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Qualitative Study of a Class of Nonlinear Boundary Value Problems at Resonance

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1.

In this paper we continue our earlier studies (cf., e.g., [1–4]) and consider the nonlinear boundary value problem

$$Eu = Nu \tag{1}$$

where N is a nonlinear operator over a real Hilbert space S and E is a linear differential operator over a bounded domain G of R^ν with homogeneous boundary conditions and possessing a finite-dimensional kernel.

The particular case when $Nu = g(u) + f$ has been the subject of much study in recent years and sufficient conditions in terms of $g(\infty)$, $g(-\infty)$ and f, patterned after the results of [8], have been discussed by several authors. In these studies a key hypothesis is that

$$g(-\infty) < g(\infty). \tag{2}$$

We have established in [1] that these results may be treated as particular cases of a general abstract theorem [3] and have extended these ideas to the case when E has an infinite-dimensional kernel [4].

We study in this paper the case when (2) is not true; more specifically we assume that

$$g(-\infty) = g(\infty). \tag{3}$$

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Existence results for (1) when (3) is assumed have been obtained in [5–7]. In this paper we show that these results may be once again derived as particular cases of the abstract theorem. We obtain in the process existence results for problem (1) when (3) is true which are not covered by the earlier work. In particular, we do not restrict f to be orthogonal to $\ker E$ as, for instance, is done in [5].

2.

We first recall an abstract existence theorem, and present a variant of it, which relates to the case of hypothesis (2). Thus, let $S = L_2(G)$ be the direct sum of orthogonal subspaces S_0 and S_1, and let $P: S \to S$ be the projection operator with nullspace S_1 and $PS = S_0$, $(I-P)S = S_1$, $S_0 = \ker E$, $S_1 = \operatorname{range} E$ and $E: \mathfrak{D}(E) \subset S \to S$. We assume that $S_0 = \ker E$ is nontrivial so that (1) is a problem at resonance. Since E restricted to $S_1 \cap \mathfrak{D}(E)$ is one-to-one and onto S_1, its partial inverse $H: S_1 \to S_1 \cap \mathfrak{D}(E)$ is a single-valued linear operator. We assume that the following natural hypotheses hold: (h_1) $H(I-P)\,Eu = (I-P)u$, (h_2) $EPu = PEu$ and (h_3) $EH(I-P)\,Nu = (I-P)\,Nu$. Then (1) is equivalent (cf. [1]) to the system of equations

$$u = Pu + H(I-P)\,Nu, \qquad PNu = 0.$$

Since every $u \in S$ has a unique decomposition $u = u^* + u_1$, $u^* \in S_0$, $u_1 \in S_1$, $u^* = Pu$, $u_1 = (I-P)u$, then the above system can also be written in the form

$$u_1 = H(I-P)\,N(u^* + u_1), \tag{4}$$

$$0 = PN(u^* + u_1), \tag{5}$$

or, equivalently, as a single operator equation

$$u_1 = H(I-P)\,N(u^* + u_1) + PN(u^* + u_1), \qquad u = u^* + u_1. \tag{6}$$

Let $(\ ,\ )$ and $\|\cdot\|$ denote the inner product and norm in S. We shall denote by L the norm of $H(I-P)$ in S.

Let r, R_0 be positive numbers and let Ω denote the set

$$\Omega = \Omega_0 \times \Omega_1, \quad \Omega_0 = \{u^* \in S_0, \quad \|u^*\| \leqslant R_0\}, \quad \Omega_1 = \{u_1 \in S_1, \quad \|u_1\| \leqslant r\}.$$

Thus, by using the Leray–Schauder principle, we can state: if there are numbers r, R such that the equation

$$u_1 = \gamma H(I-P)\,Nu + \gamma PNu = 0, \qquad u = u^* + u_1, \tag{7}$$

has no solution on $\partial\Omega$ for any $0<\gamma<1$, then (1) has at least one solution $u=u^*+u_1$ in Ω.

A simple set of sufficient conditions to satisfy the above is the following: let there exist numbers r and R such that

$$\|H(I-P)\,Nu\| \leqslant r \qquad \text{for} \quad \|u^*\| \leqslant R_0,\ \|u_1\|=r, \tag{8}$$

$$(Nu, u^*) \leqslant 0 \qquad (\text{or} \geqslant 0) \qquad \text{for} \quad \|u^*\|=R_0,\ \|u_1\| \leqslant r, \tag{9}$$

where $u=u^*+u_1$, $u^*\in S_0$, $u_1\in S_1$.

Then (1) has at least one solution in Ω.

For the equation

$$Eu+g(x,u(x))=f(x), \qquad x\in G, \tag{10}$$

where $f\in L_2(G)$ and $g(x,s)$ is a continuous real-valued function on $\bar{G}\times R$, any solution u_γ of (7) must satisfy

$$u_{1\gamma}=\gamma H(I-P)\,Nu_\gamma, \qquad PNu_\gamma=0.$$

Writing $g=g_0+g_1$, $g_0\in S_0$, $g_1\in S_1$ we have from

$$u_{1\gamma}=\gamma H(I-P)\,Nu_\gamma$$

that

$$\begin{aligned} u_{1\gamma} &= \gamma H(I-P)[-g(u)+f] \\ &= \gamma H(I-P)[-g_0-g_1+f_0+f_1], \end{aligned}$$

and

$$\|u_{1\gamma}\| < L(\|g_1\|+\|f_1\|). \tag{11}$$

Also, if $u_1=H(I-P)\,Nu$ we have

$$Eu_1=(I-P)\,Nu,$$

or

$$(Eu_1,u_1)=((I-P)\,Nu,u_1)=(Nu,u_1).$$

Let λ_1 be such that $(Eu_1,u_1)\geqslant\lambda_1\|u_1\|^2$, $u_1\in S_1$. Then

$$\lambda_1\|u_1\|^2 \leqslant (-g(u),u_1)+(f,u_1).$$

Then

$$\begin{aligned} (Nu,u^*) &= (-g(u)+f,u^*) = (-g(u)+f,u)-(-g(u)+f,u_1) \\ &\leqslant (-g(u),u)+(f,u)-\lambda_1\|u_1\|^2. \end{aligned}$$

Thus, if we assume that

$$(-g(u), u) + (f, u) \leqslant \lambda_1 \|u_1\|^2, \tag{12}$$

then $(Nu, u^*) \leqslant 0$ for every $u^* \in S_0$, $\|u^*\| = R_0$ and corresponding $u_1 \in S_1$, $\|u_1\| \leqslant r$ and $u_1 = H(I-P)\,Nu$ with $u = u^* + u_1$.

As a corollary of the above considerations we have the following situation from [1]: let $E: \mathfrak{D}(E) \subset S \to S_1$ be a nonnegative self-adjoint operator in the real Hilbert space $L_2(G)$ where G is an open bounded set in $\mathbf{R}^\nu$. Suppose that $\dim(\ker E) = 1$, and let $\theta \neq 0$ belong to the kernel of E. Let $V = \mathfrak{D}(E^{1/2})$ with the graph norm and let $A: V \times V \to R$ be the associated quadratic form

$$A(u, v) = (E^{1/2}u, E^{1/2}v), \qquad u, v \in V.$$

For every $u \in L_2(G)$ we consider the linear functional $l: V \to \mathbf{R}$ defined by $l(u) = (u, v)$, $v \in V$. Then $\sup_{v \in V} \|v\|^{-1} |(u, v)|$ defines a norm on $L_2(G)$. Let V^1 be the completion of $L_2(G)$ with this norm. The form $(\ ,\ )$ extends from $V \times L_2$ to $V \times V^1$ and with this pairing V and V^1 are duals. We assume that $V \cap L_\infty(G)$ is dense in V and the imbedding $V \to L_2(G)$ is compact. Let f be any element of $L_2(G)$.

Let $g(x, s)$ be a continuous real-valued function on $\bar{G} \times R$ such that

$$g(x, \infty) = \liminf_{s \to \infty} g(x, s),$$

$$g(x, -\infty) = \liminf_{s \to -\infty} g(x, s),$$

and let us assume that

$$g(x, -\infty) < g(x, \infty), \qquad x \in \bar{G}. \tag{13}$$

Further let:

for every $\varepsilon > 0$ and any continuous real-valued function

$$M(x),\ x \in \bar{G} \text{ with } g(x, \infty) > M(x) \text{ for all } x \in \bar{G}, \tag{14}$$

there exists $\rho > 0$ such that $g(x, s) > M(x) - \varepsilon$ for all $x \in \bar{G}$ and $s \geqslant \rho$. Similarly if $g(x, -\infty) < M(x)$ for all $x \in \bar{G}$ there exists $\rho > 0$ such that $g(x, s) < M(x) + \varepsilon$ for all $x \in \bar{G}$ and $s \leqslant -\rho$. It must be noted that as in [9], the case $g(x, \infty) = \infty$, $g(x, -\infty) = -\infty$ is not excluded.

Then it can be proved that the equation $Eu + g(x, u) = f(x)$ has at least one solution $u \in V$ provided

$$\int_{\theta > 0} g(x, \infty)\theta\, dx + \int_{\theta < 0} g(x, -\infty)\theta\, dx > \int_G (f, \theta)\, dx > \int_{\theta > 0} g(x, -\infty)\theta\, dx + \int_{\theta < 0} g(x, \infty)\theta\, dx \tag{15}$$

for all $\theta \in \ker E$, $\theta \neq 0$ (cf. [8, 1, 3]).

It can be shown that assumptions (13) and (14) imply (12) and thus (8), whereas (15) implies (9) and this may be seen by an argument similar to the one in [1].

3.

As remarked in Section 1, the main thrust of this paper will be the case when (13) does not hold. More particularly we will consider the case when hypothesis (3) is true. Before we go into the details, we first present an existence result for the problem $Eu + g(x, u) = f$ which will be utilized in the following discussions. This existence result states sufficient conditions for (8) and (9) to be satisfied when $\dim(\ker E) < \infty$.

LEMMA 1. *Let H be compact and let the continuous real-valued function $g(x, s)$ on $\bar{G} \times R$ be such that:*

(i) *there exist positive constants a, ρ such that*

$$(g(x, \rho\omega), \rho\omega) \geqslant a\rho \tag{16}$$

for all $\omega \in \ker E$ with $\|\omega\|_{L_2} = 1$;

(ii) *there exist positive constants c and k such that*

$$|g(x, s)| \leqslant c, \tag{17}$$

$$|g(x, s + h) - g(x, s)| \leqslant k\,|h|$$

for all $x \in G$ and for all $s, h \in R$;

(iii) $$L[c(\text{meas } G)^{1/2} + \|f\|_{L_2}] \leqslant r; \tag{18}$$

(iv) $$kr + \|f\|_{L_2} \leqslant a. \tag{19}$$

Then the equation $Eu + g(x, u(x)) = f(x)$ has at least one solution $u \in L_2(G)$ with $u = u^* + u_1$, $u^* \in \ker E$, $u_1 \in (\ker E)^\perp$, $\|u^*\| \leqslant \rho$, $\|u_1\| \leqslant r$.

Proof. It suffices to verify (8) and (9). Thus we have $Nu = -g(u) + f$ and with $u = \rho\omega + u_1$, $\omega \in \ker E$, $u_1 \in (\ker E)^\perp$, $\|u_1\| \leqslant r$ we have

$$\begin{aligned}
\|\bar{u}_1\| &= \|H(I - P)[-g(x, \rho\omega + u_1) + f]\| \\
&\leqslant L(\|g(x, \rho\omega + u_1)\|_{L_2} + \|f\|_{L_2}) \\
&\leqslant L(c(\text{meas } G)^{1/2} + \|f\|_{L_2}) \\
&\leqslant r.
\end{aligned}$$

Also,

$$
\begin{aligned}
(Nu, u^*) &= (-g(\rho\omega + u_1) + f, \rho\omega) \\
&= (-g(\rho\omega), \rho\omega) + (f, \rho\omega) - ([g(\rho\omega + u_1) - g(\rho\omega)], \rho\omega) \\
&\leqslant -a\rho + \int_G k\,|u_1|\,|\rho\omega|\,dx + \|f\|\,\|\rho\omega\| \\
&\leqslant -a\rho + k\rho\,\|u_1\|_2\,\|\omega\|_2 + \rho\,\|f\|_2\,\|\omega\|_2 \\
&\leqslant -a\rho + k\rho r + \rho\,\|f\| \\
&\leqslant 0.
\end{aligned}
$$

Thus (8) and (9) are satisfied with $R_0 = \rho$, and hence $Eu + g(x, u) = f$ has at least one solution.

An example to illustrate the applicability of the above lemma to the case when $g(-\infty) = g(\infty) = 0$ is the following.

EXAMPLE. Consider the nonlinear differential problem

$$
\begin{aligned}
u'' + u + g(u) &= f, \\
u(0) = u(\pi) &= 0.
\end{aligned} \tag{20}
$$

Then $Eu = u'' + u$, $G = (0, \pi)$, $\ker E = \{\sin x\}$ and we can take $\omega = (2/\pi)^{1/2} \sin x$ so that $\omega \in \ker E$ and $\|\omega\|_{L_2} = 1$. Thus, meas $G = \pi$ and it is easy to see that $L = \frac{1}{3}$.

Let $g(s) = 1 - \cos s$, $s \geqslant 0$ and $g(s) = -1 + \cos s$ for $s \leqslant 0$, so that $c = 2$. Then we have

$$
\begin{aligned}
(g(\rho\omega), \rho\omega) &= \int_0^\pi [1 - \cos(\rho(2/\pi)^{1/2} \sin x)]\, \rho(2/\pi)^{1/2} \sin x\, dx \\
&= 2\rho(2/\pi)^{1/2} - \int_0^\pi \cos(\rho(2/\pi)^{1/2} \sin x)\, \rho(2/\pi)^{1/2} \sin x\, dx.
\end{aligned}
$$

We now estimate the second term I on the right-hand side. It can be written as $I = 2\int_0^\pi \cos(\sigma \sin x)\sigma \sin x\, dx$ where $\sigma = \rho(2/\pi)^{1/2}$. Thus, setting $\sin x = t$, we have

$$
I = 2\sigma \int_0^1 \cos(\sigma t)\, t(1 - t^2)^{-1/2}\, dt.
$$

For σ large, let k_m be the largest integer with $\sigma^{-1}(k_m\pi + \pi/2) \leqslant 1$ and let $t_s = \sigma^{-1}(s\pi + \pi/2)$, $s = 0, 1, \ldots, k_m$. Then

$$
I = 2\sigma \left[\int_0^{t_0} + \sum_{s=0}^{k_m - 1} \int_{t_s}^{t_{s+1}} + \int_{t_{k_m}}^1\right] \cos(\sigma t)\, t(1 - t^2)^{-1/2}\, dt.
$$

Here $t_0 \to 0$ and $1 - t_{k_m} \to 0$ as $\sigma \to \infty$. Also $t(1-t^2)^{-1/2}$ is an increasing function of t. Thus $I \to 0$ as $\sigma \to \infty$. Hence,

$$(g(\rho\omega), \rho\omega) > 2\rho(2/\pi)^{1/2} - \varepsilon \qquad \text{for}$$

ρ sufficiently large when $\varepsilon > 0$ is given. Relations (18) and (19) now reduce to

$$3^{-1}(2\pi^{1/2} + \|f\|) \leqslant r \qquad \text{and} \qquad r + \|f\| \leqslant 2(2/\pi)^{1/2} - \varepsilon.$$

Solving the above inequalities one obtains estimates on $\|f\|$ and r. Hence, by the preceding lemma we conclude that

$$u'' + u + g(u) = f,$$
$$u(0) = u(\pi) = 0,$$

has at least one solution provided $\|f\|_{L_2}$ satisfies the estimate obtained above. We conclude this example by noting that for the $g(x, s)$ discussed here, $g(-\infty) = g(\infty) = 0$.

4.

Another example where $g(-\infty) = g(\infty) = 0$ and the problem $Eu + g(u) = f$ has infinitely many solutions is as follows:

$$u'' + g(u) = f(x),$$
$$u'(0) = u'(a) = 0. \tag{21}$$

In this case $\ker E = \{\omega\}$ where ω is the constant function $\omega(x) = a^{-1/2}$, $0 \leqslant x \leqslant a$, and $\|\omega\|_{L_2} = 1$. We define $g(s)$ as in the previous example so that $g(-\infty) = g(\infty) = 0$. Also, $g(s) = 0$ for all $s = 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$, and $|g(s)| \leqslant 2$. For the function $f(x)$ we choose a constant f_0, $0 < f_0 < 2$. Then, f_0 belongs to $\ker E$, and further there exists λ such that $g[(2k+1)\pi \pm \lambda] = f_0$, $0 < \lambda < \pi$. Then the constant functions

$$u_k(x) = (2k+1)\pi \pm \lambda, \qquad 0 \leqslant x \leqslant a,\ k = 1, 2, \ldots,$$

are all solutions of the nonlinear problem (21). Thus (21) has infinitely many solutions u with $\|u\|$ as large as we want no matter how small $f = f_0$ is in $\ker E$ with $\|f_0\| < 2a^{1/2}$.

We now consider the case when in (21) $f = f_0 + f_1$, $f_0 \in \ker E$ and $f_1 \in (\ker E)^{\perp}$, both not identically zero. Also, instead of the specific function $g(s)$ chosen in (21), we assume now that g is any continuous bounded function with $|g(s)| \leqslant c$, $sg(s) \geqslant 0$, $g(0) = 0$, $g(s) = 0$ also at countably many points $s = s_k$, $k = \pm 1, \pm 2, \ldots$, with $ks_k > 0$, $s_k \to \pm\infty$ as $k \to \pm\infty$, and g is not equal to zero otherwise. Clearly the function g of (21) has all the properties assumed here.

First, a remark concerning the eigenvalue problem $u'' + \lambda u = 0$, $0 \leqslant x \leqslant a$, with $u'(0) = u'(a) = 0$. The first eigenvalue $\lambda_0 = 0$ has the normalized constant eigenfuction $v_0 = \omega(x) = a^{-1/2}$. The other eigenvalues $\lambda_k = (\pi/a)^2 k^2$, $k = 1, 2, \dots$, have normalized eigenfunctions $v_k = (2/a)^{1/2} \cos(\pi k x/a)$, $0 \leqslant x \leqslant a$. Thus the smallest nonzero eigenvalue is $\lambda_1 = (\pi/a)^2$, and $L = \|H\| = \lambda_1^{-1} = a^2/\pi^2$.

Now we consider the decompositions $f = f_0 + f_1$, $g(\rho\omega + u_1) = g_0 + g_1$, $f_0, g_0 \in \ker E$, $f_1, g_1 \in (\ker E)^\perp$. From $PNu = 0$, that is, $P(-g + f) = 0$, we derive $f_0 = g_0$.

Again as in Section 2, we consider the equation

$$u_1 = \gamma H(I - P) Nu + \gamma PNu, \qquad 0 < \gamma < 1. \tag{22}$$

Let $\Sigma = [s \in R,\ g(s) = 0]$ be the set of zeros of g and let $\Sigma_\sigma = [s \in R,\ |g(s)| \leqslant \sigma]$ where $\sigma > 0$. We assume that for all $\sigma > 0$ sufficiently small there are infinitely many $\rho = \rho_k > 0$ such that $\mathrm{dist}(\rho_k \omega, \Sigma_\sigma) \geqslant d > 0$ for a fixed $d > 0$. For any $u_1 \in (\ker E)^\perp$, $\|u_1\| \leqslant r$ and $\beta > 0$, let $B_\beta = [x \in G,\ |u_1(x)| \geqslant \beta]$ so that

$$\beta^2 \operatorname{meas}(B_\beta) \leqslant \int_G u_1^2\, dx \leqslant r^2.$$

Thus $\operatorname{meas}(B_\beta) \leqslant r^2 \beta^{-2}$. We assume $r^2 \beta^{-2} < a$ and then $|u_1(x)| < \beta$ for all $x \in G - B_\beta$ where $\operatorname{meas}(G - B_\beta)$ is greater than or equal to $a - r^2 \beta^{-2}$.

If we now let $\beta = d$, then $\rho\omega + u_1(x) \notin \Sigma_\sigma$ for $\rho = \rho_k$ and all $x \in G - B_\beta$. Hence,

$$\begin{aligned} a g_0 = \int_G g(\rho\omega + u_1(x))\, dx &= \int_{G - B_\beta} + \int_{B_\beta} \\ &> \sigma \operatorname{meas}(G - B_\beta) - \mathrm{c} \operatorname{meas}(B_\beta) \\ &\geqslant \sigma a - \sigma r^2 \beta^{-2} - c r^2 \beta^{-2}. \end{aligned} \tag{23}$$

Also by (11) we have that for any solution of (22),

$$\begin{aligned} \|u_1\| &\leqslant L(c(\operatorname{meas} G)^{1/2} + \|(I - P) f\|) \\ &= (a^2/\pi^2)(c a^{1/2} + \|f_1\|). \end{aligned} \tag{24}$$

Thus for fixed σ, $\beta = d$ and

$$\begin{gathered} r^2 d^{-2} < a, \qquad \sigma a > (\sigma + c) r^2 d^{-2}, \qquad (a^2/\pi^2) c a^{1/2} < r, \\ \|f_0\| < a^{-1/2}(\sigma a - (\sigma + c) r^2 d^{-2}), \qquad (a^2/\pi^2) \|f_1\| < r - c a^{5/2}/\pi^2, \end{gathered} \tag{25}$$

we have

$$\|g_0\| = (\text{meas } G)^{1/2} \, |g_0| > a^{-1/2}[\sigma a - (\sigma + c)\, r^2 d^{-2}] > \|f_0\|,$$

and then (22) cannot have a solution $u(x) = \rho\omega + u_1$ for $\rho = \rho_k$, $\|u_1\| \leqslant r$ and any $0 < \gamma < 1$. Thus (21) has at least one solution $u = \rho\omega + u_1$ with $\rho \leqslant \rho_k$, $\|u_1\| \leqslant r$ provided (25) is satisfied.

In the particular case of (21) where $g(s)$ is defined by $g(s) = 1 - \cos s$, $s \geqslant 0$ and $g(s) = -1 + \cos s$ for $s \leqslant 0$ and a is chosen to be 1, we can verify (25) as follows. Here the zeros of g are $2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$, and with $0 < \sigma < 1$ we take $\rho_k = (2k + 1)$, $d = \pi - \tau$, $\tau = \text{arc}\cos(1 - \sigma)$, $0 < \tau < \pi/2$, and relations (25) become

$$r^2(\pi - \tau)^{-2} < 1, \qquad \sigma > (\sigma + 2)(\pi - \tau)^{-2}\, r^2, \qquad (1/\pi^2)2 < r,$$

$$\|f_0\| < [\sigma - (\sigma + 2)\, r^2(\pi - \tau)^{-2}], \qquad (1/\pi^2)\, \|f_1\| < r - 2/\pi^2.$$

By choosing σ, r, $\|f_0\|$, $\|f_1\|$ suitably it can be shown that (21) has at least one solution for $f = f_0 + f_1$.

5.

In the proof of existence of solutions using (13), (14) and (15) that we have sketched in Section 2, it is not restrictive to assume $g(x, s) < 0 < g(x, -\sigma)$ for all $x \in G$ and all $s, \sigma \geqslant k$ for k suitably large so that

$$\operatorname*{Inf}_{(x,s)} sg(x, s) = -\mu, \qquad 0 \leqslant \mu < \infty, \tag{26}$$

and this property is relevant in the proof.

We now assume that $g(x, -\infty) = g(x, \infty)$. Without loss of generalitly we can then assume that

$$g(x, -\infty) = g(x, \infty) = 0, \qquad x \in \bar{G}. \tag{27}$$

It does not then automatically follow that (26) holds, e.g., consider the function $g(x, s) = -s^{1/2}(1 + s)^{-1}$ for $s \geqslant 0$, $g(x, -s) = g(x, s)$, and then $g(x, -\infty) = g(x, \infty) = 0$ but $\text{Inf}\, sg(x, s) = -\infty$. However, if we know that for some $k > 0$ sufficiently large we have $sg(s) \geqslant 0$ for $s \geqslant k$ and $s \leqslant -k$, then clearly (26) follows.

Thus assuming (26) and (27) as hypotheses we consider the problem

$$Eu + g(x, u) = f(x), \qquad x \in G, f \in L_2(G). \tag{28}$$

Here we assume that E is a nonnegative operator, namely, we assume that

$\lambda_0 = 0$ is the smallest eigenvalue and $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ are the remaining eigenvalues with $\lambda_k \to \infty$, so that

$$(Eu_1, u_1) \geqslant \lambda_1 \|u_1\|^2$$

for all $u_1 \in (\ker E)^\perp$. In place of (26) we assume

$$\operatorname*{Inf}_{(x,s)} sg(x,s) \geqslant -\mu, \qquad 0 \leqslant \mu < \infty, x \in \bar{G}. \tag{29}$$

Then, as in Section 2, we consider for $0 < \gamma < 1$ the corresponding equations

$$u_1 = \gamma H(I-P)Nu, \qquad PNu = 0. \tag{30}$$

LEMMA 2. *For any solution $u = \rho\omega + u_1$ of these equations with $0 < \gamma < 1$ we have*

$$\|u_1\| < c(\rho^{1/2} + 1)$$

where $c = c(\lambda_1, \|f\|, \mu$, meas $G)$.

Proof. For any solution $u = \rho\omega + u_1$ of (30) we have

$$Eu_1 = \gamma(I-P)Nu.$$

Thus, by (29) and $u = \rho\omega + u_1$, $\|\omega\| = 1$, we have

$$\begin{aligned}\lambda_1 \|u_1\|^2 &\leqslant (Eu_1, u_1) = (\gamma(I-P)Nu, u_1)\\ &= (\gamma(I-P)Nu, \rho\omega + u_1) = (\gamma Nu, u)\\ &= \gamma[(-g(u), u) + (f, u)]\\ &\leqslant \gamma\left[-\int_G g(\rho\omega + u_1)(\rho\omega + u_1)\,dx + \|f\|\,\|u\|\right]\\ &< [\mu \text{ meas } G + \|f\|\,(\rho + \|u_1\|)].\end{aligned}$$

Hence,

$$\lambda_1 \|u_1\|^2 - \|f\|\,\|u_1\| - (\mu \text{ meas } G + \|f\|\,\rho) < 0, \tag{31}$$

and this implies the existence of a constant $c > 0$ such that for $\|u_1\| \geqslant c(\rho^{1/2} + 1)$ the above inequality (31) is not solvable and $c = c(\lambda_1, \|f\|, \mu$, meas $G)$.

Lemma 2 may also be seen in [9]. However, for our purpose, we can obtain the following stronger result.

LEMMA 3. *For* $0 < \rho \leqslant \rho_0$ *and* $r > (\lambda_1^{-1}\mu \text{ meas } G)^{1/2}$ *there exists* $\delta > 0$ *such that* $u_1 = \gamma H(I-P) Nu$ *has no solution on the boundary of the ball* $\|u_1\| \leqslant r$ *for* $\|f\| \leqslant \delta = \delta(\rho_0, \lambda_1, \mu, \text{meas } G, b)$.

Proof. Following Lemma 2, if u_1 is a solution of $u_1 = \gamma H(I-P) Nu$ then (31) holds and this can be rewritten as

$$(\|u_1\| - 2^{-1}\lambda_1^{-1}\|f\|)^2 - 4^{-1}\lambda_1^{-2}[\|f\|^2 + 4\lambda_1\mu \text{ meas } G + 4\lambda_1 \|f\|\rho] < 0. \quad (32)$$

If $\|f\| = 0$ then we get

$$\|u_1\|^2 - \lambda_1^{-1}\mu \text{ meas } G < 0 \quad (33)$$

and this is clearly impossible if $\|u_1\| > (\lambda_1^{-1}\mu \text{ meas } G)^{1/2}$.

Now let $\|u_1\| \geqslant (\lambda_1^{-1}\mu \text{ meas } G)^{1/2} + c$ for some $c > 0$. Also let δ_1 be such that for $\|f\| \leqslant \delta_1$ and $0 \leqslant \rho \leqslant \rho_0$ we have

$$4^{-1}\lambda_1^{-2}[\|f\|^2 + 4\lambda_1\mu \text{ meas } G + 4\lambda_1 \|f\|\rho] < \lambda_1^{-1}\mu \text{ meas } G + c^2/4. \quad (34)$$

For $0 < \delta_2 < c/2$ we have

$$\begin{aligned}[(\lambda_1^{-1}\mu \text{ meas } G)^{1/2} + c - \delta_2]^2 &> ((\lambda_1^{-1}\mu \text{ meas } G)^{1/2} + c/2)^2 \\ &\geqslant \lambda_1^{-1}\mu \text{ meas } G + c^2/4.\end{aligned}$$

Thus, for $\|f\| \leqslant 2\lambda_1\delta_2$ we have

$$\begin{aligned}(\|u_1\| - 2^{-1}\lambda_1^{-1}\|f\|)^2 &\geqslant ((\lambda_1^{-1}\mu \text{ meas } G)^{1/2} + c - \delta_2)^2 \\ &> \lambda_1^{-1}\mu \text{ meas } G + c^2/4. \quad (35)\end{aligned}$$

We now choose $\|f\|$ such that $\|f\| < \delta = \min[\delta_1, 2\lambda_1\delta_2]$ and $0 < \rho \leqslant \rho_0$. Then both (34) and (35) are true for all $0 < \rho \leqslant \rho_0$, and hence

$$\begin{aligned}(\|u_1\| - 2^{-1}\lambda^{-1}\|f\|)^2 &\geqslant \lambda_1^{-1}\mu \text{ meas } G + c^2/4 \\ &> 4^{-1}\lambda_1^{-2}[\|f\|^2 + 4\lambda_1\mu \text{ meas } G + 4\lambda_1 \|f\| \rho].\end{aligned}$$

In other words, for such choice of $\|u_1\|$, (31) could not be true. Hence we conclude that for

$$\|f\| \leqslant \delta, \qquad 0 < \rho \leqslant \rho_0, \qquad r \geqslant (\lambda_1^{-1}\mu \text{ meas } G)^{1/2} + c,$$

there cannot be any solution of $u_1 = \gamma H(I-P) Nu + \gamma PNu$ on the boundary of ball $\|u_1\| = r$.

Before we consider the existence of solutions of (28) under (26) and (27), we examine the above estimates in the case $\mu = 0$. In this case (32) reduces to

$$(\|u_1\| - 2^{-1}\lambda_1^{-1}\|f\|)^2 - 4^{-1}\lambda_1^{-2}[\|f\|^2 + 4\lambda_1\|f\|\rho] < 0. \tag{36}$$

For $\|f\| = 0$ this reduces to $\|u_1\|^2 < 0$, which is impossible. Now let $\|u_1\| \geqslant c$ for some $c > 0$. Choose $\delta > 0$ such that for $\|f\| \leqslant \delta_1$ and $0 < \rho \leqslant \rho_0$ we have

$$4^{-1}\lambda_1^{-2}[\|f\|^2 + 4\lambda_1\|f\|\rho] < c^2/4. \tag{37}$$

Also, given $c > 0$, let $0 < \delta_2 < c/2$.

Then we have $(c - \delta_2)^2 > (c/2)^2$ and hence for $\|f\| \leqslant 2\lambda_1\delta_2$ we have

$$(\|u_1\| - 2^{-1}\lambda_1^{-1}\|f\|)^2 \geqslant (c - \delta_2)^2 > c^2/4. \tag{38}$$

Finally, for $\|f\| \leqslant \delta = \min[\delta_1, 2\lambda_1\delta_2]$ and $0 < \rho \leqslant \rho_0$, both (37) and (38) are true so that

$$(\|u_1\| - 2^{-1}\lambda_1^{-1}\|f\|)^2 > c^2/4 > 4^{-1}\lambda_1^{-2}[\|f\|^2 + 4\lambda_1\|f\|\rho],$$

and this relation contradicts (36). The same argument holds as before.

We now return to an analysis of (28) under hypotheses (26) and (27). For $x \in \bar{G}$ and any $\sigma > 0$ let $\Sigma_\sigma(x)$ be the set defined by

$$\Sigma_\sigma(x) = \{s \in R: sg(x, s) \leqslant 0, \text{ or } g(x, s) < \sigma \text{ for } s > 0,$$
$$\text{or } g(x, s) > -\sigma \text{ for } s < 0\}.$$

Also, for given $\rho > 0$ and $\omega \in \ker E$, $\|\omega\| = 1$, let $d(x) = \text{dist}\{\rho\omega(x), \Sigma_\sigma(x)\} \geqslant 0$. We now assume ($\Gamma$): there exist positive numbers λ, h, σ, a with $0 < \lambda < \text{meas}\, G$ and infinitely many numbers $\rho_k > 0$, $\rho_k \to \infty$ as $k \to \infty$, such that for $\rho = \rho_k$, for any ω in $\ker E$ with $\|\omega\| = 1$ and any measurable subset S of G with meas $S \geqslant \lambda$, we have

$$\int_S d(x)\, dx \geqslant h, \tag{39}$$

and for any measurable subset Σ of G of measure $\geqslant(\text{meas}\, G) - \lambda$ we have $\int_\Sigma |\omega(x)|\, dx \geqslant a$.

The function $g(s)$ of the examples discussed in the previous sections clearly satisfies (Γ). For the sake of simplicity we shall write below $g(s)$ instead of $g(x, s)$.

Now, for any L_2-integrable function $u_1 \in (\ker E)^\perp$ with $\|u_1\| \leqslant r$ let

$$S_\sigma = \{x \in G: \rho\omega + u_1(x) \in \Sigma_\sigma\}.$$

Then $\rho\omega(x) + u_1(x) \in \Sigma_\sigma$ and $x \in S_\sigma$ with meas $S_\sigma \geqslant \lambda$ is possible if and only if $|u_1(x)| \geqslant d(x)$ for $x \in S_\sigma$ and

$$(\text{meas } G)^{1/2}\, r \geqslant (\text{meas } G)^{1/2} \|u_1\| \geqslant \int_G |u_1(x)|\, dx \geqslant \int_{S_\sigma} |u_1(x)|\, dx$$
$$\geqslant \int_{S_\sigma} d(x)\, dx \geqslant h.$$

This implies that for $\|u_1\| < h(\text{meas } G)^{-1/2}$ we can have $\rho\omega(x) + u_1(x) \in \Sigma_\sigma$ only in a set S_σ of measure $<\lambda$. Thus $\rho\omega(x) + u_1(x) \notin \Sigma_\sigma$ in the set $G - S_\sigma$ and hence $|g(\rho\omega(x) + u_1(x))| > \sigma$ in the set $G - S_\sigma$ of measure $>(\text{meas } G) - \lambda$. Moreover in $G - S_\sigma$ we certainly have $g(\rho\omega + u_1(x))(\rho\omega(x) + u_1(x)) > 0$, i.e., $g(\rho\omega + u_1(x))$ and $\rho\omega + u_1(x)$ have the same sign and their product is $\geqslant \sigma[\rho\omega(x) + u_1(x)]$. Thus

$$\int_G g(\rho\omega(x) + u_1(x))(\rho\omega + u_1)\, dx$$
$$= \int_{G-S_\sigma} + \int_{S_\sigma}$$
$$\geqslant -\mu(\text{meas } S_\sigma) + \sigma \int_{G-S_\sigma} |\rho\omega + u_1|\, dx$$
$$\geqslant -\mu(\text{meas } G) + \sigma\rho \int_{G-S_\sigma} |\omega|\, dx - \sigma \int_{G-S_\sigma} |u_1|\, dx$$

where $\text{meas}(G - S_\sigma) \geqslant \text{meas } G - \lambda$. Hence,

$$\int_G g(\rho\omega(x) + u_1(x))(\rho\omega + u_1)\, dx$$
$$\geqslant -\mu(\text{meas } G) + \sigma\rho a - \sigma \|u_1\|\, (\text{meas } G)^{1/2}.$$

Finally,

$$(Nu, u) = \int_G [-g(\rho\omega + u_1) + f](\rho\omega + u_1)\, dx$$
$$\leqslant -\sigma\rho a + \mu(\text{meas } G) + \sigma \|u_1\|\, (\text{meas } G)^{1/2} + \|f\|\, (\rho + \|u_1\|)$$
$$= -(\sigma a - \|f\|)\rho + \mu \text{ meas } G + (\sigma(\text{meas } G)^{1/2} + \|f\|)\, \|u_1\|.$$

Thus, we need to show that for $\rho = \rho_k$ sufficiently large, for $\|u_1\| \leqslant (\lambda_1^{-1} \mu \text{ meas } G)^{1/2} + \varepsilon$ for $\varepsilon > 0$ sufficiently small and for $\|f\| \leqslant \delta_0$ for $\delta_0 > 0$ sufficiently small, we have

$$-(\sigma a - \|f\|)\rho + \mu \text{ meas } G + (\sigma(\text{meas } G)^{1/2} + \|f\|) \|u_1\| \leqslant \lambda_1 \|u_1\|^2, \quad (40)$$

$$\|u_1\| < (\text{meas } G)^{-1/2} h. \quad (41)$$

For $\mu = 0$ these conditions imply

$$\|u_1\| \leqslant \varepsilon, \qquad \|u_1\| < (\text{meas } G)^{-1/2} h,$$

$$(\lambda_1 \|u_1\| - \sigma(\text{meas } G)^{1/2} - \|f\|) \| u_1 \| \geqslant -(\sigma a - \|f\|)\rho.$$

These relations are clearly compatible: for one could take $\varepsilon < (\text{meas } G)^{-1/2} h$, $\|u_1\| \leqslant \varepsilon$, $\|f\| \leqslant a\sigma/2$. Then we verify that

$$(-\sigma(\text{meas } G)^{1/2} - \sigma a/2)\varepsilon > -(\sigma a - \sigma a/2)\rho,$$

or

$$(\sigma a/2)\rho > \sigma\varepsilon((\text{meas } G)^{1/2} + a/2).$$

This is achieved by taking $\rho = \rho_k$ sufficiently large. Then we determine δ given by Lemma 3 for $0 < \rho \leqslant \rho_k$ and we take $\|f\| \leqslant \delta_0 = \min[\sigma a/2, \delta]$.

The discussion when $\mu > 0$ proceeds similarly. Thus for $\mu > 0$ we first ensure that

$$(\lambda_1^{-1}\mu \text{ meas } G)^{1/2} < h(\text{meas } G)^{-1/2}$$

or

$$\mu < \lambda_1(\text{meas } G)^{-2} h^2.$$

Then we choose $c < (\text{meas } G)^{-1/2} h - (\lambda_1^{-1}\mu \text{ meas } G)^{1/2}$ and $\|u_1\| \leqslant (\lambda_1^{-1}\mu \text{ meas } G)^{1/2} + c$, $\|f\| \leqslant \sigma a/2$. Now we have to verify that

$$(-\sigma(\text{meas } G)^{1/2} - \sigma a/2)[(\lambda_1^{-1}\mu \text{ meas } G)^{1/2} + c] > -(\sigma a - \sigma a/2)\rho,$$

or

$$(\sigma a/2)\rho > \sigma[(\text{meas } G)^{1/2} + a/2][\lambda_1^{-1}\mu(\text{meas } G)^{1/2} + c],$$

and again this is achieved by taking $\rho = \rho_k$ sufficiently large. As above, once ρ_k is fixed we choose δ from Lemma 3 and we take $\|f\| \leqslant \delta_0 = \min[\sigma a/2, \delta]$.

Thus we conclude that:

THEOREM. *If* $g(x, s)$ *is a continuous function on* $\bar{G} \times R$ *satisfying* (26), (27) *and* (Γ) *and* $0 \leqslant \mu < \lambda_1(\text{meas}\, G)^{-2}h^2$, *then there is some* $\delta_0 > 0$ *such that*

$$Eu + g(x, u) = f(x)$$

has at least one solution provided $\|f\| \leqslant \delta_0$.

6.

We conclude this paper with the related case in which

$$g(x, -\infty) = 0 = g(x, \infty), \tag{42}$$

and there are numbers $C > 0$, $0 < \alpha \leqslant 1$, $h > 0$, $M > 0$ such that

$$|g(x, s)| \leqslant C \qquad \text{for all} \quad x \in G,\ s \in R, \tag{43}$$

$$g(x, \sigma)\, \text{sgn}\, \sigma \geqslant hs^{-\alpha} \qquad \text{for all} \quad M \leqslant |\sigma| \leqslant s,\ s \geqslant M. \tag{44}$$

We shall denote by $|A|$ the measure of any measurable set A. Under assumption (43) the auxiliary equation $u_1 = H(I-P)N(\rho\omega + u_1)$, $\omega \in \ker E$, $\|\omega\| = 1$, $\rho > 0$, $u_1 \in (\ker E)^\perp$, is solvable for every ρ and ω, with $\|u_1\| \leqslant LC = r$. Let us prove that there are numbers γ, $R_0 > 0$ such that for every $\rho = R_0$, $\omega \in \ker E$, $\|u_1\| \leqslant LC$, $\|f\| \leqslant \gamma$, we have

$$\int_G [g(x, \rho\omega(x) + u_1(x)) - f(x)]\, \rho\omega(x)\, dx \geqslant 0.$$

To prove this we shall assume that:

There is a number $\omega_0 > 0$ such that $|\omega(x)| \leqslant \omega_0$ for all $x \in G$ and $\omega \in \ker E$, $\|\omega\| = 1$. (45)

Given $0 < \varepsilon < \omega_0$ there is $k > 0$ such that, for every $\omega \in \ker E$, $\|\omega\| = 1$, and $\Sigma_2 = [x \in G \,|\, |\omega(x)| \leqslant \varepsilon]$, then $|\Sigma_2| \leqslant k\varepsilon$. (46)

We can take ω_0 sufficiently large so that $\|\omega\| \leqslant \omega_0$ and then

$$\left| \int_G f(x)\, \rho\omega(x)\, dx \right| \leqslant \rho\omega_0 \|f\|.$$

First, for every $N > 0$ let $\Sigma_1 = [x \in G: | |u_1(x)| \geqslant N]$ so that $N^2 |\Sigma_1| \leqslant \|u_1\|^2 \leqslant L^2C^2$, and hence $|\Sigma_1| \leqslant L^2C^2N^{-2}$. For the sake of simplicity we shall write $g(\rho\omega + u_1)$ for $g(x, \rho\omega(x) + u_1(x))$. Then

$$I = \int_G g(\rho\omega + u_1)\,\rho\omega\,dx = \left(\int_{G-\Sigma_1-\Sigma_2} + \int_{\Sigma_1} + \int_{\Sigma_2}\right) g(\rho\omega + u_1)\,\rho\omega\,dx$$
$$= I_0 + I_1 + I_2.$$

For $x \in G - \Sigma_1 - \Sigma_2$ we have $\varepsilon \leqslant |\omega(x)| \leqslant \omega_0$, $|u_1(x)| \leqslant N$, and hence

$$|\rho\omega(x) + u_1(x)| \leqslant \rho\omega_0 + N, \qquad |\rho\omega(x) + u_1(x)| \geqslant \rho\varepsilon - N,$$

and for $\rho\varepsilon - N \geqslant M$, we also have $\operatorname{sgn}(\rho\omega + u_1) = \operatorname{sgn}\omega$, and

$$g(\rho\omega + u_1)\operatorname{sgn}\omega \geqslant h(\rho\omega_0 + N)^{-\alpha},$$
$$g(\rho\omega + u_1)\,\rho\omega \geqslant h(\rho\omega_0 + N)^{-\alpha}\,\rho\varepsilon,$$
$$I_0 \geqslant |G - \Sigma_1 - \Sigma_2|\,h\rho\varepsilon(\rho\omega_0 + N)^{-\alpha},$$

with $|\Sigma_1| \leqslant L^2C^2N^{-2}$, $|\Sigma_2| \leqslant k\varepsilon$. Hence

$$I_0 \geqslant (|G| - L^2C^2N^{-2} - k\varepsilon)\,h\rho\varepsilon(\rho\omega_0 + N)^{-\alpha},$$

and for

$$L^2C^2N^{-2} \leqslant 4^{-1}|G|, \qquad k\varepsilon \leqslant 4^{-1}|G|, \tag{47}$$

also

$$I_0 \geqslant 2^{-1}|G|\,h\rho\varepsilon(\rho\omega_0 + N)^{-\alpha}.$$

On Σ_1 we have $|u_1| \geqslant N$, $|g(s)| \leqslant C$, $|\Sigma_1| \leqslant L^2C^2N^{-2}$, and hence

$$|I_1| \leqslant |\Sigma_1|\,C\rho\omega_0 \leqslant C\rho\omega_0(L^2C^2N^{-2}).$$

On Σ_2 we have $|\omega(x)| \leqslant \varepsilon$, $|g(s)| \leqslant C$, $|\Sigma_2| \leqslant k\varepsilon$, and hence

$$|I_2| \leqslant |\Sigma_2|\,C\rho\varepsilon \leqslant Ck\rho\varepsilon^2.$$

Thus,

$$I \geqslant \rho[2^{-1}|G|\,h\varepsilon(\rho\omega_0 + N)^{-\alpha} - C\omega_0(L^2C^2N^{-2}) - Ck\varepsilon^2].$$

We shall now take

$$\begin{aligned} C\omega_0(L^2C^2N^{-2}) &\leqslant 2^{-3}|G|\,h\varepsilon(\rho\omega_0 + N)^{-\alpha}, \\ Ck\varepsilon^2 &\leqslant 2^{-3}|G|\,h\varepsilon(\rho\omega_0 + N)^{-\alpha}. \end{aligned} \tag{48}$$

We must show that relations (47), (48) and $\rho\varepsilon - N \geqslant M$ are compatible. First, we can write these relations as follows:

$$\begin{aligned} k\varepsilon &\leqslant 4^{-1}|G|, & k\varepsilon &\leqslant 2^{-3}|G|\,C^{-1}h(\rho\omega_0+N)^{-\alpha}, \\ L^2C^2N^{-2} &\leqslant 4^{-1}|G|, & L^2C^2N^{-2} &\leqslant 2^{-3}|G|\,C^{-1}\omega_0^{-1}h\varepsilon(\rho\omega_0+N)^{-\alpha}, \\ & & \rho\varepsilon - N &\geqslant M. \end{aligned} \tag{49}$$

We shall assume

$$2^{-1}C^{-1}h(\rho\omega_0+N)^{-\alpha} \leqslant 1, \tag{50}$$

so that of the relations (49) the second one is stronger than the first one, and the fourth is stronger than the third one. Then the fourth relation (49) yields

$$\begin{aligned} L^2C^2N^{-2} &\leqslant 2^{-3}|G|\,C^{-1}\omega_0^{-1}h(\rho\omega_0+N)^{-\alpha}\cdot 2^{-3}k^{-1}|G|\,C^{-1}h(\rho\omega_0+N)^{-\alpha} \\ &= 2^{-6}|G|^2\,C^{-2}h^2(k\omega_0)^{-1}\,(\rho\omega_0+N)^{-2\alpha} \end{aligned}$$

or

$$N^{-2} \leqslant 2^{-6}|G|^2\,L^{-2}C^{-4}h^2(k\omega_0)^{-1}\,(\rho\omega_0+N)^{-2\alpha},$$

or

$$(\rho\omega_0+N)^{\alpha} \leqslant 2^{-3}|G|\,L^{-1}C^{-2}h(k\omega_0)^{-1/2}\,N. \tag{51}$$

Thus, for $0<\alpha<1$, we may take $\rho\omega_0 \geqslant N$ and

$$k\varepsilon = 2^{-3}|G|\,C^{-1}h(\rho\omega_0+N)^{-\alpha}.$$

Thus, $k\varepsilon \leqslant 2^{-3}|G|\,C^{-1}hN^{-\alpha}$,

$$2^{-1}C^{-1}h(\rho\omega_0+N)^{-\alpha} \leqslant 2^{-1-\alpha}C^{-1}hN^{-\alpha},$$

and we can take N sufficiently large so that this last expression is $\leqslant 1$, and (51) holds. Moreover

$$\rho\varepsilon = 2^{-3}k^{-1}|G|\,C^{-1}h\rho(\rho\omega_0+N)^{-\alpha}$$

and for any N we can take $\rho \geqslant \omega_0^{-1}N$ sufficiently large so that the last expression is $\geqslant M+N$, that is, $\rho\varepsilon - N \geqslant M$. Now, for $0<\alpha<1$, we have satisfied all five relations (49), and

$$I \geqslant i = 2^{-2}|G|\,h\varepsilon(\rho\omega_0+N)^{-\alpha}\,\rho \equiv \rho\omega_0\gamma.$$

For $\alpha = 1$ relation (51) can be satisfied only if

$$\Lambda = 2^{-3}|G|\,L^{-1}C^{-2}h(k\omega_0)^{-1/2} > 1.$$

Actually, under this restriction, relation (51) reduces to $\rho\omega_0 \leqslant (\Lambda - 1)N$. If we take

$$\rho\omega_0 = (\Lambda - 1)N,$$

then $\rho\omega_0 + N = \Lambda N$, and relation (50) becomes $2^{-1}C^{-1}h\Lambda^{-1}N^{-1} \leqslant 1$, which we satisfy by taking

$$N = 2^{-1}C^{-1}h\Lambda^{-1}.$$

Finally, we take

$$\varepsilon = 2^{-3}k^{-1}\,|G|\,C^{-1}h(\rho\omega_0 + N)^{-1}.$$

Then

$$\begin{aligned}
\varepsilon &= 2^{-3}k^{-1}\,|G|\,C^{-1}h\Lambda^{-1}N^{-1} \\
&= 2^{-3}k^{-1}\,|G|\,C^{-1}h\Lambda^{-1} \cdot (2^{-1}C^{-1}h\Lambda^{-1})^{-1} \\
&= 2^{-2}k^{-1}\,|G|, \\
\rho\varepsilon &= \omega_0^{-1}(\Lambda - 1) \cdot 2^{-1}C^{-1}h\Lambda^{-1} \cdot 2^{-2}k^{-1}\,|G| \\
&= \omega_0^{-1}(\Lambda - 1) \cdot 2^{-1}C^{-1}h\Lambda^{-1} \cdot 2^{-2}k^{-1} \cdot 2^3 ALC^2h^{-1}(k\omega_0)^{1/2} \\
&= \omega_0^{-1}(\Lambda - 1)\,Ck^{-1}L(k\omega_0)^{1/2}.
\end{aligned}$$

Finally, relation $\rho\varepsilon - N \geqslant M$ becomes

$$(\Lambda - 1)\,\omega_0^{-1}LCk^{-1}(k\omega_0)^{1/2} \geqslant 2^{-1}C^{-1}h\Lambda^{-1} + M.$$

For $A = \omega_0^{-1}LCk^{-1}(k\omega_0)^{1/2}$, $B = 2^{-1}C^{-1}h$, this relation becomes

$$A(\Lambda - 1)\Lambda \geqslant B + M\Lambda,$$

or

$$A\Lambda^2 - (A + M)\Lambda - B \geqslant 0.$$

If λ_0 denotes the positive root of this equation, and $\Lambda_0 = \max[1, \lambda_0]$, then for

$$\Lambda = 2^{-3}\,|G|\,L^{-1}C^{-2}h(k\omega_0)^{-1/2} \geqslant \Lambda_0,$$

all relations (49) are satisfied, and again

$$I \geqslant i = 2^{-2}\,|G|\,h\varepsilon(\rho\omega_0 + N)^{-1}\,\rho \equiv \rho\omega_0\gamma.$$

We conclude with the following theorem.

THEOREM. *Under assumptions* (42), (43), (44), (45), (46) *and* $0 < \alpha < 1$, *there is* $\gamma > 0$ *such that for* $\|f\| < \gamma$, *Eq.* (28) *has at least one solution. For* $\alpha = 1$, *the same is true provided* $2^{-3}|G|\, L^{-1}C^{-2}h(k\omega_0)^{-1/2} \geqslant \Lambda_0$.

Indeed, in either case, we have, for $\|f\| < \gamma$,

$$\int_G [g(\rho\omega + u_1)\rho\omega - f]\,\rho\omega\, dx \geqslant \rho\omega_0\gamma - \|f\|\,\rho\omega_0 \geqslant 0$$

and the statement follows by [3].

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