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Approximations of relations by continuous functions

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Abstract

Let X be a Tychonoff space, $C(X)$ be the space of all continuous real-valued functions defined on X and $CL(X \times R)$ be the hyperspace of all nonempty closed subsets of $X \times R$. We prove the following result. Let X be a countably paracompact normal space. The following are equivalent: (a) $\dim X = 0$; (b) the closure of $C(X)$ in $CL(X \times R)$ with the Vietoris topology consists of all $F \in CL(X \times R)$ such that $F(x) \neq \emptyset$ for every $x \in X$ and F maps isolated points into singletons; (c) each usco map which maps isolated points into singletons can be approximated by continuous functions in $CL(X \times R)$ with the locally finite topology. From the mentioned result we can also obtain the answer to Problem 5.5 in [L’. Holá, R.A. McCoy, Relations approximated by continuous functions, Proc. Amer. Math. Soc. 133 (2005) 2173–2182] and to Question 5.5 in [R.A. McCoy, Comparison of hyperspace and function space topologies, Quad. Mat. 3 (1998) 243–258] in the realm of normal, countably paracompact, strongly zero-dimensional spaces. Generalizations of some results from [L’. Holá, R.A. McCoy, Relations approximated by continuous functions, Proc. Amer. Math. Soc. 133 (2005) 2173–2182] are also given.

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1. Introduction

Let X be a Tychonoff space, $C(X)$ be the space of all continuous real-valued functions defined on X and $CL(X \times R)$ be the hyperspace of all nonempty closed subsets of $X \times R$, where R is the space of real numbers. In our paper we will identify mappings with their graphs.

The fundamental result concerning approximations of relations by continuous functions is due to Cellina [4], who studied the approximation of relations in the Hausdorff metric (see also [3,9,10]). In [11] approximations of relations in the Vietoris and the locally finite topologies were studied. In fact, it was proved in [11] that if X is a countably paracompact normal space without isolated points and $F \in CL(X \times R)$ is the graph of cusco map, then F can be

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approximated in the Vietoris topology by continuous functions and if moreover X is also a q-space, then F can be approximated in the locally finite topology by continuous functions.

In our paper we prove some improvements of the above results and we also give the solution of Problem 5.5 in [11] and Question 5.5 in [13] in the realm of normal, countably paracompact, strongly zero-dimensional spaces. The following theorem is the main result of our paper.

Theorem. *Let X be a countably paracompact normal space. The following are equivalent*:

(a) $\dim X = 0$;
(b) *Each closed subset F of $X \times R$, satisfying $F(x) \neq \emptyset$ for every $x \in X$ and F maps isolated points into singletons is in the closure of $C(X)$ in $CL(X \times R)$ with the Vietoris topology*;
(c) *The closure of $C(X)$ in $CL(X \times R)$ with the Vietoris topology consists of all $F \in CL(X \times R)$ such that $F(x) \neq \emptyset$ for every $x \in X$ and F maps isolated points into singletons.*

There are other theorems like this theorem in the literature; see [2] about relations that are approximated in the Hausdorff distance by Baire class one functions, and [13] about the closure of densely continuous forms in the Vietoris topology. Moreover there is a rich literature concerning an approximation of a multifunction from above by a decreasing sequence of "continuous" multifunctions (see [12,5,6]).

2. Preliminaries

We refer to Beer [1] and Engelking [8] for basic notions. If X and Y are nonempty sets, a *set-valued mapping* or *multifunction* from X to Y is a mapping that assigns to each element of X a (possibly empty) subset of Y. If T is a set-valued mapping from X to Y, then its graph is $\{(x, y)\colon\ y \in T(x)\}$.

If F is a subset of $X \times Y$ and $x \in X$, define $F(x) = \{y\colon\ (x, y) \in F\}$. Then we can assign to each subset F of $X \times Y$, a set-valued mapping which takes the value $F(x)$ at each point $x \in X$. Then F is the graph of the set-valued mapping.

Let X and Y be topological spaces, and let T be a set-valued mapping from X to Y. Following Christensen [7], T is a usco map if T is upper semicontinuous and $T(x)$ is a nonempty compact set for all $x \in X$. Similarly, a cusco map is a usco map such that $T(x)$ is a connected set for all $x \in X$.

To describe the hypertopologies with which we will work, we need to introduce the following notation. Let (X, τ) be a topological space and $CL(X)$ be the hyperspace of all nonempty closed subsets of X. For $U \subset X$, define

$$U^+ = \{A \in CL(X)\colon\ A \subset U\} \quad \text{and} \quad U^- = \{A \in CL(X)\colon\ A \cap U \neq \emptyset\}.$$

If $\mathcal{U}$ is a family of open sets in X, define $\mathcal{U}^- = \bigcap\{U^-\colon\ U \in \mathcal{U}\}$.

A subbase for the Vietoris (resp., locally finite) topology on $CL(X)$ (see [1]) are the sets of the form U^+ with $U \in \tau$ and the form $\mathcal{U}^-$ with $\mathcal{U} \subset \tau$ finite (resp., locally finite).

Throughout the paper X will be a Hausdorff topological space. We use $CL_V(X \times R)$ and $CL_{LF}(X \times R)$ to denote the hyperspace of nonempty closed subsets of $X \times R$ with the Vietoris topology and the locally finite topology, respectively.

$I(X)$ will denote the set of all isolated points in X.

3. Relations approximated by continuous functions in the Vietoris topology

First, we note that if F is in the closure of $C(X)$ in $CL_V(X \times R)$ then $F(x) \neq \emptyset$ for every $x \in X$, and also that F maps isolated points of X to singletons (see Remark 3.1 in [11]).

The following theorem is the main result of our paper.

Theorem 3.1. *Let X be a countably paracompact normal space. The following are equivalent*:

(a) $\dim X = 0$;
(b) *Each closed subset F of $X \times R$, satisfying $F(x) \neq \emptyset$ for every $x \in X$ and F maps isolated points into singletons is in the closure of $C(X)$ in $CL_V(X \times R)$*;

(c) *The closure of $C(X)$ in $CL_V(X \times R)$ consists of all $F \in CL(X \times R)$ such that $F(x) \neq \emptyset$ for every $x \in X$ and F maps isolated points into singletons.*

Proof. (b) $\Rightarrow$ (a) Let $U_1, U_2, \dots, U_n$ be a finite functionally open cover of X. W.l.o.g. we can suppose that all U_i $(i = 1, 2, \dots, n)$ are different from X. The normality of X implies that there is a finite closed cover $F_1, F_2, \dots, F_n$ of X with $F_i \subset U_i$ for every $i \in \{1, 2, \dots, n\}$. W.l.o.g. we can suppose that there is no F_i $(i = 1, 2, \dots, n)$ such that F_i is covered by a union of other elements of $\{F_j\colon j \neq i\}$ (otherwise we change the family $\{F_1, F_2, \dots, F_n\}$ by the removing of some elements and of course we change also the indexed system). Put

$$H = \bigcup\bigl\{F_i \times \{i\}\colon 1 \leqslant i \leqslant n\bigr\}.$$

Then H is a closed set in $X \times R$, and for every $x \in X$, $H(x)$ has finitely many values. For each $x \in I(X)$ choose and fix some $i(x) \in \{1, 2, \dots, n\}$ with $x \in F_{i(x)}$. Let

$$L = \Bigl(\bigcup\bigl\{\{x\} \times H(x)\colon x \notin I(X)\bigr\}\Bigr) \cup \Bigl(\bigcup\bigl\{\bigl(x, i(x)\bigr)\colon x \in I(X)\bigr\}\Bigr).$$

Then of course L is a closed set in $X \times R$, satisfying $L(x) \neq \emptyset$ for every $x \in X$ and L maps isolated points into singletons.

Put $I_0 = \{i \in \{1, 2, \dots, n\}\colon F_i \subset I(X)\}$. For every $i \in I_0$ choose $x_i \in F_i$ such that all x_i $(i \in I_0)$ are different and define

$$M = \Bigl(L \setminus \bigcup\bigl\{\{x_i\} \times R\colon i \in I_0\bigr\}\Bigr) \cup \bigl\{(x_i, i)\colon i \in I_0\bigr\}.$$

Then also M is a closed set in $X \times R$, $M(x) \neq \emptyset$ for every $x \in X$, M maps isolated points into singletons and $M \cap (U_i \times O_i) \neq \emptyset$ for every $i \in \{1, 2, \dots, n\}$, where $O_i = (i - 1/2, i + 1/2)$. Now put

$$\mathcal{V} = \bigcap_{1 \leqslant i \leqslant n} (U_i \times O_i)^- \cap \Biggl[\bigcup_{1 \leqslant i \leqslant n} U_i \times O_i\Biggr]^+.$$

Then of course $M \in \mathcal{V}$. Thus by (b) there must exist $f \in C(X) \cap \mathcal{V}$. For every $k \in \{1, 2, \dots, n\}$, $f^{-1}(O_k)$ is nonempty and open. We claim that $f^{-1}(O_k) \subset U_k$ for every $k \in \{1, 2, \dots, n\}$. Suppose this is not true. Then there would be $k \in \{1, 2, \dots, n\}$ and $x \in f^{-1}(O_k) \setminus U_k$. Thus $x \in \bigcup_{i \neq k} U_i$; i.e., $f(x) \in \bigcup_{i \neq k} O_i$, a contradiction.

(a) $\Rightarrow$ (b) Let $F \in CL(X \times R)$ be such that $F(x) \neq \emptyset$ for every $x \in X$ and F maps isolated points into singletons. Let $W_0, V_i, i \in I$, I finite, be open sets in $X \times R$ with

$$F \in W_0^+ \cap \bigcap\bigl\{V_i^-\colon i \in I\bigr\}.$$

Without loss of generality we can suppose that $V_i \subset W_0$ for every $i \in I$ and $V_i = U_i \times T_i$ for every $i \in I$, where U_i, T_i are open sets in X and R, respectively.

Put $I_0 = \{i \in I\colon U_i \subset I(X)\}$. For every $i \in I_0$ choose $(x_i, F(x_i)) \in F \cap (U_i \times T_i)$. For every $i \in I \setminus I_0$ choose different points $x_i \in U_i$ (it is possible since for $i \in I \setminus I_0$ U_i contains also a non-isolated point and I is finite). For every $i \in I \setminus I_0$ choose $y_i \in T_i$. For every $i \in I \setminus I_0$ let $O(x_i)$ be an open-and-closed neighborhood of x_i such that $O(x_i) \cap O(x_j) = \emptyset$ for $i \neq j$, $i, j \in I \setminus I_0$. Put

$$S = X \setminus \Bigl(\bigcup\bigl\{O(x_i)\colon i \in I \setminus I_0\bigr\} \cup \{x_i\colon i \in I_0\}\Bigr).$$

Thus S is also open-and-closed; i.e., $\dim S = 0$ and S is a countably paracompact normal space.

Let $\{I_n\colon n \in \omega\}$ be an enumeration of open intervals with rational endpoints. Let $x \in S$. Choose $y_x \in F(x)$. There are open sets U_x in S and $n \in \omega$ such that $y_x \in I_n$ and $U_x \times I_n \subset W_0$. For every $n \in \omega$ put

$$U_n = \{U_x\colon U_x \times I_n \subset W_0\}.$$

The countable paracompactness of S implies that there is a locally finite open cover $\{G_i\colon i \in \omega\}$ with $G_i \subset U_i$ for every $i \in \omega$.

We will define by the induction a sequence $\{V_i : i \in \omega\}$ of pairwise disjoint open-and-closed sets such that

(1) $V_n \subset G_n$ for every $n \in \omega$, $S = \bigcup\{V_n\colon n \in \omega\}$;
(2) $\bigcup\{G_k \setminus \bigcup_{i\leqslant n} V_i\colon k \geqslant n+1\}$ is open-and-closed for every $n \in \omega$;
(3) $\bigcup\{G_k \setminus \bigcup_{i\leqslant n} V_i\colon k \geqslant n+1\} \cup \bigcup_{i\leqslant n} V_i = S$ for every $n \in \omega$.

The sets G_1 and $\bigcup\{G_k\colon k \geqslant 2\}$ form an open cover of S. Since $\dim S = 0$ and S is normal, there is a finite open refinement $\{V_i^1\colon i \in I_1\}$, I_1 finite, such that $V_i^1 \cap V_j^1 = \emptyset$ for $i \neq j$, $i, j \in I_1$. Thus all V_i^1 are open-and-closed sets. Put

$$V_1 = \bigcup\{V_i^1\colon V_i^1 \subset G_1,\ i \in I_1\}.$$

Thus also V_1 is an open-and-closed set, $\bigcup\{G_k \setminus V_1\colon k \geqslant 2\}$ is open-and-closed too, and $V_1 \cup \bigcup\{G_k \setminus V_1\colon k \geqslant 2\} = S$.

Suppose $V_1, V_2, \ldots, V_n$ were defined such that $V_1, V_2, \ldots, V_n$ are open-and-closed, $V_i \cap V_j = \emptyset$ for $i \neq j$, $i, j \leqslant n$, $\bigcup\{G_k \setminus \bigcup_{i\leqslant n} V_i\colon k \geqslant n+1\}$ is open-and-closed and $\bigcup_{i\leqslant n} V_i \cup \bigcup\{G_k \setminus \bigcup_{i\leqslant n} V_i\colon k \geqslant n+1\} = S$.

The set $H = \bigcup\{G_k \setminus \bigcup_{i\leqslant n} V_i\colon k \geqslant n+1\}$ is normal, $\dim H = 0$ and sets $G_{n+1} \setminus \bigcup_{i\leqslant n} V_i$, $\bigcup\{G_k \setminus \bigcup_{i\leqslant n} V_i\colon k \geqslant n+2\}$ form a finite open cover of H.

Since $\dim H = 0$ and H is normal there is a finite open refinement $\{V_i^{n+1}\colon i \in I_{n+1}\}$, I_{n+1} finite, of $G_{n+1} \setminus \bigcup_{i\leqslant n} V_i$ and $\bigcup\{G_k \setminus \bigcup_{i\leqslant n} V_i\colon k \geqslant n+2\}$ with $V_i^{n+1} \cap V_j^{n+1} = \emptyset$ for $i \neq j$, $i, j \in I_{n+1}$. Put

$$V_{n+1} = \bigcup\left\{V_i^{n+1}\colon V_i^{n+1} \subset G_{n+1} \setminus \bigcup_{i\leqslant n} V_i,\ i \in I_{n+1}\right\}.$$

Then of course V_{n+1} is open-and-closed, $V_1, V_2, \ldots, V_{n+1}$ are pairwise disjoint and $\bigcup_{i\leqslant n+1} V_i \cup \bigcup\{G_k \setminus \bigcup_{i\leqslant n+1} V_i\colon k \geqslant n+2\} = S$.

To show that $\bigcup\{V_i\colon i \in \omega\} = S$, let $x \in S$. There are $G_{i_1}, G_{i_2}, \ldots, G_{i_n}$ $(i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n)$ with $x \in \bigcap\{G_{i_k}\colon k \leqslant n\}$. Thus $x \notin \bigcup\{G_i\colon i \geqslant i_n + 1\}$, i.e., $x \in \bigcup\{V_i\colon i \leqslant i_n + 1\}$.

For every $n \in \omega$ choose $s_n \in I_n$. Define the function f as follows: $f(s) = s_n$ for $s \in V_n$, $f(x_i) = F(x_i)$ for $i \in I_0$ and $f(x) = y_i$ for $x \in O(x_i)$, $i \in I \setminus I_0$. Then of course f is a continuous function and $f \in W_0^+ \cap \bigcap\{V_i^-\colon i \in I\}$. □

From Theorem 3.1 we can also obtain the answer to Problem 5.5 in [11] and to Question 5.5 in [14] concerning the closure of $C(X)$ in $CL_V(X \times R)$ for countably paracompact normal spaces X with $\dim X = 0$.

4. Relations approximated by continuous functions in the locally finite topology

We prove an analog of Theorem 3.1 for locally finite topology.

Theorem 4.1. *Let X be a countably paracompact normal space. The following are equivalent*:

(a) $\dim X = 0$;
(b) *Each usco map which maps isolated points into singletons is in the closure of $C(X)$ in $CL_{LF}(X \times R)$.*

To prove Theorem 4.1 we will need the following lemma.

Lemma 4.2. *Let X be a countably paracompact space and let $F \in CL(X \times R)$ be the graph of a usco map. If $\mathcal{G}$ is a locally finite family in $X \times R$ with $F \in \mathcal{G}^-$ then there is a locally finite family $\mathcal{V} = \{U_\lambda \times G_\lambda\colon \lambda \in \Lambda\}$ in $X \times R$ with $F \in \mathcal{V}^- \subset \mathcal{G}^-$ and $\{U_\lambda\colon \lambda \in \Lambda\}$ locally finite in X.*

Proof. There is a locally bounded set W in $X \times R$ such that $F \subset W$ (W is locally bounded provided that for every $x \in X$, there exists a neighborhood O of x in X and a compact set $K \subset R$ with $W(z) \subset K$ for every $z \in O$).

Put $\mathcal{V} = \{O \cap W\colon O \in \mathcal{G}\}$. Without loss of generality we can suppose that $\mathcal{V} = \{U_\lambda \times G_\lambda\colon \lambda \in \Lambda\}$ and of course $(U_\lambda \times G_\lambda) \subset W$ for every $\lambda \in \Lambda$ and $\mathcal{V}$ is a locally finite family in $X \times R$. To prove that $\{U_\lambda\colon \lambda \in \Lambda\}$ is a locally finite family in X, let $x \in X$. There are a neighborhood O of x and a compact set $K \subset R$ with $W(z) \subset K$ for every $z \in O$. Since $\{U_\lambda \times G_\lambda\colon \lambda \in \Lambda\}$ is a locally finite family in $X \times R$ there is a neighborhood $H \subset O$ of x such that the set $\{\lambda \in \Lambda\colon (U_\lambda \times G_\lambda) \cap (H \times K) \neq \emptyset\}$ is finite. We claim that $\{\lambda \in \Lambda\colon U_\lambda \cap H \neq \emptyset\}$ is finite. Suppose there is an infinite set $I \subset \Lambda$ with $U_n \cap H \neq \emptyset$ for every $n \in I$. Let $x_n \in U_n \cap H$ for each $n \in I$. Then, for each $n \in I$, we have

$$\{x_n\} \times G_n \subset U_n \times G_n \subset W.$$

Thus $G_n \subset W(x_n) \subset K$. It follows that $(U_n \times G_n) \cap (H \times K) \neq \emptyset$ for every $n \in I$, a contradiction. □

Proof of Theorem 4.1. (a) ⇒ (b) Let W be an open set in $X \times R$ and $\mathcal{V}$ be a locally finite family of open sets in $X \times R$ with

$$F \in W^+ \cap \mathcal{V}^-.$$

Let $\mathcal{V} = \{V_\lambda\colon \lambda \in \Lambda\}$. Without loss of generality we can suppose that $V_\lambda = U_\lambda \times G_\lambda$ for every $\lambda \in \Lambda$. By Lemma 4.2 we can suppose that $\{U_\lambda\colon \lambda \in \Lambda\}$ is a locally finite family in X. We can suppose that $|\Lambda| \geqslant \aleph_0$; the finite case is known (Theorem 3.1). Put

$$\beta = \big\{\lambda \in \Lambda\colon U_\lambda \subset I(X)\big\}.$$

For every $\lambda \in \beta$ let $(x_\lambda, F(x_\lambda)) \in U_\lambda \times G_\lambda$. Put $L = \{x_\lambda\colon \lambda \in \beta\}$. Then of course L is closed-and-open. For every $\lambda \in \Lambda \setminus \beta$, U_λ contains a non-isolated point. There is a closed discrete set $E \subset X \setminus I(X)$ with

$$E \in \bigcap\{U_\lambda^-\colon \lambda \in \Lambda \setminus \beta\} \quad \text{and with} \quad |E| = |\lambda \setminus \beta|.$$

There is a discrete family $\{O(e)\colon e \in E\}$ of open and closed neighborhoods of E (X is zero-dimensional). For every $e \in E$ put $A(e) = O(e) \cap \bigcap\{U_\lambda \mid e \in U_\lambda\}$ and $\Lambda(e) = \{\lambda \in \Lambda \setminus \beta \mid e \in U_\lambda\}$ and choose different $x_e^\lambda \in A(e)$ for every $\lambda \in \Lambda(e)$. For every $\lambda \in \Lambda \setminus \beta$ choose also $y_\lambda \in G_\lambda$.

Now for every $e \in E$ and $\lambda \in \Lambda(e)$ let $P(x_e^\lambda) \subset A(e)$ be an open-and-closed neighborhood of x_e^λ such that $\{P(x_e^\lambda) | \lambda \in \Lambda(e)\}$ is a pairwise disjoint family. Thus the family

$$\big\{P\big(x_e^\lambda\big)\colon e \in E,\ \lambda \in \Lambda(e)\big\}$$

is a discrete family. Put $S = X \setminus L \setminus \bigcup\{P(x_e^\lambda)\colon e \in E, \lambda \in \Lambda(e)\}$. Then S is closed-and-open. Thus $\dim S = 0$ and S is a countably paracompact normal space. By Theorem 3.1 there is

$$g \in \big(W \cap (S \times R)\big)^+ \cap C(S).$$

Define the function f as follows: $f(s) = g(s)$ for every $s \in S$, $f(x) = F(x)$ for $x \in L$ and $f(x) = y_\lambda$ for every $x \in P(x_e^\lambda)$, $e \in E, \lambda \in \Lambda(e)$. It is clear that f is continuous and $f \in W^+ \cap \mathcal{V}^-$.

(b) ⇒ (a) Suppose that (a) does not hold. By Theorem 3.1 there is $F \in CL(X \times R)$ such that $F(x) \neq \emptyset$ for every $x \in X$, F maps isolated points into singletons and there are open sets $W_0, W_1, \ldots, W_n$ in $X \times R$ with

$$F \in W_0^+ \cap W_1^- \cap \cdots \cap W_n^- \quad \text{and} \quad C(X) \cap W_0^+ \cap W_1^- \cap \cdots \cap W_n^- = \emptyset.$$

We will define a usco map H which maps isolated points into singletons and $H \in W_0^+ \cap W_1^- \cap \cdots \cap W_n^-$.

For every $i = 1, 2, \ldots, n$ choose $(x_i, y_i) \in F \cap W_i$. There is an open set W in $X \times R$ with $F \subset W \subset \overline{W} \subset W_0$.

Let $\{I_n\colon n \in \omega\}$ be an enumeration of open intervals with rational endpoints. For every $x \in X$ choose $y_x \in F(x)$, an open set O_x and $n \in \omega$ such that $y_x \in I_n$, $(O_x \times I_n) \subset W$. For every $n \in \omega$ put

$$O_n = \bigcup\big\{O_x\colon (O_x \times I_n) \subset W\big\}.$$

The countable paracompactness of X implies that there is a locally finite open cover $\{V_i\colon i \in \omega\}$ with $V_i \subset O_i$ for every $i \in \omega$. Put

$$L = \bigcup\{V_i \times I_i\colon i \in \omega\} \quad \text{and} \quad L_0 = \overline{L}.$$

Then $L_0 \subset W_0$ and L_0 is an usco map, since for every $x \in X$ there is a neighborhood O_x^* and a compact set K_x such that $L_0(z) \subset K_x$ for every $z \in O_x^*$. Now put

$$H = \bigg(L_0 \cup \bigcup_{1 \leqslant i \leqslant n} \big\{(x_i, y_i)\big\}\bigg) \setminus \Big(\bigcup\big\{\{x\} \times \big(R \setminus F(x)\big)\colon x \in I(X)\big\}\Big).$$

Then H is an usco map which maps isolated points into singletons and $H \in W_0^+ \cap W_1^- \cap \cdots \cap W_n^-$, a contradiction. □

The following result is an improvement of Lemma 3.5 in [11] and shows that for q-spaces to be a usco map is a necessary condition for an element $F \in CL(X \times R)$ to be in the closure of $C(X)$ in $CL_{LF}(X \times R)$; where a q-space is a space in which every point has a sequence (U_n) of neighborhoods such that if $x_n \in U_n$ for each n, then (x_n) has a cluster point. This concept of q-space was introduced in [16] and has been useful, among other things, for studying function spaces (see [15]).

Proposition 4.3. *Let X be a q-space. If $F \in CL(X \times R)$ is in the closure of $C(X)$ in $CL_{LF}(X \times R)$, then F is locally bounded*; *i.e., F is the graph of a usco map.*

Proof. Suppose F is not locally bounded; i.e., there is $x \in X$ such that for every open neighborhood O of x, $F(O)$ is an unbounded set in R. Let $\{O_n\colon n \in \omega\}$ be a sequence of neighborhoods of x from the definition of the q-space. Choose $x_1 \in O_1$ and $y_1 \in F(x_1)$ with $|y_1| > 1$. Define by the induction points $x_n \in O_n, n \in \omega$, and points $y_n \in F(x_n)$ with $|y_{n+1}| > \max\{|y_n|, n\}$. The family $\{O_n \times (y_n - 1/n, y_n + 1/n)\colon n \in \omega\}$ is a locally finite family and

$$F \in \{O_n \times (y_n - 1/n, y_n + 1/n)\colon n \in \omega\}^-.$$

Thus there must exist $f \in C(X) \cap \{O_n \times (y_n - 1/n, y_n + 1/n)\colon n \in \omega\}^-$. Let $(b_n, f(b_n)) \in O_n \times (y_n - 1/n, y_n + 1/n)$ for every $n \in \omega$. By the definition of the q-space, the sequence $\{b_n\colon n \in \omega\}$ must have a cluster point $b \in X$. Now we have a contradiction with the continuity of f at b. □

The following example shows that the condition of a q-space in Proposition 4.3 is essential.

Example 4.4. Let W be the set of all ordinal numbers less than or equal to the first uncountable ordinal number ω_1 with the usual topology. Let L be the set of all limit ordinal numbers different from ω_1. Put $X = W \setminus L$ and equip X with the induced topology from W. X is not a q-space. Let $F \in CL(X \times R)$ be such that $F(\omega_1) = R$ and $F(x) = \{0\}$ otherwise. Then of course F cannot be locally bounded, but F is approximated by continuous functions in the locally finite topology.

We have the following description of the closure of $C(X)$ in $CL_{LF}(X \times R)$ for strongly zero-dimensional countably paracompact normal q-spaces X.

Theorem 4.5. *Let X be a countably paracompact normal q-space with* $\dim X = 0$, *and let $F \in CL(X \times R)$. Then the following are equivalent*:

(a) *F is in the closure of $C(X)$ in $CL_{LF}(X \times R)$*;
(b) *F is the graph of a usco map which maps isolated points into singletons.*

The following theorem is an improvement of Theorem 4.6 in [11]. We omit the proof of it since it is based on ideas from Lemmas 4.1, 4.3 from [11] and Theorem 4.1 and Lemma 4.2 of our paper.

Theorem 4.6. *Let X be a countably paracompact normal space. If $F \in CL(X \times R)$ is the graph of a cusco map which maps isolated points into singletons, then F is in the closure of $C(X)$ in $CL_{LF}(X \times R)$.*

We finish with the following improvement of Theorem 4.7 in [11].

Theorem 4.7. *Let X be a countably paracompact normal q-space which is locally connected, and let $F \in CL(X \times R)$. Then the following are equivalent*:

(a) *F is in the closure of $C(X)$ in $CL_{LF}(X \times R)$*;
(b) *F is the graph of a cusco map which maps isolated points into singletons.*

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