Heat content and Hardy inequality for complete Riemannian manifolds

M. van den Berg*

School of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, UK

Received 1 July 2005; accepted 31 August 2005
Communicated by D. Stroock
Available online 4 November 2005

Abstract

Upper bounds are obtained for the heat content of an open set $D$ in a complete Riemannian manifold, provided the Dirichlet–Laplace–Beltrami operator satisfies a strong Hardy inequality, and the distance function on $D$ satisfies an integrability condition.

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Keywords: Riemannian manifold; Heat content; Hard inequality

1. Introduction

Let $D$ be an open subset with boundary $\partial D$ in a $m$-dimensional, geodesically complete Riemannian manifold $M$, and let $u : D \times [0, \infty) \to [0, 1]$ be the unique weak solution of

$$\Delta u = \frac{\partial u}{\partial t}, \quad x \in D, t > 0,$$

where $\Delta$ is the Laplace–Beltrami operator, with initial condition

$$u(x; 0) = 1, \quad x \in D,$$
and with boundary condition

\[ u(x; t) = 0, \quad x \in \partial D, \quad t > 0. \quad (3) \]

The asymptotic behaviour of \( u(x; t) \) for \( t \to 0 \) and for \( t \to \infty \) have been the subject of a thorough investigation. The treatise of Carslaw and Jaeger [13] deals with a range of choices for regions \( D \) in Euclidean space \( \mathbb{R}^m \) where classical tools like separation of variables and Laplace transforms are available.

Probabilistic techniques were used in [10] to obtain accurate estimates for \( u \) near \( \partial D \) as \( t \to 0 \) in case \( D \) is an arbitrary open, bounded and connected subset of \( \mathbb{R}^m (m \geq 2) \) with \( C^3 \) boundary. These pointwise estimates were used to prove that the heat content, defined by

\[ Q_D = \int_D u(x; t) \, dx, \quad (4) \]

has the following asymptotic behaviour for \( t \to 0 \):

\[ Q_D(t) = |D| - 2\pi^{-1/2} |\partial D|^{1/2} t^{1/2} + 2^{-1} (m - 1) t \int_{\partial D} H + O(t^{3/2}), \quad (5) \]

where \( |D| \) is the volume of \( D \), \( |\partial D|^{m-1} \) is the perimeter of \( \partial D \), and \( H : \partial D \to \mathbb{R} \) is the mean curvature map if \( \partial D \) is oriented with an inward unit normal vector field.

Asymptotic formulae like (5) can be obtained in the much more general setting of Laplace-type operators acting on smooth vector bundles over compact Riemannian manifolds with smooth boundary [17]. It turns out that the heat content in this setting has an asymptotic expansion as \( t \to 0 \) in \( t^{1/2} \), and that the coefficients of \( t^{j/2} (j = 0, 1, 2 \ldots) \) are locally computable geometric invariants. However, no such power series exists if the boundary of \( D \) is not smooth, and precise analysis is restricted to special cases [3–5,7,9].

It is well known [5,8,11] that the heat content can be finite for all \( t > 0 \) if \( D \) has infinite volume. This happens if the boundary of \( D \) is not too thin, and if most of the mass of \( D \) is close to its boundary. A cusp is an example where precise asymptotic analysis is possible [5]. However, no asymptotic formulae for \( Q_D(t), t \to 0 \) are available in full generality.

In this paper, we obtain bounds for the heat content in the non-classical situation where either \( D \) has finite volume and \( \partial D \) is non-smooth, or where \( D \) has infinite volume. In order to make some progress and to exclude pathological examples where the boundary, or part of it, is very thin we require some regularity of \( \partial D \). A uniform capacitary density condition for the boundary was used in [2] to obtain bounds for the heat content for regions in Euclidean space. In the general setting of an open set \( D \) in a complete Riemannian manifold \( M \) it is advantageous to assume that the Dirichlet–Laplace–Beltrami operator acting in \( L^2(D) \) satisfies a strong Hardy inequality. The proof of Theorem 3 below relies on differential inequalities for the heat content.
and avoids delicate pointwise estimates of $u$ [2]. In Corollary 4 we show that, under the additional hypothesis that $M$ has non-negative Ricci curvature, the upper bounds obtained in Theorem 3 have the right order of magnitude.

The heat content for the boundary-value problem $\Delta v = \frac{\partial v}{\partial t}$, $x \in D, t > 0$ with initial condition $v(x; 0) = 0$, $x \in D$ and boundary condition $v(x; t) = 1$, $x \in \partial D$, $t > 0$ is closely related to some probabilistic quantities. First note that

$$v = 1 - u. \tag{6}$$

Define the corresponding heat content by

$$E_D(t) = \int_D v(x; t) \, dx. \tag{7}$$

Let $K$ be a compact, non-polar set in $\mathbb{R}^m$, $D = \mathbb{R}^m \setminus K$, and denote by $W^1_K(t), \ldots, W^k_K(t)$ $k$ independent Wiener sausages up to time $t$ and associated to $K$ i.e.

$$W^i_K(t) = \{B^i(s) + y : 0 \leq s \leq t, y \in K\}, \quad i = 1, \ldots, k, \tag{8}$$

where $(B^i(s), s \geq 0; \mathbb{P}, x \in \mathbb{R}^m), i = 1, \ldots, k$ are $k$ independent Brownian motions with generator $\Delta$. Then for $k \in \mathbb{N}$

$$E_0^1 \otimes \cdots \otimes E_0^k \left[\bigcap_{i=1}^k W^i_K(t) \right] = \int_{\mathbb{R}^m \setminus K} v(x; t)^k \, dx + |K|, \tag{9}$$

and

$$E_0[|W^i_K(t)|] = E_{\mathbb{R}^m \setminus K}(t) + |K| \tag{10}$$

for a single Wiener sausage. The asymptotic behaviour of (9) for $t \to \infty$ has been investigated in [6] for $m \geq 3$ and in [23] for $m = 2$.

In general we note that if $|D| < \infty$ then, by (6),

$$Q_D(t) + E_D(t) = |D|, \tag{11}$$

and all information for $E_D(t)$ can be read-off from the corresponding information for $Q_D(t)$. The following is a refinement of the observation that if $D$ has infinite volume then at most one of $Q_D(t), E_D(t)$ may be finite.

**Theorem 1.** Suppose $M$ is a smooth, geodesically complete, and stochastically complete Riemannian manifold. Let $D$ be an open subset of $M$ with infinite volume, and let

$$t_D = \inf\{s > 0 : Q_D(s) < \infty\}. \tag{12}$$
Then precisely one of (i)–(iii) holds:

(i) $t_D = \infty$ and $E_D(t) = \infty$ for all $t > 0$.

(ii) $t_D = \infty$ and there exists $t_1 > 0$, such that $E_D(t_1) < \infty$, and

$$E_D(t) \leq \left(1 + \frac{t}{t_1}\right) E_D(t_1), \quad t > 0.$$  \hfill (13)

(iii) $t_D < \infty$ and $E_D(t) = \infty$ for all $t > 0$.

An example of (ii) is contained in the following.

**Corollary 2.** Suppose $M$ is a smooth, geodesically complete Riemannian manifold with infinite measure. Suppose there exist $p \in M, C_1 < \infty, C_2 < \infty$, such that for all $r > 0$

$$|B(p; r)| \leq C_1 e^{C_2 r^2},$$  \hfill (14)

where $B(p; r)$ is the closed geodesic disc with centre $p$ and radius $r$. Let $K$ be compact and let $D = M \setminus K$. Suppose $v$ satisfies a uniform principle of not feeling the boundary: there exists $C < \infty$, such that

$$v(x; t) \leq Ce^{-\delta(x)^2/Ct}, \quad x \in D, t > 0,$$  \hfill (15)

where

$$\delta(x) = \min\{d(x, y) : y \in M \setminus D\},$$  \hfill (16)

and where $d$ denotes the Riemannian distance. Then (13) holds for any $t_1 \in (0, 1/(2CC_2)).$

We refer to [21] for a discussion of the validity of estimate (15).

The main result of this paper is an upper bound for $Q_D(t)$, assuming that the Dirichlet–Laplace–Beltrami operator acting in $L^2(D)$ satisfies a strong Hardy inequality in the following sense:

There exists $h : [0, \infty) \to [0, \infty)$ continuous, increasing and with $h(0) = 0$, such that for all $f \in C_c^\infty(D)$

$$\int_D |\nabla f|^2 \geq \int_D f^2/h(\delta).$$  \hfill (17)

The validity and applications of (17) have been investigated by many authors for a variety of different functions $h$ (see [12,14–16] and the references therein).
In [11], it was shown that if (17) holds with
\[ h(\delta) = c \delta^{\gamma} \]  
(18)
for positive constants \( c \) and \( \gamma \), and if \( \delta \in L^{\beta}(D) \) for some \( \beta \in (0, 2\gamma] \) then \( t_D = 0 \), and
\[ Q_D(t) \leq C t^{-\beta/\gamma}, \quad t > 0 \]  
(19)
for some constant \( C \) depending on \( c, \beta \) and on \( \gamma \).

The main result below does not assume power behaviour of \( h \) in terms of \( \delta \). Moreover, it gives some further information on the heat contained in a neighbourhood \( A \) of \( \partial D \).

**Theorem 3.** Suppose \( D \) is an open set in a smooth, geodesically complete Riemannian manifold, and suppose that (17) holds for some continuous, increasing \( h : [0, \infty) \to [0, \infty) \) with \( h(0) = 0 \).

(i) Suppose \( D \) has infinite measure, and suppose there exists \( \beta > 0 \) such that
\[ \int_D h(\delta)^\beta < \infty. \]  
(20)
Then for any Borel subset \( A \) of \( D \) with \( |D \setminus A| < \infty \)
\[ Q_D(t) \leq \frac{\beta + 1}{\beta} |D \setminus A| + (2\beta)^\beta \int_A (h(\delta)/t)^\beta, \quad t > 0. \]  
(21)
Moreover
\[ Q_D(t) \leq \beta^\beta \int_D (h(\delta)/t)^\beta, \quad t > 0, \]  
(22)
and
\[ Q_D(t) \leq \frac{\beta + 1}{\beta} |\{ x \in D : h(\delta(x)) > t \}| + (2\beta)^\beta \int_{\{ x \in D : h(\delta(x)) \leq t \}} dx \frac{(h(\delta(x))/t)^\beta}. \]  
(23)

(ii) Suppose \( D \) has finite measure. Then
\[ Q_D(t) \leq |D| - \frac{1}{2} |\{ x \in D : h(\delta(x)) \leq t \}|. \]  
(24)
Below we will show that the bounds obtained in (23) and (24) are sharp for a wide class of geometries.

**Corollary 4.** Suppose $M$ is a geodesically complete Riemannian manifold with non-negative Ricci curvature. Suppose $D$ is an open set in $M$ for which (17) holds with $h(\delta) = c\delta^2$ and some $c \in (0, \infty)$.

(i) If $D$ has infinite measure and for some $\alpha > 0$

\[ |\{ x \in D : \delta(x) > \varepsilon \}| \asymp \varepsilon^{-\alpha}, \quad \varepsilon \to 0, \quad (25) \]

then

\[ Q_D(t) \asymp t^{-\alpha/2}, \quad t \to 0. \quad (26) \]

(ii) If $D$ has finite measure and for some $d \in [m - 1, m)$

\[ |\{ x \in D : \delta(x) < \varepsilon \}| \asymp \varepsilon^{m-d}, \quad \varepsilon \to 0, \quad (27) \]

then

\[ |D| - Q_D(t) \asymp t^{(m-d)/2}, \quad t \to 0. \quad (28) \]

The remaining part of this paper is organized as follows: in Section 2 we prove Theorem 1 and Corollary 2. In Section 3 we prove Theorem 3 and Corollary 4, and in Section 4 we give two examples.

2. Proofs of Theorem 1 and Corollary 2

**Proof of Theorem 1.** It suffices to prove that if $E_D(t_1) < \infty$ for some $t_1 > 0$ then (13) holds. Let $p_D(x, y; t)$ and $p(x, y; t)$ denote the Dirichlet heat kernel for $D$, and the heat kernel for $M$, respectively. Recall that for $x \in D, t > 0$

\[ 0 \leq p_D(x, y; t) \leq p(x, y; t). \quad (29) \]

It is convenient to extend $p_D(x, y; t)$ to $M \times M \times [0, \infty)$ by putting $p_D(x, y; t) = 0$ if $x \in M \setminus D$ or $y \in M \setminus D$. By definition of stochastic completeness of $M$

\[ \int_M p(x, y; t) \, dy = 1, \quad x \in M, t > 0. \quad (30) \]
By (30) and the semigroup property we have that

$$u(x; t_1) - u(x; 2t_1) = \int_D dy \int_M dz p_D(x, y; t_1)(p(z, y; t_1) - p_D(z, y; t_1)).$$  \hfill (31)

The integrand in the right-hand side of (31) is non-negative by (29). By Fubini–Tonelli’s theorem and (29)–(31)

$$\int_D dx (u(x; t_1) - u(x; 2t_1))$$

$$\leq \int_D dy \int_M dz \int_M dx p(x, y; t_1)(p(z, y; t_1) - p_D(z, y; t_1))$$

$$= \int_D dy \int_M dz (p(z, y; t_1) - p_D(z, y; t_1))$$

$$= \int_D dy (1 - u(y; t_1)).$$  \hfill (32)

But the left-hand side of (32) equals $E_D(2t_1) - E_D(t_1)$, while the right-hand side equals $E_D(t_1)$. Hence $E_D(2t_1) \leq 2E_D(t_1)$. We proceed by induction to obtain that for $j \in \mathbb{N}$

$$E_D((j + 1)t_1) - E_D(jt_1)$$

$$= \int_D dx \int_D dy \int_M dz p_D(x, y; jt_1)(p(z, y; t_1) - p_D(z, y; t_1))$$

$$\leq \int_D dy \int_M dz \int_M dx p(x, y; jt_1)(p(z, y; t_1) - p_D(z, y; t_1))$$

$$= E_D(t_1).$$  \hfill (33)

Hence $E_D(jt_1) \leq jE_D(t_1)$. Suppose $t \in [jt_1, (j + 1)t_1)$ for some $j \in \mathbb{N}$. Then

$$E_D(t) \leq E_D((j + 1)t_1) \leq (j + 1)E_D(t_1) \leq \left(1 + \frac{t}{t_1}\right)E_D(t_1).$$  \hfill (34)

This proves (13). \hfill \Box

It is convenient to define for an open set $D$ in a Riemannian manifold $M$ $\mu : (0, \infty) \to [0, \infty]$ by

$$\mu(\varepsilon) = |\{x \in D : \delta(x) < \varepsilon\}|.$$  \hfill (35)
Proof of Corollary 2. Since $K$ is compact

$$\rho = \sup\{d(p, x) : x \in K\} < \infty. \quad (36)$$

By (14), (35) and (36)

$$\mu(\varepsilon) \leq |B(p; \rho + \varepsilon)| \leq C_1 e^{2C_2 \rho^2 + 2C_2 \varepsilon^2}. \quad (37)$$

By (15) and (37) we have for $t < 1/(2CC_2)$

$$E_D(t) \leq C\int e^{-\varepsilon^2/(Ct)} d\mu(\varepsilon) \leq CC_1 e^{2C_2 \rho^2} (1 - 2CC_2 t)^{-1}. \quad (38)$$

Since $M$ is geodesically complete, (14) implies stochastic completeness of $M$ by Theorems 1 in [18]. Corollary 2 follows by (38) and (13). $\square$

The $t$ dependence in estimate (13) of Theorem 1 can in general not be improved: for $K$ compact and non-polar in $\mathbb{R}^m, m \geq 3$

$$E_{\mathbb{R}^m \setminus K}(t) = C(K)t + o(t), \quad t \to \infty, \quad (39)$$

where $C(K)$ is the Newtonian capacity of $K$ [24].

3. Proofs of Theorem 3 and Corollary 4

Proof of Theorem 3. (i) Let $f : D \to [0, 1]$ be in $L^1(D) \cap L^\infty(D)$ with $\|f\|_\infty = 1$, and let $u_f : D \times [0, \infty) \to [0, 1]$ be the unique weak solution of (2), (4) with initial condition

$$u(x; 0) = f(x), \quad x \in D. \quad (40)$$

Then

$$u_f(x; t) = \int_D p_D(x, y; t) f(y) dy. \quad (41)$$

Since

$$\int_D p_D(x, y; t) dy \leq 1, \quad (42)$$
we have by (41)–(42) that
\[
0 < \int_D u_f(x; t) \, dx = \int_D \int_D p_D(x, y; t) f(y) \, dy \, dx \leq \|f\|_1 < \infty. \tag{43}
\]

Let \( p > 1 \) be a such that \( h(\delta)^{1/(p-1)} \in L^1(D) \). Then by Hölder’s inequality
\[
\int_D u_f^2 \leq \left( \int_D u_f^2 h(\delta)^{-1} \right)^{1/p} \left( \int_D h(\delta)^{1/(p-1)} \right)^{(p-1)/p}. \tag{44}
\]

Since
\[
u_f(x; t) \leq \int_D p_D(x, y; t) \|f\|_\infty \, dy \leq 1, \tag{45}\]
we have by Hardy’s inequality and (44) that
\[
\int_D u_f^2 \leq \left( \int_D \frac{\partial u_f}{\partial t} h(\delta)^{-1} \right)^{1/p} \left( \int_D h(\delta)^{1/(p-1)} \right)^{(p-1)/p} \leq \left( \int_D |\nabla u_f|^2 \right)^{1/p} \left( \int_D h(\delta)^{1/(p-1)} \right)^{(p-1)/p}. \tag{46}
\]

Since
\[-\frac{d}{dt} \int_D u_f(x; t)^2 \, dx = -2 \int_D u_f(x; t) \frac{\partial u_f}{\partial t}(x; t) \, dx = 2 \int_D |\nabla u_f|^2, \tag{47}\]
we have that (46) can be rewritten as
\[
\left( \int_D u_f^2 \right)^p \leq - \frac{1}{2} \frac{d}{dt} \left( \int_D u_f^2 \right) \left( \int_D h(\delta)^{1/(p-1)} \right)^{p-1}. \tag{48}\]

Hence
\[
1 \leq (2(p-1))^{-1} \frac{d}{dt} \left( \int_D u_f^2 \right)^{1-p} \left( \int_D h(\delta)^{1/(p-1)} \right)^{p-1}. \tag{49}\]

Integrating (49) with respect to \( t \) over the interval \([0, t]\) gives
\[
t \leq (2(p-1))^{-1} \left( \int_D u_f^2 \right)^{1-p} \left( \int_D h(\delta)^{1/(p-1)} \right)^{p-1}. \tag{50}\]
This implies that
\[
\int_D u^2 \leq (2(p-1)t)^{-1/(p-1)} \int_D h(\delta)^{1/(p-1)}.
\] (51)

An application of the monotone convergence theorem gives that
\[
\int_D u^2 \leq (2(p-1)t)^{-1/(p-1)} \int_D h(\delta)^{1/(p-1)}.
\] (52)

But the left-hand side of (52) equals \(Q_D(2t)\) by the semigroup property and Fubini’s theorem. We conclude, choosing \(p = (\beta + 1)/\beta\), that (22) holds.

To prove (21) we assume that \(|D\setminus A| < \infty\). Since \(|D| = \infty\) we have that \(|A| = \infty\). By (52)
\[
0 < \int_A u^2 < \infty.
\] (53)

Similarly to (46)–(48) we arrive at
\[
\left( \int_A u^2 \right)^p \leq -\frac{1}{2} \frac{d}{dt} \left( \int_D u^2 \right) \left( \int_A h(\delta)^{1/(p-1)} \right)^{p-1}.
\] (54)

Rewriting (54) gives
\[
1 \leq (2(p-1))^{-1} \frac{d}{d\tau} \left( \int_A u(x; \tau)^2 \, dx \right)^{1-p} \left( \int_A h(\delta)^{1/(p-1)} \right)^{p-1}
- \frac{1}{2} \left( \int_A u(x; \tau)^2 \, dx \right)^{-p} \frac{d}{d\tau} \left( \int_{D\setminus A} u(x; \tau)^2 \, dx \right) \left( \int_A h(\delta)^{1/(p-1)} \right)^{p-1}.
\] (55)

Since \(\tau \to u(x; \tau)\) is decreasing we have that
\[
\int_A u(x; \tau)^2 \, dx \geq \int_A u(x; t)^2 \, dx, \quad 0 < \tau \leq t,
\] (56)

and
\[
\frac{d}{d\tau} \int_{D\setminus A} u(x; \tau)^2 \, dx < 0, \quad \tau > 0.
\] (57)
Using (55)–(57) we obtain that

\[ t \leq \left( 2(p - 1) \right)^{-1} \left( \int_A u^2 \right)^{1-p} + 2^{-1} \left( \int_A u^2 \right)^{-p} |D \backslash A| \left( \int_A h(\delta)^{1/(p-1)} \right)^{p-1}. \]  

(58)

Put

\[ \int_A u^2 = B, \quad \int_A (h(\delta)/t)^{1/(p-1)} = C. \]  

(59)

Then (58) reads

\[ B^p \leq \left( 2(p - 1) \right)^{-1} BC^{p-1} + 2^{-1}|D \backslash A|C^{p-1}. \]  

(60)

If \( B > (p - 1)|D \backslash A| \), then by (60)

\[ B \leq (p - 1)^{-1/(p-1)}C. \]  

(61)

By (60)–(61)

\[ B \leq (p - 1)|D \backslash A| + (p - 1)^{-1/(p-1)}C, \]  

(62)

and, by (59) and (62)

\[ \int_D u^2 \leq p|D \backslash A| + (p - 1)^{-1/(p-1)} \int_A (h(\delta)/t)^{1/(p-1)}. \]  

(63)

This implies (21) by noting that the left-hand side of (63) equals \( Q_D(2t) \), and by choosing \( p = (\beta + 1)/\beta \). Finally, (23) follows by choosing \( A = \{ x \in D : h(\delta(x)) \leq t \} \).

To prove part (ii) of Theorem 3 we use an integration by parts and (17) to get that

\[ -\frac{d}{dt} Q_D(2t) = -2 \int_D u \frac{\partial u}{\partial t} = 2 \int_D |\nabla u|^2 \geq 2 \int_D u^2 / h(\delta). \]  

(64)

Integrating (64) with respect to \( t \) yields

\[ |D| - Q_D(2t) \geq 2 \int_D dx h(\delta(x))^{-1} \int_0^t u(x; \tau)^2 d\tau. \]  

(65)

Since \( u(x; \tau) \geq u(x; t), 0 \leq \tau \leq t \) inequality (65) gives

\[ |D| - Q_D(2t) \geq 2t \int_D h(\delta)^{-1} u^2 \geq \int_{\{ x \in D : h(\delta(x)) \leq 2t \}} u(x; t)^2 dx. \]  

(66)
By (66)
\[
\int_{\{x \in D : h(\delta(x)) \leq 2t\}} u(x ; t) \, dx \leq \frac{1}{2} \left( |D| - \int_{\{x \in D : h(\delta(x)) > 2t\}} u(x ; t)^2 \, dx \right)
\]  
(67)
and
\[
\int_{D} u^2 \leq |D| - \frac{1}{2} \left( |D| - \int_{\{x \in D : h(\delta(x)) > 2t\}} u(x ; t)^2 \, dx \right) .
\]  
(68)

Since \( u(x ; t) \leq 1 \) we have by (68) that
\[
\int_{D} u^2 \leq |D| - \frac{1}{2} |\{x \in D : h(\delta(x)) \leq 2t\}| .
\]  
(69)

This completes the proof of Theorem 3. \( \square \)

In order to prove Corollary 4 we will need a lower bound on the solution \( u \) of (2)–(4) and on \( Q_D(t) \).

**Lemma 5.** Let \( M \) be a smooth, geodesically complete Riemannian manifold with non-negative Ricci curvature, and let \( D \) be an open subset of \( M \) with boundary \( \partial D \). Then for \( x \in D, t > 0 \)
\[
u(x ; t) \geq 1 - 2^{(2+m)/2} e^{-\delta(x)^2/(8t)}.\]  
(70)
If \( D \) has infinite measure, then
\[
Q_D(t) \geq 2^{-1} |\{x \in D : \delta(x) \geq 2((m + 4)t \log 2)^{1/2}\}| .
\]  
(71)
If \( D \) has finite measure, then
\[
|D| - Q_D(t) \geq 2^{(m-2)/2} t^{-1} \int_{0}^{\infty} \varepsilon e^{-\varepsilon^2/(8t)} \mu(\varepsilon) \, d\varepsilon.
\]  
(72)

**Proof.** Since \( M \) is geodesically complete and since \( M \) has non-negative Ricci curvature, \( M \) satisfies a relative Faber–Krahn inequality [19]. By Proposition 5.2 in [20] the volume function \( (x, r) \rightarrow |B(x; r)|, x \in M, r > 0 \) satisfies the doubling property. Then \( M \) is stochastically complete by Theorem 1 in [18]. Let \( (B(s), s \geq 0; \mathbb{P}_x, x \in M) \) be a Brownian motion on \( M \) with generator \( \Delta \). Since \( M \) is stochastically complete, the lifetime of a Brownian path on \( M \) is infinite. Define the first exit time of \( D \) by
\[
T_D = \inf\{s \geq 0 : B(s) \in M \setminus D\}.
\]  
(73)
Then, for \( x \in D, t > 0 \)

\[
u(x; t) = \mathbb{P}_x[T_D > t] \geq \mathbb{P}_x[T_{B_0(x; \delta(x))} > t], \tag{74}\]

where \( B_0(x; r) \) is the interior of \( B(x; r) \). Since \( M \) has non-negative Ricci curvature the drift of the radial process on \( M \) is bounded from above by the drift of the radial process on a flat manifold. Therefore, there is a pathwise comparison of the radial process on \( M \) and the radial process on \( \mathbb{R}^m \). In particular the first exit time of a geodesic disc in \( M \) is bounded from below by the first exit time of the corresponding disc in \( \mathbb{R}^m \). For details we refer to Theorem 3.5.3 of Hsu \[22\]. Thus, if \( (\beta(s), s \geq 0; \mathbb{P}_\xi, \xi \in \mathbb{R}^m) \) is a Brownian motion on \( \mathbb{R}^m \) with the flat Laplacian as generator, then

\[
\mathbb{P}_x[T_{B_0(x; \delta(x))} > t] \geq \mathbb{P}_0[\inf\{s \geq 0 : |\beta(s)| \geq \delta(x)\} > t]. \tag{75}\]

The proof of (70) is, by (73)–(75), complete if we can show that

\[
\mathbb{P}_0[\inf\{s \geq 0 : |\beta(s)| \geq \delta(x)\} > t] \geq 1 - 2^{(2+m)/2} e^{-\delta(x)^2/(8t)}. \tag{76}\]

But this is precisely Corollary 6.4 in \[8\]. By (70) we have that \( u(x; t) \geq 1/2 \) on the set \( \{x \in D : \delta(x) \geq 2((m+4)t \log 2)^{1/2}\} \). This proves (71). If \( D \) has finite volume then (72) follows from (70) and definition (35). \( \square \)

**Proof of Corollary 4.** The lower bound in (26) follows directly from (71) and (25). Similarly, the upper bound in (28) follows from (72) and (27). To prove the upper bound in (26) we note that, by (25), (20) holds for all \( \beta > \alpha/2 \). Let \( \beta > \alpha/2 \) be arbitrary. By (25)

\[
|[x \in D : \delta(x) > (ct)^{1/2}]| \asymp t^{-3/2}, \tag{77}\]

and

\[
\int_{\{x \in D : \delta(x) \leq (ct)^{1/2}\}} (\delta(x)^2/(ct))^\beta \, dx \asymp t^{-2/2}. \tag{78}\]

The upper bound in (26) follows from (77), (78) and Theorem 3(i). Finally, the lower bound in (28) follows by (27) and Theorem 3(ii). \( \square \)

Note that two sided bounds like (28) have been obtained in the Euclidean setting in Corollary 1.5 of \[2\] assuming (27) and a uniform capacitary density condition.

Estimate (70) has been proved in the Euclidean setting using probabilistic methods with constant \( 2^{(m+2)/2} \) (\[8, Corollary 6.4\] and \[2, p. 444\]).
4. Examples

Example 6. Let $1 < a < 2$ and define in polar coordinates the spiny urchin in $\mathbb{R}^2$ by

$$U_a = \bigcup_{k \in \mathbb{N}} \bigcup_{j=1}^{2^{k+1}} \{(r, \theta) : r \geq a^k, \theta = \pi j 2^{-k}\}. \quad (79)$$

Then $\mathbb{R}^2 \setminus U_a$ is an open set with boundary $U_a$. Since $D_a = \mathbb{R}^2 \setminus U_a$ is simply connected (18) holds with $\gamma = 2, c = 16 \ [1]$. Moreover (25) holds with

$$\alpha = \frac{2 \log a}{\log 2 - \log a}. \quad (80)$$

By Corollary 4 we conclude that for $1 < a < 2$

$$Q_{D_a}(t) \asymp t^{-\frac{\log a}{\log 2 - \log a}}. \quad (81)$$

The following illustrates the limitations of Theorem 3(i) and provides an example where $0 < t_D < \infty$.

Example 7. Let $u_{D_a}$ be the solution of (2)–(4) with $D = D_a$, where

$$D_a = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < \pi(\log(3 + x_1) + a \log \log(3 + x_1))^{-1/2}\}, \quad (82)$$

and $a > 0$. Using Lemmas 5.7 and 5.8 of [8] one can show that for $q \geq 1$

$$u_{D_a} \in L^q(D_a) \text{ if and only if } t > q^{-1} \text{ or } t = q^{-1} \text{ and } a > q/2. \quad (83)$$

We see from (83) that $t_{D_a} = 1$ and that the integrability properties of $u_{D_a}$ improve as $t$ increases on $(0, t_{D_a})$. Theorem 3(i) fails to give non-trivial upper bounds since $\delta^{2\beta}$ fails to be in $L^1(D_a)$ for any $\beta > 0$. However, one can use Theorem 5.4 of [8] to see that $Q_{D_a}(1) < \infty$ for $a > \frac{1}{2}$, and that for $0 < a \leq 1/2$ and $t \downarrow 1$

$$Q_{D_a}(t) = \begin{cases} 8 \pi^{-2} \Gamma(\frac{1}{2} - a)(t - 1)^{a - \frac{1}{2}}(1 + o(1)), & 0 < a < \frac{1}{2}, \\ 8 \pi^{-2}(\log \frac{1}{t-1})(1 + o(1)), & a = \frac{1}{2}. \end{cases} \quad (84)$$

While we do not have a statement like (83) for arbitrary open sets $D$ with $0 < t_D < \infty$ the result below provides a bound in one direction and shows that $u \in L^n(D)$ implies $t_D \leq nt$.

**Proposition 8.** Let $D$ be an open set in the complete Riemannian manifold $M$, and let $u$ be the solution of (2)–(4). Then

$$Q_D(2t) = \|u\|_2^2. \quad (85)$$
Suppose $n \in \mathbb{N}$, $n \geq 3$. Then

$$Q_D(nt) \leq c_n \|u\|_n^n,$$

where

$$c_n = 2^{n-2}(n-1)^{-1} \prod_{j=1}^{n-2} j^{-1/(j+1)}. \quad (87)$$

**Proof.** Equality (85) follows directly by the semigroup property of $p_D(x, y; t)$. To prove (86) we use the semigroup property and obtain that

$$Q_D(nt) = \int_D \cdots \int_D dx_0 \cdots dx_n \prod_{j=1}^n p_D(x_{j-1}, x_j; t)$$

$$= \int_D \cdots \int_D dx_1 \cdots dx_{n-1} u(x_1; t) u(x_{n-1}; t) \prod_{j=2}^{n-1} p_D(x_{j-1}, x_j; t)$$

$$\leq \frac{1}{2} \int_D \cdots \int_D dx_1 \cdots dx_{n-1} (u(x_1; t)^2 + u(x_{n-1}; t)^2) \prod_{j=2}^{n-1} p_D(x_{j-1}, x_j; t)$$

$$= \int_D \cdots \int_D dx_1 \cdots dx_{n-1} u(x_{n-1}; t)^2 \prod_{j=2}^{n-1} p_D(x_{j-1}, x_j; t). \quad (88)$$

If $n = 3$ then the right-hand side of (88) equals $\|u\|_3^3$. If $n > 3$ then we proceed by induction. Using the inequality

$$ab^n \leq \frac{n}{n+1} n^{-1/(n+1)} (a^{n+1} + b^{n+1}), \quad a > 0, b > 0, \quad (89)$$

we obtain that

$$Q_D(nt) \leq \prod_{j=1}^{n-2} \left( \frac{2j}{j+1} j^{-1/(j+1)} \right) \|u\|_n^n. \quad (90)$$

**Acknowledgements**

It is a pleasure to thank A. Grigor’yan and E.P. Hsu for helpful discussions.
References