A Distribution Invariant for Association Schemes and Strongly Regular Graphs

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ABSTRACT

In an association scheme $X$ with symmetric classes the projections $f_i(x)$ of the points $x \in X$ into the given Eigenspace $E_i$ are considered. In particular the smallest number of points whose projection vectors are strictly on one side of a hyperplane is called the distribution invariant (with respect to $E_i$) of the association scheme $X$. Examples are given where the distribution invariants can be calculated, as for the triangular graphs and for all strongly regular graphs with less than 17 points.

INTRODUCTION

An invariant for association schemes is introduced that depends on the geometry of the eigenspaces of the scheme over the field of real numbers. There is a connection with $T$-designs in the scheme as defined by Delsarte, and in the case of net graphs the invariant measures the deviation of adjoining a "transversal" to the given graph.

The original definition of this invariant arose in a problem of real algebra and topology concerning the construction of nonsingular bilinear maps (cf. [1]), but this is not to be discussed here.

In Section 1 we define the invariants and give some elementary lower bounds. In Section 2 we define and analyze $R_k$-regular sets and their connection to $T$-designs as defined by Delsarte, and obtain certain upper bounds on the invariants. The applications to strongly regular graphs are given in Corollary 2. In Section 3 we give specific calculations for strongly regular graphs, in particular for netgraphs and the triangular graphs. The invariants are calculated for strongly regular graphs with less than 17 points.
Let \((X, R)\) be an association scheme with \(s + 1\) symmetric classes on the finite set \(X\) with \(v = |X|\) elements. On the space \(\mathbb{R}[X]\) we have the canonical inner product defined by
\[
\langle e_x, e_y \rangle = \delta_{xy},
\]
where \(e_x, e_y\) are the basis vectors corresponding to \(x, y \in X\).

One has a decomposition
\[
\mathbb{R}[X] = E_0 \perp E_1 \perp \cdots \perp E_s
\]
of \(\mathbb{R}[X]\) into the eigenspaces of the incidence algebra of \((X, R)\) (see [5]). In particular \(E_0 = \mathbb{R} \cdot u\), \(u\) the all-one vector, and so \(\dim E_0 = 1\). Denote by \(J_0, J_1, \ldots, J_s\) the minimal orthogonal idempotents of the incidence algebra of \((X, R)\). Let \(J_i(x) = J_i(e_x)\) be the projection of \(e_x \in \mathbb{R}[X]\) into the \(i\)th eigenspace \(E_i\). In particular \(J_0 = J\), the all-one matrix, and \(J_0(e_x) = J_0(x) = u\) for all \(x \in X\).

**Lemma 1.** For \(i = 1, 2, \ldots, s\)
\[
\sum_{x \in X} J_i(x) = 0.
\]

**Proof.** \(\sum_{x \in X} J_i(x) = J_i(u) = J_i J_0(e_x) = 0\), since \(J_0, J_1, \ldots, J_s\) are orthogonal idempotents. \(\blacksquare\)

Let \(e \in E_i\) for a fixed but arbitrary \(i \geq 1\). Call the vector \(e\) general iff \(\langle e, J_i(x) \rangle \neq 0\) for all \(x \in X\).

Note that the vectors \(J_i(x)\) define an arrangement of hyperplanes \((H_x | x \in X)\), by \(H_x \subseteq E_i\) and \(H_x = J_i(x)\). A vector \(e \in E_i\) is general iff \(e\) is in the interior of a region of the arrangement [8].

Any general vector \(e \in E_i\) determines a partition \(X = X_+(e) \cup X_-(e)\) as follows:
\[
X_+(e) = \{x \in X | \langle e, J_i(x) \rangle > 0\}
\]
\[
X_-(e) = \{x \in X | \langle e, J_i(x) \rangle < 0\}
\]

In view of Lemma 1 this partition is nontrivial, i.e. \(X_+(e) \neq \emptyset\) and \(X_-(e) \neq \emptyset\).
The \( i \)th distribution invariant \( v_{t_i}(X) \) is the minimal cardinality of the sets \( X_+(e) \), where \( e \) ranges over all general vectors of \( E_i \), i.e.

\[
v_{t_i}(X) = \min \left\{ |X_+(e)| \mid e \in E_i \text{ and } e \text{ general} \right\}.
\] (5)

Let us call \( X_+(e) \) a distributed set (with respect to the \( i \)th eigenspace); any \( e \) as above could be called a distributing vector.

Note that if \( X_+(e) \) is a distributed set, then so is \( X_-(e) = X_+(-e) \).

Keep \( i \ (1 \leq i \leq s) \) and \( E_i \) fixed. Following [5], we denote by \( D_k \) the incidence matrix of the \( k \)th relation \( R_k \subset X \times X \) of \( (X, R) \), and by \( P_k(i) \) the (real) eigenvalue of \( D_k \) for the vectors \( e \in E_i \), \( k = 0, 1, \ldots, s \). In particular \( P_0(i) = 1 \) for all \( i \).

Let

\[
\text{Pos}(i) = |\{k|0 \leq k \leq s, P_k(i) > 0\}| \quad \text{and} \quad \text{Neg}(i) = |\{k|0 \leq k \leq s, P_k(i) < 0\}|.
\]

Also call a relation \( R_k \) positive if \( P_k(i) > 0 \) and negative if \( P_k(i) < 0 \).

**Lemma 2.** For \( i = 1, \ldots, s \)

\[
v_{t_i}(X) \geq \text{Pos}(i) \quad \text{(6a)}
\]

\[
v_{t_i}(X) \geq \text{Neg}(i) \quad \text{(6b)}
\]

**Proof.** We have \( I_k D_k(e_z) = D_k I_k(e_z) = P_k(i) I_k(e_z), \) \( k = 0, 1, \ldots, s \). Here \( e_z \) is a canonical basis vector of \( \mathbb{R}[X] \). By expanding \( D_k(e_z) \) it follows that

\[
\sum_{y \in \Gamma_k(z)} I_k(y) = P_k(i) \cdot I_k(z), \quad (7)
\]

where \( \Gamma_k(z) = \{y \in X|(y, z) \in R_k\} \).

For an arbitrary general vector \( e \in E_i \), by Lemma 1 there exists at least one \( z \) with \( \langle e, I_k(z) \rangle > 0 \). Then for each positive relation \( R_k \) with \( 1 \leq k \leq s \) there exists at least one further \( y_k \in \Gamma_k(z) \) such that \( \langle e, I_k(y_k) \rangle > 0 \) by (7). Since for fixed \( z \) the sets \( \Gamma_k(z) \) are disjoint, the \( y_k 's \) and \( z \) give at least \( \text{Pos}(i) \) vectors in any distributed set \( X_+(e) \). This proves (6a). (6b) is proved similarly. \( \blacksquare \)
For strongly regular graphs, i.e. $s = 2$, (6) gives $vt_1(X) \geq 2$, $vt_2(X) \geq 2$.

**Lemma 3.** For a strongly regular graph with $\lambda > 0$, then $vt_1(X) \geq 3$. If $\rho - 2k + \mu - 2 > 0$, then $vt_2(X) \geq 3$.

**Proof.** In the notation of [7], $P_1(1) = \rho$ and $P_2(1) = s$. For $z \in X$ there exists $y \in \Gamma_1(z)$ with $\langle e, I_1(z) \rangle > 0$ and $\langle e, I_1(y) \rangle > 0$, by (7) and $P_1(1) = \rho > 0$.

Since $\lambda > 0$, there exists $x \in \Gamma_1(z) \cap \Gamma_1(y)$. By (7) $\sum_{w \in \Gamma_2(x)} I_1(w) = s \cdot I_1(x)$, so we have a relation

$$\sum_{w \in \Gamma_2(x)} I_1(w) + ( - s ) I_1(x) = 0 \quad (8)$$

with only positive coefficients not involving $y$ and $z$, so that $vt_1(X) \geq 3$. The second statement is proved similarly by considering the complementary graph.  

**Remark.** Obviously, the invariants only depend on the spherical design $X_i = \{ I_i(x) | x \in X \} \subset E_i$. It seems to be an interesting problem to use the $t$-design property for $t > 1$ and improve the above lemmas.

2. REGULAR SETS AND T-DESIGNS

Let $H \subseteq X$ and $1 \leq k \leq s$, $k$ fixed. We call $H$ an $R_k$-regular set (or $k$-regular set) iff in the relation $R_k$ each $x \in H$ is connected with a fixed number $d_k$ of points of $H$, and each $z \in H$ is connected with a fixed number $e_k$ of points of $H$. Here the numbers $d_k$ and $e_k$ are not to depend on the particular choice of the points $y$ and $z$.

In the case $s = 2$ a set $H$ is $R_1$-regular iff it is $R_2$-regular, and will just be called a regular set in accordance with [6]. However, a $R_k$-regular set is not the same as a regular subset of [5, p. 25], except in the case $s = 2$.

Let $v_k$ be the valency of $R_k$, so $v_k = | \Gamma_k(z) |$ for any $z \in X$. For a subset $H \subseteq X$, let $c(H) = \sum_{h \in H} e_h \in \mathbb{R} [X]$ be the characteristic vector of $H$.

**Lemma 4.** Let $H$ be an $R_k$-regular set. Then the vector $w_k(H) := (v_k - d_k)c(H) - e_kc(X - H)$ is an eigenvector of the matrix $D_k$ with the eigenvalue $d_k - e_k$ of $D_k$. 

Proof.

\[ D_k(w_k(H)) = (v_k - d_k)D_k(c(H)) - e_kD_k(c(X - H)) \]

\[ = (v_k - d_k)d_kc(H) + (v_k - d_k)e_kc(X - H) \]

\[ - e_k(v_k - d_k)c(H) - e_k(v_k - e_k)c(X - H) \]

\[ = (d_k - e_k)(v_k - d_k)c(H) - (d_k - e_k)e_kc(X - H) \]

\[ = (d_k - e_k)w_k(H). \]

In particular \( d_k - e_k \) is an eigenvalue of \( D_k \). ■

Since \( d_k - e_k \) is an eigenvalue of \( D_k \), there exists an \( i \) such that \( P_k(i) = d_k - e_k \). However, the value of \( i \) is not uniquely determined, since it can and does happen that for certain \( k \) there exist several \( i_1, \ldots, i_d \) such that \( P_k(i_1) = \cdots = P_k(i_d) = d_k - e_k \). Denote this set of indices by \( I(H) = \{i_1, \ldots, i_d\} \). Also let \( E(I(H)) = \oplus_{j=1}^d E_{i_j} \subseteq \mathbb{R}[X] \).

Note that \( w_k(H) \in E(I(H)) \).

THEOREM 1. Let \( H \) be a \( R_k \)-regular set, and let \( v_k \neq d_k - e_k \). Then for \( j \notin I(H), j > 0, \)

\[ \sum_{y \in H} J_j(y) = 0. \] (9)

Proof. \[ [v_k - (d_k - e_k)] \cdot c(H) = w_k(H) + e_kc(X) \]. Applying \( J_j \) gives

\[ J_j([v_k - (d_k - e_k)] c(H)) = J_j w_k(H) + e_k J_j c(X) = 0. \]

The term \( J_j w_k(H) \) is 0, since \( j \notin I(H) \) but \( w_k(H) \in E(I(H)) \); the second term \( J_j c(X) \) is 0 by Lemma 1. Since \( v_k \neq (d_k - e_k), (9) \) follows. ■

Recall [5, pp. 32, 33] that for \( T \subset \{1, \ldots, s\} \) a set \( H \) is a \( T \)-design iff \( \sum_{y \in H} J_j(y) = 0 \) for all \( j \in T \).

COROLLARY 1. Let \( H \) be a \( R_k \)-regular set, \( v_k \neq d_k - e_k \), and \( T = \{1, \ldots, s\} - I(H) \). Then \( H \) is a \( T \)-design.
Theorem 2. Let $H$ be a $R_k$-regular set with $I(H) = \{i\}$. Let $v_k > d_k$ and $e_k > 0$. Then $H = X_{+}(w_k(H))$ is a distributed set with respect to the $i$th eigenspace. In particular, $v_{ti}(X) \leq |H|$.

Proof. By assumption, $E(I(H)) = E_{i} \supset w_k(H)$. So

$$\langle w_k(H), J_i(x) \rangle = \langle w_k(H), e_x \rangle = \begin{cases} v_k - d_k & \text{if } x \in H, \\ -e_k & \text{if } x \notin H. \end{cases}$$

(10)

In the case of strongly regular graphs, recall from [N] that a regular set is positive iff $d_1 - e_1 = r \geq 0$, negative iff $d_1 - e_1 = s < 0$.

Corollary 2. Let $X$ be a connected strongly regular graph, $H \subset X$ a regular set.

(i) If $H$ is positive, then $H$ is a distributed set with respect to the first eigenspace, and $\sum_{h \in H} J_2(h) = 0$ in the second eigenspace. In particular, $v_{t_1}(X) \leq |H|$.

(ii) If $H$ is negative, then $\sum_{h \in H} J_1(h) = 0$ in the first eigenspace, and $H$ is a distributed set with respect to the second eigenspace. In particular, $v_{t_2}(X) \leq |H|$.

Proof. The assumption $v_1 \neq d_1 - e_1$ of Theorem 1 and the assumptions $v_1 > d_1$ and $e_1 > 0$ of Theorem 2 apply, since $X$ is connected.

From the above, we may extract the following

Corollary 3. Let $X$ be a connected strongly regular graph, and assume $X$ contains $c$ disjoint positive (negative) regular sets. Then $v_{t_2}(X) \geq c$ ($v_{t_1}(X) \geq c$).

3. SOME CALCULATIONS FOR STRONGLY REGULAR GRAPHS

Example 1. Recall [4] that a net $N$ of degree $R$ and order $n$ is a set $X$ of $n^2$ points and $R \cdot n$ $n$-subsets of $X$, called lines, satisfying the following two axioms:

(i) the lines of $N$ are divided into $R$ classes of $n$ lines each such that the lines in each class partition the set $X$;

(ii) two lines belonging to different classes intersect in a unique common point.
The net graph on $X$ is defined by connecting $x$ and $y$ iff they are on a common line. It is easy to show that $X$ is a strongly regular graph on $v = n^2$ points of valency $k = R \cdot (n - 1)$. The eigenvalues of $X$ are given by $r = n - R$ and $s = - R$. The lines are positive regular sets with $d_1 = n - 1$ and $e_1 = R - 1$, i.e. regular cliques.

A transversal for $N$ is a set $T \subseteq X$ with $|T| = n$ and $|T \cap L| = 1$ for each line $L$. Any transversal for $X$ is a negative regular set with $d_1 = 0$, $e_1 = R$, i.e. a regular coclique.

By Corollary 3, $v_{t_2}(X) > n$. If a transversal $T$ exists, then $v_{t_2}(X) = n$ by Corollary 2(ii). On the other hand, if $v_{t_2}(X) = n$, then any distributed set with $n$ points must be a transversal, again by Corollary 2(i). So $v_{t_2}(X) = n$ is equivalent to the existence of a transversal for $N$. Similarly $v_{t_1}(X) \leq n$, and if there exist $n$ disjoint transversals $T_1, \ldots, T_n$, then $v_{t_1}(X) = n$.

For a nontrivial case look at the net of degree $R = 3$ defined by a cyclic group of even order, $n = 2m > 2$. The three classes of lines are the horizontal lines, the vertical lines, and the sets of pairs $(i, j)$, $0 \leq i, j < 2m$, with $i + j = \text{constant mod } 2m$.

The set $\{(0,0),(0,1),\ldots,(0,2m-2)\}$ is a distributed set with respect to the first eigenspace. Therefore $v_{t_1} \leq 2m - 1$ and there does not exist a parallel class of transversals to this net. But it is easily seen that any transversal for the net of degree 3 of a group gives by parallel translation a whole parallel class of transversals. Therefore there does not exist any transversal to this net, and as shown above, $v_{t_2} > 2m$.

On the other hand the set $\{(0,0),(1,1),\ldots,(m-1,m-1),(m,m+1),\ldots,(2m-2,2m-1),(2m-1,i)\}$ for any $i < 2m$ is a distributed set with respect to the second eigenspace. Therefore one gets for this net graph $v_{t_2} = 2m + 1$.

Example 2 (The triangular graphs $T(n)$ [3]). In case $n = 0 \ (2)$, $n = 2m$ say, one gets $v_{t_1} = 2m - 1$ and $v_{t_2} = m$. Distributed sets are given by

$$W = \{\{1,2\},\ldots,\{1,2m\}\}$$

for the first eigenspace,

$$C = \{\{1,2\},\{3,4\},\ldots,\{2m-1,2m\}\}$$

for the second eigenspace.

In case $n = 1 \ (2)$, $n = 2m + 1$ say, the cases $m = 2$ and $m = 3$ are exceptional. In case $m = 2$, $X = T(5)$ is the Petersen graph on 10 vertices, and $v_{t_1} = 3$, $v_{t_2} = 3$. Distributed sets are given by

$$W' = \{\{1,2\},\{1,3\},\{2,3\}\},$$

$$C' = \{\{1,2\},\{3,4\},\{1,5\}\}.$$
For $m \geq 3$, one obtains $v_{t_1} = 2m$ and $v_{t_2} = m + 1$. Distributed sets are given by

$$W' = \{(1,2),\{1,3\},\ldots,\{1,2m+1\}\},$$

$$C' = \{(1,2),\{3,4\},\ldots,\{2m-1,2m\},\{1,2m+1\}\}.$$

In case $m = 3$, another distributed set of minimal size is

$$W''' = \{(1,2),\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}.$$

The proofs of these assertions for $n \equiv 0 \pmod{2}$ are not difficult, and in the case of the first eigenspace generalize nicely to the general case of the Johnson schemes $J(n, k)$ with $n \equiv 0 \pmod{k}$; see [2]. The proofs of the case $n \equiv 1 \pmod{2}$ require more detailed analysis, and will not be given here.

**Example 3** (The Payley graph on 13 points). Take $X = \mathbb{Z}/13$ and $(x, y) \in R_1$ iff $X - Y = a^2$ for some $a \in \mathbb{Z}/13$. A distributed set for the first eigenspace is given by $\{0,1,4\}$. Since this graph is self-complementery, we get $v_{t_1} = v_{t_2} = 3$.

**Example 4** [The strongly regular graph $(v, k, \lambda, \mu) = (16,5,0,2)$]. Take $X = F_2^4$, $Q(x) = x_1x_2 + x_3x_4 + x_3^2 + x_4^2$. Let $x, y \in X$ be connected iff $Q(x + y) = 0$. Then in the resulting strongly regular graph $X$, $\{(0,0,0,0), (1,0,0,0), (0,1,0,0), (1,1,0,0), (1,1,0,1), (1,1,1,1)\}$ is a distributed set for the first eigenspace and $\{(1,0,0,0), (0,1,0,0), (1,1,0,0), (1,1,0,0), (1,1,0,1), (1,1,1,1)\}$ is a distributed set for the second eigenspace. Although Lemma 3 does not apply here, still $v_{t_1} = 3$ and also $v_{t_2} = 5$.

By combining the above examples, Lemma 3 implies

**Theorem 3.** The distribution invariants for strongly regular graphs with less than 17 points are given by the following table:

<table>
<thead>
<tr>
<th>$(v, k, \lambda, \mu)$</th>
<th>$v_{t_1}$</th>
<th>$v_{t_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(5,2,0,1)$</td>
<td>$v_{t_1} = v_{t_2} = 2$</td>
<td></td>
</tr>
<tr>
<td>$(9,4,1,2)$</td>
<td>$v_{t_1} = v_{t_2} = 3$</td>
<td></td>
</tr>
<tr>
<td>$(10,3,0,1)$</td>
<td>$v_{t_1} = v_{t_2} = 3$</td>
<td></td>
</tr>
</tbody>
</table>
AN INVARIANT FOR ASSOCIATION SCHEMES 113

(13, 6, 2, 3) \( v_{t_1} = v_{t_2} = 3 \)

(15, 6, 1, 3) \( v_{t_1} = 5, \ v_{t_2} = 3 \)

(16, 5, 0, 2) \( v_{t_1} = 3, \ v_{t_2} = 5 \)

(16, 9, 4, 6) \( v_{t_1} = v_{t_2} = 4 \) for \( L(\mathbb{Z}/2 \oplus \mathbb{Z}/2) \)

\( v_{t_1} = 3, \ v_{t_2} = 5 \) for \( L(\mathbb{Z}/4) \).

**Remark.** There are other examples where graphs with the same parameters can be distinguished by their distribution invariants. As seen above, for \( X = T(8) \) one has \( v_{t_1} = 7 \), while it is possible to show for the three other graphs \((28, 12, 6, 4)\), the Chang graphs, that \( v_{t_1} = 9 \).

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