0, \ldots, \lambda_{k-1}\} \) where \( A \cap \{\lambda_0, \ldots, \lambda_{k-1}\} = \emptyset \) and where every parameter \( \lambda_i \) occurs at least once in \( f \). To avoid ambiguity, it is required that the first occurrence of \( \lambda_i \) is before the first occurrence of \( \lambda_{i+1} \), for \( 0 \leq i < k - 1 \). If we also require the last occurrence of \( \lambda_i \) to be before the first occurrence of \( \lambda_{i+1} \) for \( 0 \leq i < k - 1 \), \( f \) is called an ascending \( k \)-parameter word. We denote the set of all \( k \)-parameter words (respectively ascending \( k \)-parameter words) of length \( n \) over \( A \) by \([A](n)\) (respectively \([A]_{\text{<}}(n)\)). If \( f \) is an \( m \)-parameter word of length \( n \) and \( g \) a \( k \)-parameter word of length \( m \), the composition \( f \circ g \) is a \( k \)-parameter word of length \( n \) constructed from \( f \) and \( g \) by replacing the \( i \)th parameter in \( f \) with the \( i \)th letter of \( g \). Formally, if \( g \in [A](n) \) and \( f \in [A](m) \) then \( f \circ g \) is defined as

\[
(f \circ g)(i) = \begin{cases} 
    f(i) & \text{if } f(i) \in A, \\
    g(j) & \text{if } f(i) = \lambda_j.
\end{cases}
\]

It is clear that \( f \circ g \) is ascending if \( f \) and \( g \) are. A function \( \chi: [A]_{\text{<}}(n) \rightarrow [r] \) (where \( [r] := \{1, 2, \ldots, r\} \)) is called an \( r \)-colouring of \([A]_{\text{<}}(n)\). A word \( f \in [A]_{\text{<}}(n) \) is called monochromatic if \( \chi(f \circ g) = \chi(f \circ h) \) for all \( g, h \in [A]_{\text{<}}(n) \).

The Shelah proof of the Hales–Jewett theorem for one-parameter words as discussed in [4] uses induction on the size of the alphabet. The result is then generalized to an arbitrary number of parameters by a natural correspondence between \([A^m](n)\) and \([A](m^n)\). However, one can modify the proof by fixing the number \( m \) of parameters for any natural number \( m \) and still induct on the cardinality of the alphabet. In this case the constructed monochromatic word is an ascending parameter word. Hence, the Shelah proof is also a proof of the following version of the Hales–Jewett theorem for ascending parameter words.

**Theorem 1.1.** Given an alphabet \( A \) and natural numbers \( m \) and \( r \), there exists a primitive recursive function \( HJ_{\leq}(|A|, m, r) \) such that for every \( r \)-colouring \( \chi \) of \([A]_{\text{<}}(N)\), where \( N = HJ_{\leq}(|A|, m, r) \), there exists a monochromatic \( m \)-parameter word \( f \in [A]_{\text{<}}(N) \).
A $*$-version of the Hales–Jewett theorem was formulated and proved by Voigt in 1980 [6].

For $\ast$ any symbol not contained in $A \cup \{ \lambda_0, \ldots, \lambda_{m-1} \}$ we define $[A]^* \binom{n}{m}$ to be the set of all ascending $m$-parameter words $f$ of length $n$ over $A \cup \{ \ast \}$ such that if $f(i) = \ast$ for some $i < n$ then $f(j) = \ast$ for all $i \leq j < n$. For $f \in [A]^* \binom{n}{m}$ and $g \in [A]^* \binom{m}{k}$ the composition $f \circ g \in [A]^* \binom{m}{k}$ is defined by

$$(f \circ g)(i) = \begin{cases} \ast & \text{if there exists a } j < i \text{ such that } (f \circ g)(j) = \ast, \\ f(i) & \text{if } f(i) \in A \cup \{ \ast \} \text{ and } (f \circ g)(j) \neq \ast \text{ for all } j < i, \\ g(j) & \text{if } f(i) = \lambda_j. \end{cases}$$

With the obvious adjustments the proof of the $\ast$-version of the Hales–Jewett theorem (see e.g. [4, Theorem 3.3]) is also a proof of the following version for ascending parameter words.

**Theorem 1.2.** Given an alphabet $A$ and natural numbers $m$ and $r$, there exists a primitive recursive function $HJ^*_r(|A|, m, r)$ such that for every $r$-colouring $\chi$ of $[A]^* \binom{n}{0}$, where $N = HJ^*_r(|A|, m, r)$, there exists a monochromatic $f \in [A]^* \binom{n}{N}$.

A $\ast$-word $f_0 f_1 \cdots f_{n-1} \ast \cdots \ast \in [A]^* \binom{n}{0}$ can be interpreted as the equivalence class of all words in $A^n$ with left factor $f_0 \cdots f_{n-1}$. In the $\ast$-version of the Hales–Jewett theorem all such equivalence classes are coloured.

We generalize this idea to the case where an equivalence class is determined by a factor and its position in the word. To this end we introduce $\dagger \ast$-words. A word $\dagger \cdots \dagger a_0 \cdots a_{n-1} \ast \cdots \ast$, $a_i \in A$, for example, represents the equivalence class of all words $f = f_0 \cdots f_{n-1}$ in $A^n$ such that the factor $f_{n_0} \cdots f_{n_{i-1}+n_0}$ is the word $a_0 \cdots a_{n_1-1}$.

Formally, let $\dagger$ and $\ast$ be two symbols not in $A \cup \{ \lambda_0, \ldots, \lambda_{m-1} \}$ and define $\dagger [A]^* \binom{n}{m}$ to be the set of all $m$-parameter words of length $n$ over $A \cup \{ \dagger, \ast \}$ such that

- if $f(i) = \ast$ for some $i < n$ then $f(j) = \ast$ for all $i \leq j < n$ and
- if $f(i) = \dagger$ for some $i < n$ then $f(j) = \dagger$ for all $0 \leq j < i$.

It is natural to regard substitution with respect to $\dagger$ as the dual of $\ast$-substitution, which requires rigidity from both sides, i.e. the first (dually last) occurrence of $\lambda_i$ is before that of $\lambda_{i+1}$, $0 \leq i < m - 1$. However, there is ambiguity when substituting words consisting of $\dagger$'s and $\ast$'s only into parameter-words with rigidity from both sides. This necessitates the use of ascending parameter words when dealing with $\dagger \ast$-words. We shall denote the set of ascending $\dagger \ast$ $m$-parameter words of length $n$ over $A$ by $\dagger [A]^* \binom{n}{m}$.
Substitution (composition) is defined as follows. If \( f \in [A]_{\leq}^n \) and \( g \in [A]_{\leq}^m \) the word \( f \circ g \in [A]_{\leq}^n \) is defined by

\[
(f \circ g)(i) = \begin{cases} 
\dagger & \text{if } f(i) = \dagger \text{ or if there exists a } j > i \text{ such that } (f \circ g)(j) = \dagger, \\
\star & \text{if } f(i) = \star \text{ or if there exists a } j < i \text{ such that } (f \circ g)(j) = \star, \\
f(i) & \text{if } f(i) \in A \text{ and } (f \circ g)(j) \neq \star \text{ for all } j < i \text{ and } (f \circ g)(j) \neq \dagger \text{ for all } j > i, \\
g(j) & \text{if } f(i) = \lambda_j.
\end{cases}
\]

We are now in a position to state a \( \dagger \star \)-version of the Hales–Jewett theorem for ascending parameter words. The proof is again similar to the proof of the \( \star \)-version of the Hales–Jewett theorem (see e.g. [4, Theorem 3.3]).

**Theorem 1.3.** Given an alphabet \( A \) and natural numbers \( m \) and \( r \), there exists a primitive recursive function \( \dagger HJ_{\leq}(|A|m,r) \) such that for every \( r \)-colouring \( \chi \) of \( [A]_{\leq}^n \), where \( N = \dagger HJ_{\leq}(|A|m,r) \), there exists a monochromatic \( f \in [A]_{\leq}^n \).

### 2. A Hales–Jewett theorem for three-alphabet ascending parameter words

In the \( \dagger \star \)-version (Theorem 1.3) words with identical factors are identified. In this section we keep a trace of the remaining left and right factors by introducing ‘mirror-images’ of these factors. This is done by introducing disjoint copies \( \overline{A} \) and \( \overline{A} \) of the alphabet \( A \). We set out by considering the partitioning of the left factors of all words of length \( n \) over an alphabet \( A \). To be precise, let \( \overline{A} \) be a disjoint copy of \( A \) and for every \( a \in A \) denote the corresponding letter in \( \overline{A} \) by \( \overline{a} \). The set of parameter symbols is disjoint from both \( A \) and \( \overline{A} \). Define \( [A]_{\leq}^n (m) \) to be the set of all ascending \( m \)-parameter words \( f \) of length \( n \) over \( A \cup \overline{A} \) such that if \( f(i) = \overline{a} \) for some \( i < n \) and \( a \in A \), then for every \( i < j < n \), \( f(j) \in A \).

Substitution (composition) is defined as follows. If \( f \in [A]_{\leq}^n (m) \) and \( g \in [A]_{\leq}^m (k) \) then \( f \circ g \in [A]_{\leq}^n (m) \) is defined by

\[
(f \circ g)(i) = \begin{cases} 
\overline{a} & \text{if } f(i) = a \text{ and there exists a } j < i \text{ such that } \overline{(f \circ g)(j)} \in A \text{ or if } f(i) = \overline{a}, \\
f(i) & \text{if } f(i) \in A \text{ and } \overline{(f \circ g)(j)} \notin \overline{A} \text{ for all } j < i, \\
g(j) & \text{if } f(i) = \lambda_j.
\end{cases}
\]

Note that \( [A]_{\leq}^n (m) \subset [A]_{\leq}^n (m) \) and that \( [\overline{A}]_{\leq}^n (0) \subset [A]_{\leq}^n (0) \).

**Example.** The word 0110100111 is an example of a word in \( \{(0,1)\}_{\leq}^n (10) \) and the words \( 0\lambda_01\lambda_10\lambda_10 \) and \( 0\overline{\lambda}_00\lambda_001\lambda_1\lambda_10 \) are 2-parameter words in \( \{(0,1)\}_{\leq}^n (10) \).
Note that monochromacity cannot be attained here as is demonstrated by the following 2-colouring of $[\{0,1\}]^{\infty}_{\leq N}$: assign the colour 1 to all words where the first occurrence of a letter of the second alphabet is 0 and 2 otherwise.

We state the following $\Rightarrow$-version of the Hales-Jewett theorem for ascending words. The proof is similar to that of Theorem 3.3.

**Theorem 2.1.** Given an alphabet $A$ and natural numbers $m$ and $r$, there exists a primitive recursive function $HJ^r_{\leq}(|A|, m, r)$ such that for every $r$-colouring $\chi$ of $[A]^r_{\leq N}$, where $N = HJ^r_{\leq}(|A|, m, r)$, there exists an $m$-parameter word $f \in [A]^r_{\leq \binom{m}{r}}$ with the following property: If $g, h \in [A]^r_{\leq \binom{m}{r}}$ and the first letter from $A$ in $g$ is the same as the first letter from $A$ in $h$, then $\chi(f \circ g) = \chi(f \circ h)$.

Let us call the phenomenon in Theorem 2.1 where the colour only depends on the first occurrence of a letter from the second alphabet (i.e. where the transition takes place) transition-chromatic. As was noted before the statement of Theorem 2.1, monochromacity cannot be attained. Thus, transition-chromacity is the best that can be expected.

For $|A| = 1$ the $\Rightarrow$-version of the Hales-Jewett theorem for ascending parameter words is a restatement of the $\star$-version of the Hales-Jewett theorem for ascending parameter words over a one-alphabet (see Theorem 1.1).

It follows immediately that the dual of the $\Rightarrow$-version of the Hales-Jewett theorem holds. The following definitions are dual restatements of those for the $\Rightarrow$-version. Again we make use of a disjoint copy $\overline{A}$ of the alphabet $A$. For every $a \in A$, denote the corresponding letter in $\overline{A}$ by $\overline{a}$. Both $A$ and $\overline{A}$ are disjoint from the set of parameter symbols. Define $\overline{\star}[A]_{\leq \binom{n}{m}}$ to be the set of all ascending $m$-parameter words $f$ of length $n$ over $A \cup \overline{A}$ such that if $f(i) = a$ for some $i < n$ and $a \in A$, then for every $0 \leq j < i$, $f(j) \in A$.

Substitution (composition) is defined as follows. If $f \in \overline{\star}[A]_{\leq \binom{n}{m}}$ and $g \in \overline{\star}[A]_{\leq \binom{m}{k}}$ then $f \circ g \in \overline{\star}[A]_{\leq \binom{k}{r}}$ is defined by

\[
(f \circ g)(i) = \begin{cases} 
\overline{a} & \text{if } f(i) = a \in A \text{ and there exists a } j > i \text{ such that } (f \circ g)(j) \in A \text{ or } f(i) = a, \\
\overline{f}(i) & \text{if } f(i) \in A \text{ and } (f \circ g)(j) \notin A \text{ for all } j > i, \\
g(j) & \text{if } f(i) = \lambda_j.
\end{cases}
\]

**Example.** The words $100111010$ and $0101010101$ are examples of words in $\overline{\star}[\{0,1\}]_{\leq \binom{10}{8}}$ and the words $010010101010101 \lambda_1 \lambda_1 0 \lambda_1 1$ and $101010101010101 \lambda_1 \lambda_1 0 \lambda_1 1$ are examples of 2-parameter words in $\overline{\star}[\{0,1\}]_{\leq \binom{10}{2}}$.

The following is a so-called $\Rightarrow$-version of the Hales-Jewett theorem for ascending words.

**Theorem 2.2.** Given an alphabet $A$ and natural numbers $m$ and $r$, there exists a primitive recursive function $\overline{HJ}^r_{\leq}(|A|, m, r)$ such that for every $r$-colouring $\chi$ of $\overline{\star}[A]_{\leq \binom{N}{0}}$, where $N = \overline{HJ}^r_{\leq}(|A|, m, r)$, there exists an $m$-parameter word $f \in \overline{\star}[A]_{\leq \binom{m}{r}}$ with the following property: If $g, h \in \overline{\star}[A]_{\leq \binom{m}{r}}$ and the first letter from $A$ in $g$ is the same as the first letter from $A$ in $h$, then $\chi(f \circ g) = \chi(f \circ h)$.
where \( N = \text{HJ}_x(\{A\}, m, r) \), there exists an \( m \)-parameter word \( f \in \text{HJ}_x(\{A\}_{\leq m}^N) \) with the following property: If \( g, h \in \text{HJ}_x(\{A\}_{\leq 0}^n) \) and the last letter from \( \Delta \) in \( g \) is the same as the last letter from \( \Delta \) in \( h \), then \( \chi(f \circ g) = \chi(f \circ h) \).

For the proof of the main result we need a \( \star \)-version of the \( \vdash \)-version of Theorem 2.2.

Let \( \star \) be a symbol not in \( A \cup \{\lambda_0, \ldots, \lambda_{m-1}\} \) and define \( \text{HJ}_x(\{A\}_{\leq m}^n) \) to be the set of all \( m \)-parameter words of length \( n \) over \( A \cup \{\star\} \) such that

- if \( f(i) = \star \) for some \( i < n \) then \( f(j) = \star \) for all \( i \leq j < n \), and
- if \( f(i) = a \) for some \( i < n \) and \( a \in A \) then for every \( 0 \leq j < i \), \( f(j) \in \Delta \).

Substitution (composition) is defined as follows. If \( f \in \text{HJ}_x(\{A\}_{\leq m}^n) \) and \( g \in \text{HJ}_x(\{A\}_{\leq k}^n) \) then \( f \circ g \in \text{HJ}_x(\{A\}_{\leq k}^n) \) is defined by

\[
(f \circ g)(i) =
\begin{cases}
  a & \text{if } f(i) = a \text{ or } f(i) = a \text{ and there exists a } j > i \text{ such that } (f \circ g)(j) \in \Delta, \\
  \star & \text{if } f(i) = \star \text{ or } f(i) = \star \text{ and there exists a } j < i \text{ such that } (f \circ g)(j) = \star, \\
  f(i) & \text{if } f(i) \in A \text{ and } (f \circ g)(j) \notin A \text{ for all } j > i \text{ and } (f \circ g)(j) \neq \star \text{ for all } j < i, \\
  g(j) & \text{if } f(i) = \lambda_j.
\end{cases}
\]

**Example.** Let \( f = 011\lambda_0\lambda_1\lambda_2\star\star\star \in \text{HJ}_x(\{0,1\}_{\leq 1}^1) \) and \( g = \emptyset\mu\star \in \text{HJ}_x(\{0,1\}_{\leq 3}^3) \). Then \( f \circ g = 01101\mu\star\star\star \in \text{HJ}_x(\{0,1\}_{\leq 1}^1) \).

We give the following \( \star \)-version of Theorem 2.2. The proof is essentially the same as for the \( \vdash \)-version of the Hales–Jewett theorem [4, Theorem 3.3].

**Theorem 2.3.** Given an alphabet \( A \) and natural numbers \( m \) and \( r \), there exists a primitive recursive function \( \text{HJ}_x(\{A\}_{\leq m}^N) \) such that for every \( r \)-colouring \( \chi \) of \( \text{HJ}_x(\{A\}_{\leq m}^N) \), where \( N = \text{HJ}_x(\{A\}_{\leq m}^N) \), there exists a transition-chromatic \( f \in \text{HJ}_x(\{A\}_{\leq m}^N) \).

We now address the partitioning of the factors of words in their context. In Theorem 1.3 we disregarded the complement of the factors. Here factors are taken to be different even if they only differ in their complements. We make use of three mutually disjoint copies \( A, \overline{A} \) and \( A \) of the alphabet.

Define \( \text{HJ}_x(\{A\}_{\leq m}^N) \) to be the set of all ascending \( m \)-parameter words \( f \) of length \( n \) over \( A \cup \overline{A} \) such that if \( f(i) \in \overline{A} \) for some \( i < n \) then for every \( i \leq j < n \), \( f(j) \in \overline{A} \), and if \( f(i) \in A \) for some \( i < n \) then for every \( 0 \leq j \leq i \), \( f(j) \in A \).

Substitution is defined as follows. If \( f \in \text{HJ}_x(\{A\}_{\leq m}^N) \) and \( g \in \text{HJ}_x(\{A\}_{\leq k}^N) \) then \( f \circ g \in \text{HJ}_x(\{A\}_{\leq k}^N) \) is defined by
In the following \( \Leftrightarrow \)-version of the Hales-Jewett theorem for ascending parameter words the colour will eventually depend only on the last occurrence of a letter from the first alphabet and the first of the last alphabet. Again we use the term transition-chromatic.

**Theorem 2.4.** Given an alphabet \( A \) and natural numbers \( m \) and \( r \), there exists a primitive recursive function \( \text{HJ}_{\leq}^+(|A|, m, r) \) such that for every \( r \)-colouring \( \chi \) of \( [A]_{\leq}^+(N) \), where \( N = \text{HJ}_{\leq}^+(|A|, m, r) \), there exists a word \( f \in [A]_{\leq}^+(N) \) which is transition-chromatic with respect to \( \chi \).

### 3. A Graham–Rothschild theorem for three-alphabet ascending parameter words

We now turn to the partitioning of all the \( k \)-parameter factors of all \( k \)-parameter words (where the complement of the factor does not contain any parameter). We show that results similar to that in Section 2 hold. We need a version of the Graham–Rothschild theorem for ascending parameter words recently proved by Fouché.

**Theorem 3.1** (Fouché [1], Theorem A). There exists a primitive recursive function \( \text{GR}_{\leq}(t, k, m, r) \) such that for \( k, m, r \in \mathbb{N} \) and for an alphabet \( A \) with \( |A| = t \), we have for every \( r \)-colouring

\[
\chi: [A]_{\leq}(M/k) \to \{1, \ldots, r\}
\]

of ascending \( k \)-parameter words of length \( M = \text{GR}_{\leq}(t, k, m, r) \) over \( A \), that there is some \( f \in [A]_{\leq}(M/k) \) which is monochromatic with respect to \( \chi \).

As in the case of the Hales–Jewett theorem in Section 2 we add a \( * \)-version.

**Lemma 3.2.** For an alphabet \( A \) and natural numbers \( k, m \) and \( r \), there exists a primitive recursive function \( \text{GR}^*(|A|, k, m, r) \) such that for every \( r \)-colouring \( \chi \) of \( [A]_{\leq}^*(N) \), with \( N = \text{GR}^*(|A|, k, m, r) \), there exists a monochromatic word \( f \in [A]_{\leq}^*(N) \).
We are now ready to prove a $k$-parameter version of Theorem 2.1.

**Theorem 3.3.** Given an alphabet $A$ and natural numbers $k, m$ and $r$, there exists a primitive recursive function $GR_x^r(|A|, k, m, r)$ such that for every $r$-colouring $\chi$ of $[A]_k^r$ with $N = GR_x^r(|A|, k, m, r)$, there exists an $m$-parameter word $f \in [A]_k^r$ with the following property: If $g, h \in [A]_k^r$ and the first letter from $A$ in $g$ is the same as the first letter from $A$ in $h$, then $\chi(f \circ g) = \chi(f \circ h)$.

**Proof.** Associate with every $f \in [A]_k^r$ the left factor $f^{-1}$ given by $f^{-1}(i) = f(i)$ if $i < \min_{x \in A} f^{-1}(x)$ and $f^{-1}$ given by $f^{-1}(i) = f(i)$ if $i \leq \min_{x \in A} f^{-1}(x)$. If $f$ does not contain any letter from $A$, let $f^{-1} := f$ and $f^{-1} := f$.

Let $x = GR_x^r(|A|, k, m, r)$ and $x = x$. For $0 \leq j < x$, let

$$n_j = GR_x^r(|A|, n_{j+1} - j - k - 1, r^{|A|}(|A|) + j + k + 1) + j + k + 1.$$

Choose $N = n_0$ and let $\chi : [A]_k^r \to [r]$ be any $r$-colouring. We now prove that there exists a word $f_x \in [A]_k^r$ such that if $g, h \in [A]_k^r$ and $g^{-1} = h^{-1}$ then $\chi(f_x \circ g) = \chi(f_x \circ h)$. The proof is by induction.

Let $\chi_0$ be the restriction of $\chi$ to the set $\{g \in [A]_k^r : |g^{-1}| = k\}$. Fix an order on $A$ and consider the product colouring

$$\chi'_0 : [A]_k^r \to \left[\frac{n_0 - k - 1}{0}\right]$$

defined by

$$\chi'_0(g) = \prod_{a \in A} \chi_0(\mu_0 \cdots \mu_{k-1} \bar{a} g_0 \cdots g_{n_0 - k - 2})$$

for every $g = g_0 \cdots g_{n_0 - k - 2} \in [A]_k^r(\frac{n_0 - k - 1}{0})$.

By the choice of $n_0$ there exists a $\chi'_0$-monochromatic word $f''_i \in [A]_k^r(\frac{n_0 - k - 1}{0})$. If we rename the $n_1 - k - 1$ parameters in $f''_i$ to $\mu_{k+1}, \mu_{k+2}, \ldots, \mu_{n_1 - 1}$, the word

$$f_i := \mu_0 \cdots \mu_k f''_i(0) \cdots f''_i(n_0 - k - 2)$$

in $[A]_k^r(\frac{n_1}{0})$ has the property that if $g, h \in [A]_k^r(\frac{n_1}{0})$ with $|g^{-1}| = |h^{-1}| = k$ and $g^{-1} = h^{-1}$ then $\chi(f_i \circ g) = \chi(f_i \circ h)$. Indeed, let $g, h \in [A]_k^r(\frac{n_1}{0})$ with

$$g = \mu_0 \cdots \mu_{k-1} \bar{a}_0 g_{k+1} \cdots g_{n_1 - 1}$$

and

$$h = \mu_0 \cdots \mu_{k-1} \bar{a}_0 h_{k+1} \cdots h_{n_1 - 1}$$

for some $a_0 \in A$. Since $f''_i$ is $\chi'_0$-monochromatic, we have that

$$\chi'_0(f''_i \circ g_{k+1} \cdots g_{n_1 - 1}) = \chi'_0(f''_i \circ h_{k+1} \cdots h_{n_1 - 1}),$$
i.e.  
\[
\prod_{a \in A} \chi_0(f_i \circ \mu_0 \cdots \mu_{k-1} \bar{a}g_{k+1} \cdots g_{n-1}) = \prod_{a \in A} \chi_0(f_i \circ \mu_0 \cdots \mu_{k-1} ah_{k+1} \cdots h_{n-1}).
\]

In particular, it means that  
\[
\chi_0(f_i \circ g) = \chi_0(f_i \circ h),
\]

i.e.  
\[
\chi(f_i \circ g) = \chi(f_i \circ h).
\]

Suppose for \(0 < j < x\) that \(f_j \in [A]^e_\leq (\binom{n}{k})\) is such that for \(g, h \in [A]^e_\leq (\binom{n}{k})\) if \(|g| < k + j\) and \(h = h^{-}\) then \(\chi(f_j \circ g) = \chi(f_j \circ h)\).

Define an induced \(r\)-colouring \(\chi_j\) on the set \(\{g \in [A]^e_\leq (\binom{n}{k}) : |g| = k + j\}\) as follows:

\[
\chi_j(g) = \chi(f_j \circ g)
\]

for every \(g \in [A]^e_\leq (\binom{n}{k})\).

Consider the colouring

\[
\chi'_j: [A]^e_\leq (\binom{n}{k}) \rightarrow [r^{|A^e_\leq (\binom{n}{k})| + |A|}]
\]

defined by

\[
\chi'_j(g) = \prod_{w \in [A]^e_\leq (\binom{j}{k})} \chi_j(w_0 \cdots w_{j-1} \bar{a}g_0 \cdots g_{n-k-j-2}),
\]

where a fixed order is chosen for the words in \([A]^e_\leq (\binom{j}{k})\) and the letters of \(A\).

By the choice of \(n_j\) there exists a \(\chi'_j\)-monochromatic \(n_{j+1} - k - j - 1\)-parameter word

\[
f''_{j+1} \in [A]^e_\leq (\binom{n_{j+1} - k - j - 1}{n_{j+1} - k - j - 1}).
\]

After renaming the \(n_{j+1} - k - j - 1\) parameters in \(f''_{j+1}\) to \(\mu_{k+j+1}, \ldots, \mu_{n_{j+1}-1}\) the word

\[
f'_{j+1} = \mu_0 \mu_1 \cdots \mu_{k+j} f''_{j+1}(0) \cdots f''_{j+1}(n_j - k - j - 2)
\]

is an \(n_{j+1}\)-parameter word in \([A]^e_\leq (\binom{n_{j+1}}{n_{j+1}})\) satisfying the following properties with respect to words \(g, h \in [A]^e_\leq (\binom{n_{j+1}}{k})\):

1. if \(g^{-} = h^{-}\) and \(|g| \leq k + j\) then \((f'_{j+1} \circ g)^{-} = (f'_{j+1} \circ h)^{-}\) and
2. if \(g^{-} = h^{-}\) and \(|g| = k + j\) then \(\chi_j(f'_{j+1} \circ g) = \chi_j(f'_{j+1} \circ h)\).

If we let \(f_{j+1}\) be the word \(f_j \circ f'_{j+1}\) then \(f_{j+1}\) is an \(n_{j+1}\)-parameter word in \([A]^e_\leq (\binom{n_{j+1}}{n_{j+1}})\)

such that if \(g, h \in [A]^e_\leq (\binom{n_{j+1}}{k})\) and \(g^{-} = h^{-}\) and \(|g| \leq k + j\) then

\[
\chi(f_{j+1} \circ g) = \chi(f_{j+1} \circ h).
\]
It follows by induction that a word \( f_x \in [A]_{\leq}^{n\choose k} \) exists such that for every \( g, h \in [A]_{\leq}^{n\choose k} \) with \( g^{-1} = h^{-1} \), we have that \( \chi(f_x \circ g) = \chi(f_x \circ h) \). (Note that for \( g, h \in [A]_{\leq}^{n\choose k} \) we have that if \( |g| = x \) then \( g^{-1} = h^{-1} \) if and only if \( g = h \).)

Consider the \( r^{[A]} \)-colouring \( \chi^* \) of \([A]_{\leq}^{n\choose k} \) defined by

\[
\chi^*(g) = \begin{cases} 
\prod_{a \in A} \chi(f_x \circ g_0 \cdots g_{i-1} \overline{a}a_1 \cdots a) & \text{if } g \in [A]_{\leq}^{n\choose k} \text{ and } \min g^{-1} = i, \\
\prod_{a \in A} \chi(f_x \circ g_0 \cdots g_{k-1}) & \text{if } g \text{ contains no } \ast.
\end{cases}
\]

By the choice of \( x = GR^*_\leq([A], k, m, r^{[A]}) \) there exists a \( \chi^* \)-monochromatic \( m \)-parameter word \( f^* \in [A]_{\leq}^{n\choose k} \).

Let \( a \) be any letter in \( A \) and define \( f^* \) to be the word

\[
f^*(i) := \begin{cases} 
f^*(i) & \text{if } f^*(i) \neq \ast, \\
\overline{a} & \text{otherwise}.
\end{cases}
\]

Then \( f = f_x \circ f^* \in [A]_{\leq}^{n\choose k} \) is such that if \( g, h \in [A]_{\leq}^{n\choose k} \) and \( g(|g|) = h(|h|) \) then \( \chi(f \circ g) = \chi(f \circ h) \).

Using the same strategy as in Section 2 one can derive a \( \ast \)-version and a \( \ast \)-version of it to finally prove the main result.

**Theorem 3.4.** Given an alphabet \( A \) and natural numbers \( k, m \) and \( r \), there exists a primitive recursive function \( GR^*_\leq([A], k, m, r) \) such that for every \( r \)-colouring \( \chi \) of \( [A]_{\leq}^{n\choose k} \), with \( N = GR^*_\leq([A], k, m, r) \), there exists a word \( f \in ^n[A]_{\leq}^{n\choose k} \) which is transition-chromatic with respect to \( \chi \).

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**References**