Weakly shod algebras

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Abstract

We study the class of algebras having a path of nonisomorphisms from an indecomposable injective module to an indecomposable projective module and any such path is bounded by a fixed number. We show that such an algebra is an iteration of one-point extensions starting at a product of tilted algebras. This allows us to describe, for instance, its Auslander–Reiten quiver.

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There are several important classes of algebras where there are bounds in the lengths of paths from indecomposable injective modules to indecomposable projective ones. This is true, for instance, for the class of quasitilted algebras, introduced in [9] to obtain a common treatment of both the classes of tilted algebras [10] and of canonical algebras [14]. This class has been a central object of investigation in the representation theory of finite-dimensional algebras.

Another example of such algebras is the so-called shod algebras, introduced in [5] with the aim of extending some results on quasitilted algebras, specifically the existence of a trisection in their module categories. Following [9], denote by $\mathcal{L}_A$ (or by $\mathcal{R}_A$) the full sub-category of mod\,$A$ consisting of all indecomposable $A$-modules whose predecessors (or successors) have projective dimension (or injective dimension, respectively) at most one. A shod algebra $A$ can be characterized by the property that each indecomposable $A$-module lies in one of the classes $\mathcal{L}_A$ or $\mathcal{R}_A$.

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In the present work, we shall study the class of algebras having a path of nonisomorphisms from an indecomposable injective module to an indecomposable projective module and any such path is bounded by a fixed number, and give a description of those which are not quasitilted in terms of tilted algebras and iteration of one-point extensions. To be more precise, we say that an algebra $A$ is weakly shod provided there exists a positive integer $n_0 \geq 0$ such that any path of nonzero nonisomorphisms from an indecomposable injective $A$-module to an indecomposable projective $A$-module has length bounded by $n_0$.

One important ingredient in our study is the notion of pip-bounded component as introduced in [6]. We say that a nonsemiregular component $\Gamma$ of the Auslander–Reiten quiver $\Gamma_A$ of an algebra $A$ is pip-bounded provided there is an $n_0 \geq 0$ such that the length of any path of nonzero nonisomorphisms from an indecomposable injective $A$-module in $\Gamma$ to an indecomposable projective $A$-module in $\Gamma$ is bounded by $n_0$. It was shown in [6] that a pip-bounded component is generalized standard and has no oriented cycles. Now, if $A$ is weakly shod, then clearly any nonsemiregular component is pip-bounded. Our main result can be resumed as follows. Denote by $\mathcal{P}_f^f A$ the set of all indecomposable projective $A$-modules $P$ such that there exists a path of nonzero morphisms from an indecomposable injective to $P$.

**Theorem.** Let $A$ be a connected weakly shod algebra with $\mathcal{P}_f^f A \neq \emptyset$. Then there are algebras $B = A_1, \ldots, A_0 = A$ and $A_i$-modules $M_i$ for each $i = 1, \ldots, t$ such that

(a) $B$ is a product of tilted algebras.

(b) $A_i = A_{i+1}[M_{i+1}]$ for each $i = 0, \ldots, t - 1$.

(c) For each $i = 0, \ldots, t - 1$, the extended projective $A_i$-module is a maximal element in $\mathcal{P}_f^f A_i$ with the order induced by the existence of paths between its elements.

As a consequence we get a description of the Auslander–Reiten quiver $\Gamma_A$ of a weakly shod algebra $A$ which is not quasitilted. The quiver $\Gamma_A$ has a unique distinguished component $\Gamma$ which is faithful and pip-bounded which plays a similar role in the Auslander–Reiten quiver $\Gamma_A$ of a weakly shod algebra $A$ as the connecting component of a tilted algebra. Such a component $\Gamma$ divides $\Gamma_A$ into two parts which connect each other through $\Gamma$. The other components of $\Gamma_A$ are components of tilted algebras, whose description is well known (see Section 5 below).

The proof of these results will be given along the paper. In Section 1 we establish some notations and recall results from [6] concerning pip-bounded components. As in the quasitilted and shod cases, we shall consider the two special subcategories $\mathcal{L}_A$ and $\mathcal{R}_A$ of $\text{ind} \ A$. In Section 2 we will show that weakly shod algebras can be characterized by the properties:

(i) $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in $\text{ind} \ A$; and

(ii) none of the nonsemiregular components of $\Gamma_A$ has oriented cycles.

The study of the relations between the subcategories $\mathcal{L}_A$ and $\mathcal{R}_A$ and the quiver $\Gamma_A$ is an important feature in the proofs of our main results. This will mainly be done in Section 3. Section 4 contains the description of weakly shod algebras as an iteration of one-
point extensions starting from a tilted algebra, while in Section 5 we show the uniqueness and faithfulness of the pip-bounded component when the algebra is connected. In the last section we complete the proofs of our main results, give some consequences and exhibit an example.

The theorem above can be dualize using one point coextensions. However, we shall not discuss it here, the interested reader will not have much difficulty to do so.

Versions of these results for shod algebras were presented in the congresses of Bielefeld (ICRA 8,5 – 1998) and São Paulo (CRASP – 1999). During the preparation of this work, we have learned that Reiten and Skowroński have independently characterized the shod algebras in [13].

1. Preliminaries

1.1. Along Sections 1 to 3, all algebras are connected Artin algebras. From Section 4 on, when we deal with one-point extensions, we shall restrict our discussion to finite-dimensional $K$-algebras, where $K$ is a fixed algebraically closed field. For an algebra $A$, we denote by $\text{mod}\, A$ the category of all finitely generated left $A$-modules, and by $\text{ind}\, A$ a full subcategory of $\text{mod}\, A$ having as objects a full set of representatives of the isomorphism classes of indecomposable $A$-modules. Denote by $\text{rk}(K_0(A))$ the rank of the Grothendieck group of $A$, which equals the number of simple $A$-modules [1].

For $X, Y \in \text{ind}\, A$, denote by $\text{rad}_A(X, Y)$ the set of the morphisms $f : X \rightarrow Y$ which are not isomorphisms and by $\text{rad}_A^\infty(X, Y)$ the intersection of all powers $\text{rad}_A^i(X, Y), i \geq 1$, of $\text{rad}_A(X, Y)$. We indicate by $\text{rad}_A^\infty(\text{mod}\, A)$ the ideal in $\text{mod}\, A$ generated by all morphisms in $\text{rad}_A^\infty(X, Y)$ for some $X, Y \in \text{ind}\, A$.

1.2. For an algebra $A$, denote by $\Gamma_A$ its Auslander–Reiten quiver, by $\tau_A$ the Auslander–Reiten translation $\text{DTr}$ and by $\tau_A^{-1}$ its inverse. We say that an indecomposable $A$-module $X$ is right stable, left stable or stable provided $\tau^n_A X \neq 0$ for each $n \leq 0$, $n \geq 0$ and for any $n$, respectively. For a connected component $\Gamma$ of the Auslander–Reiten quiver $\Gamma_A$ of $A$, denote by $\Gamma$ the full subquiver of $\Gamma$ generated by the stable modules in $\Gamma$. For unexplained notions on representation theory, we refer the reader to [1].

1.3. Let $A$ be an algebra and let $X$ be an $A$-module. We shall denote by $\text{pd}_A X$ and by $\text{id}_A X$ the projective dimension and the injective dimension of $X$, respectively. We shall need the following results whose proofs can be found in [1]:

(i) $\text{pd}_A X \leq 1$ if and only if $\text{Hom}_A(I, \tau_A X) = 0$ for each indecomposable injective module $I$.

(ii) $\text{id}_A X \leq 1$ if and only if $\text{Hom}_A(\tau_A^{-1} X, P) = 0$ for each indecomposable projective module $P$.

Also, $\text{gl.dim}\, A$ will denote the global dimension of $A$, that is, the supremum of the set $\{\text{pd}_A X : X \in \text{mod}\, A\}$. 
1.4. Given two modules $X$, $Y$ in $\text{ind} \ A$, a path from $X$ to $Y$ of length $t$ in $\text{ind} \ A$ is a sequence

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{t-1}} X_{t-1} \xrightarrow{f_t} X_t = Y$$

($t \geq 0$) where all $X_i$ lie in $\text{ind} \ A$, and all $f_i$ are nonzero. We write in this case $X \leadsto Y$, and say that $X$ is a predecessor of $Y$, or that $Y$ is a successor of $X$. Observe that each indecomposable module is a predecessor and a successor of itself. When all morphisms $f_i$'s in the path (*) are irreducibles, then we say that (*) is a path of irreducibles or, simply, a path in $\Gamma_A$. A path in $\Gamma_A$ starting and ending at the same module is called an oriented cycle.

A hook in (*) is a $j$, $1 \leq j \leq t - 1$, such that

$$X_{j-1} \xrightarrow{f_j} X_j \xrightarrow{f_{j+1}} X_{j+1}$$

satisfies: (i) $f_j$ and $f_{j+1}$ are irreducible maps; and (ii) $\tau_A X_{j+1} = X_{j-1}$. A path of irreducible maps without hooks is called a sectional path. A refinement of (*) is a path

$$X = Z_0 \xrightarrow{g_1} Z_1 \xrightarrow{g_2} \cdots \xrightarrow{g_u-1} Z_{u-1} \xrightarrow{g_u} Z_u = Y$$

in $\text{ind} \ A$ from $X$ to $Y$ such that there exists an order-preserving function $\sigma$ from $\{1, \ldots, t-1\}$ to $\{1, \ldots, u-1\}$ such that $X_i \cong Z_{\sigma(i)}$ for each $1 \leq i \leq t - 1$.

1.5. We recall the following result from [15] which will be very useful lately.

**Theorem.** Let $A$ be an Artin algebra and let $f$ be a nonzero morphism in $\text{rad}_A^\infty(X, Y)$. Then, for each $t \geq 1$,

(a) there exists a path

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} X_t \xrightarrow{g_t} Y$$

where $f_1, \ldots, f_t$ are irreducible maps and $g_t \in \text{rad}_A^\infty(X_t, Y)$;

(b) there exists a path

$$X \xrightarrow{g'_t} Y_t \xrightarrow{f'_t} Y_{t-1} \xrightarrow{f'_{t-1}} \cdots \xrightarrow{f'_1} Y_1 \xrightarrow{f'_1} Y_0 = Y$$

where $f'_1, \ldots, f'_t$ are irreducible maps and $g'_t \in \text{rad}_A^\infty(X, Y_t)$.

1.6. We shall now recall some results from [6] concerning an special class of components of the Auslander–Reiten quiver of an algebra. Recall first that a component $\Gamma$ of $\Gamma_A$ for an algebra $A$ is generalized standard if $\text{rad}_A^\infty(X, Y) = 0$, for all $X, Y \in \Gamma$. By [16, (2.3)], any generalized standard component has only finitely many $\tau_A$-orbits not containing modules lying in oriented cycles.
Theorem. Let $A$ be an algebra and let $\Gamma$ be a nonsemiregular component of $\Gamma_A$.

(1) The following statements are equivalent:
(a) There exists an $m_0$ such that any path in $\text{ind} A$ from an injective module in $\Gamma$ to a projective module in $\Gamma$ pass through at most $m_0$ hooks.
(b) There exists an $n_0$ such that any path in $\text{ind} A$ from an injective module in $\Gamma$ to a projective module in $\Gamma$ has length at most $n_0$.
(c) Given $X, Y \in \Gamma$, there exists an $m_1 = m_1(X, Y)$ such that any path in $\text{ind} A$ from $X$ to $Y$ pass through at most $m_1$ hooks.
(d) Given $X, Y \in \Gamma$, there exists an $n_1 = n_1(X, Y)$ such that any path in $\text{ind} A$ from $X$ to $Y$ has length at most $n_1$.

(2) If $\Gamma$ satisfies one of the above conditions, then:
(a) $\Gamma$ is generalized standard and has no oriented cycles.
(b) Given $X, Y \in \Gamma$, no path in $\text{ind} A$ from $X$ to $Y$ pass through a morphism in $\text{rad}^\infty(\text{mod} A)$.

Proof. (1) The proof of the equivalence between (a) and (b) can be found in [6, (4.1)]. The implications (c) $\Rightarrow$ (a) and (d) $\Rightarrow$ (c) are easy to see. Then the result will follow once we prove (b) $\Rightarrow$ (d).

Let $X, Y \in \Gamma$. Assume that the condition (b) holds and suppose there exist paths of arbitrary length from $X$ to $Y$.

It follows from [6, (3.4), (3.5)] that any component satisfying condition (1)(b) has only finitely many $\tau_A$-orbits and no oriented cycles. Hence, we infer that there are modules $Z_1$ and $Z_2$ such that for each $n \geq 0$, there is a path $X \xrightarrow{\Delta_n} \tau_A^{-n} Z_1 \xrightarrow{\Delta_n^+} \tau_A^n Z_2 \xrightarrow{\Delta''_n} Y$ where $\Delta_n^+$ and $\Delta''_n$ are paths in $\Gamma_A$ and $\Delta_n$ is a path in $\text{ind} A$ of length greater than $n_0$.

Observe that, in particular, $Z_1$ is right stable while $Z_2$ is left stable. Since $\Gamma$ has injective modules, let

$$I = U_0 - U_1 - \cdots - U_r = Z_1'$$

be a walk of minimal length between an injective $I \in \Gamma$ and a module $Z_1'$ in the $\tau_A$-orbit of $Z_1$. By the minimality property, we infer that $U_1, \ldots, U_r$ are right stable modules. So, by applying $\tau_A^{-1}$, we get a path from $I$ to a module in the $\tau_A$-orbit of $Z_1$, let us say $\tau_A^n Z_1$. Using a dual argument, one can show that there exists an $l_2$ such that $\tau_A^{l_2} Z_2$ is a predecessor in $\Gamma_A$ of a projective. If $l = \max\{|l_1|, |l_2|\}$, we get a path from an injective in $\Gamma$ to a projective in $\Gamma$ containing a subpath $\Delta_l$ with length greater than $n_0$, a contradiction to (1)(b).

(2) It has been proven in [6, (4.3), (3.4)] that condition (1)(b) implies (2)(a).
Suppose now that there exists a path $X \hookrightarrow Y$ in $\text{ind} A$ passing through a morphism in $\text{rad}^{\infty}(\text{mod} A)$. So, using Section 1.5, for each $t \geq 1$, there exists a path

$$X = Y_0 \to Y_1 \to \cdots \to Y_t \to Z$$

in $\text{ind} A$, contradicting (1)(d), and the result is proven. $\square$

1.7. Recall the following definition from [6].

**Definition.** Let $A$ be an Artin algebra. A nonsemiregular component $\Gamma$ of $\Gamma_A$ satisfying one of the equivalent conditions (1)(a)–(1)(d) in the above theorem is called **pip-bounded component**.

1.8. **Examples.** (a) Let $A$ be the $K$-algebra given by the following quiver $\Delta$:

\[
\begin{array}{cccccc}
1 & \alpha_1 & 2 & \beta & 3 & \gamma & 4 & \delta_1 & 5 & \delta_2 & 6 \\
\end{array}
\]

with $\alpha_1 \beta = \gamma \delta_1 = 0$, for $i = 1, 2$. The Auslander–Reiten quiver $\Gamma_A$ of $A$ consists of:

(i) the postprojective component and the family of orthogonal homogeneous tubes corresponding to the Kronecker algebra given by the full subquiver of $\Delta$ containing only the vertices 1 and 2;

(ii) the preinjective component and the family of orthogonal homogeneous tubes corresponding to the Kronecker algebra given by the full subquiver of $\Delta$ containing only the vertices 5 and 6; and

(iii) a component $\Gamma$ with the following shape:

![Diagram]

Observe that here the length of any path from an indecomposable injective to an indecomposable projective is bounded by 8.
(b) It is not difficult to construct examples of algebras with more than one pip-bounded component. Let $A$ be the $K$-algebra given by the following quiver $\Delta$:

```
\begin{tikzpicture}
\node (1) at (0,0) {$1$};
\node (2) at (1,0) {$2$};
\node (3) at (2,0) {$3$};
\node (4) at (3,0) {$4$};
\node (5) at (4,0) {$5$};
\node (6) at (5,0) {$6$};
\node (7) at (0,-1) {$7$};
\node (8) at (1,-1) {$8$};
\node (9) at (2,-1) {$9$};
\node (10) at (3,-1) {$10$};
\node (11) at (4,-1) {$11$};
\node (12) at (5,-1) {$12$};
\node (13) at (6,0) {$13$};
\draw (1) edge (2);
\draw (2) edge (3);
\draw (3) edge (4);
\draw (4) edge (5);
\draw (5) edge (6);
\draw (7) edge (8);
\draw (8) edge (9);
\draw (9) edge (10);
\draw (10) edge (11);
\draw (11) edge (12);
\draw (12) edge (13);
\end{tikzpicture}
```

with $\alpha_i\beta_1 = \gamma_1\delta_i = \delta_i\epsilon_1 = 0$ for $i = 1, 2$ and $\alpha_j\beta_2 = \gamma_2\delta_j = \delta_j\epsilon_2 = 0$ for $j = 3, 4$. The Auslander–Reiten quiver of $A$ contains two components like the one described in the item (iii) of the previous example. Unlike there, here we have paths from injective modules to projective modules of arbitrary length (for instance, from any of the injectives $I_2$ or $I_8$ to the projective $P_{13}$).

1.9. For later reference, we recall the following results proven in [7].

**Lemma.** Let $A$ be an Artin algebra and let $n$ be the rank of the Grothendieck group $K_0(A)$ of $A$. Let $\Gamma'$ be a component of $\Gamma_A$ and $\Gamma''$ be a connected component of the stable part $\overset{\cdot}{\Gamma}$ of $\Gamma$. Assume that $\Gamma''$ has infinitely many $\tau_A$-orbits and has no oriented cycles. Let $M$ be a module in $\Gamma''$ such that the length of any walk in $\Gamma$ from a nonstable module to $M$ is at least $2n$. Then, for each $r \geq 1$, there exists a path from $M$ to $\tau_r^A M$ passing through modules of $\Gamma$.

1.10. **Corollary.** Let $A$ be an Artin algebra and let $\Gamma'$ be a regular component of $\Gamma_A$ with infinitely many $\tau_A$-orbits. Then, for each $M \in \Gamma'$ and each $r \geq 1$, there exists a path in $\text{ind} A$ from $M$ to $\tau_r^A M$.

1.11. We shall also need the following result from [6].

**Lemma.** Let $A$ be an Artin algebra, let $\Gamma$ be a component of $\Gamma_A$ and let $M, N \in \Gamma$. Then

(a) there exists an integer $n$ such that $\tau_A^n N$ is a successor (by irreducible maps) of $M$ or is a successor (by irreducible maps) of an injective;
(b) there exists an integer $m$ such that $\tau_A^m N$ is a predecessor (by irreducible maps) of $M$ or is a predecessor (by irreducible maps) of a projective.

1.12. We shall end this section by recalling the notions of tilted and quasitilted algebras. The class of tilted algebras was introduced in [10]. For a given algebra $A$, an $A$-module $T$ is called tilting provided: (i) $\text{pd}_A T \leq 1$; (ii) $\text{Ext}_A^1(T, T) = 0$; and (iii) there exists a short exact sequence $0 \to A \to T' \to T'' \to 0$, where $T', T'' \in \text{add} T$, the additive full subcategory of $\text{mod} A$ generated by the indecomposable summands of $T$. There is a nice relation between the categories $\text{mod} A$ and $\text{mod}(\text{End}_A T)$ when $T$ is a tilting $A$-module,
given by the so-called tilting functors (see [2,10]). In the particular case where $A$ is hereditary, the endomorphism algebra $\text{End}_A T$ is called tilted. One of the striking features of a tilted algebra is the existence of a component, called connecting, in its Auslander–Reiten quiver containing a complete slice (see [10] for details). This component is unique unless the tilted algebra is concealed, that is, the tilting module it is defined from is either postprojective or preinjective. In this case, there are exactly two connecting components (a postprojective and a preinjective component).

1.13. Quasitilted algebras were introduced in [9] in order to give a more general approach to the tilting theory including not only the tilted algebras but also the canonical algebras. We shall not give here full definitions but quasitilted algebras are endomorphism algebras of tilting objects over locally finite hereditary Abelian $R$-categories, where $R$ is a commutative Artin ring. Recently, all such categories having tilting objects were classified by Happel in [8]. A quasitilted algebra $A$ can also be characterized by the properties: (i) $\text{gl.dim} A \leq 2$; and (ii) for each indecomposable $A$-module $X$, $\text{pd}_A X \leq 1$ or $\text{id}_A X \leq 1$.

2. Weakly shod algebras

2.1. In order to extend some results proven for quasitilted algebras, we have introduced in [5] the class of shod algebras, which we shall now recall. Let $A$ be an algebra. Following [9], denote by $\mathcal{L}_A$ and $\mathcal{R}_A$ the following subcategories of $\text{ind} A$:

$$\mathcal{L}_A = \{X \in \text{ind} A: \text{pd}_A Y \leq 1 \text{ for each predecessor } Y \text{ of } X\},$$

$$\mathcal{R}_A = \{X \in \text{ind} A: \text{id}_A Y \leq 1 \text{ for each successor } Y \text{ of } X\}.$$

We recall the following result from [5].

**Theorem.** The following statements are equivalent for an algebra $A$:

(a) For each indecomposable $A$-module $X$, $\text{pd}_A X \leq 1$ or $\text{id}_A X \leq 1$.

(b) $\mathcal{L}_A \cup \mathcal{R}_A = \text{ind} A$.

(c) Any path from an indecomposable injective module to an indecomposable projective module can be refined to a path of irreducible maps and any such refinement has at most two hooks, and, in case there are two, they are consecutive.

Moreover, for such an algebra,

$$\text{Hom}_A(\mathcal{R}_A, \mathcal{L}_A \setminus \mathcal{R}_A) = 0 = \text{Hom}_A(\mathcal{R}_A \setminus \mathcal{L}_A, \mathcal{L}_A \cap \mathcal{R}_A).$$

An algebra satisfying one of the properties in the above theorem is called shod (for small homological dimension). It follows from [9, II.1.1] that a shod algebra has global dimension at most 3 and, clearly, a quasitilted algebra is a shod algebra of global dimension.
at most 2. Also, a shod algebra of global dimension 3 is called strict shod. Observe that the above theorem shows that there exists a trisection in ind $A$ given by

$$\text{ind } A = (\mathcal{L}_A \setminus \mathcal{R}_A) \lor (\mathcal{L}_A \cap \mathcal{R}_A) \lor (\mathcal{R}_A \setminus \mathcal{L}_A)$$

with nonzero morphisms going only from left to right. This trisection was first established for quasitilted algebras in [9].

2.2. Here, we are interested in a larger class of algebras, namely those $A$ such that the length of any path in ind $A$ from an injective module to a projective module is bounded by a fixed number. We shall use the results of [6] mentioned in the last section.

2.3. Definition. An algebra $A$ is called weakly shod provided there exists a positive integer $n_0$ such that any path in ind $A$ from an injective $A$-module to a projective $A$-module has length at most $n_0$.

2.4. Remarks.
(a) Let $A$ be a weakly shod algebra. Then any path in ind $A$ from an injective module to a projective module can be refined to a path in $\Gamma_A$. Indeed, suppose $I \xrightarrow{(\ast)} P$ is a path in ind $A$ from an injective $I$ to a projective $P$. Clearly, the path $(\ast)$ does not pass through a morphism in rad$^\infty(\text{mod } A)$, otherwise it could be extended indefinitely (using Section 1.5), contradicting the hypothesis on $A$. Hence $(\ast)$ can be refined to a path of irreducible morphisms as claimed. In particular, the modules $I$ and $P$ lie in a same component of $\Gamma_A$, which is clearly a pip-bounded component.

(b) Let $A$ be a connected representation-finite algebra. By [1, VII.2.1], $\Gamma_A$ is connected and clearly finite. Then $A$ will be weakly shod if and only if $\Gamma_A$ is a pip-bounded component which is equivalent, in this case, to $\Gamma_A$ having no oriented cycles (see Section 1.6). Hence $A$ is weakly shod if and only if $A$ is a directed algebra.

2.5. The next result relates weakly shod algebras and informations in the union $\mathcal{L}_A \cup \mathcal{R}_A$.

Theorem. The following statements are equivalent for an algebra $A$:

(a) $A$ is weakly shod;
(b) (i) $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in ind $A$; and
(ii) none of the nonsemiregular components of $\Gamma_A$ has oriented cycles.

Moreover, in this case, any indecomposable module not belonging to $\mathcal{L}_A \cup \mathcal{R}_A$ lies in a pip-bounded component.

Proof. Let $X \in \text{ind } A$ and suppose $X \notin \mathcal{L}_A \cup \mathcal{R}_A$. We claim that there exists a path in ind $A$ from an injective to a projective passing through $X$. Indeed, since $X \notin \mathcal{L}_A$, there exists a path $Y \xrightarrow{(\ast)} X$ in ind $A$ such that $\text{pd}_A Y \geq 2$. By Section 1.3, we know that there exists an
indecomposable injective $A$-module $I$ and a nonzero morphism $f : I \to \tau_A Y$. Also, since $X \not\in R_A$, there exists a path $X \xrightarrow{(\alpha)} Z$ in $\text{ind} A$ such that $\text{id}_A Z \geq 2$. By Section 1.3, there exists an indecomposable projective $A$-module $P$ and a nonzero morphism $g : \tau_A^{-1} Z \to P$.

Therefore,

$$I \xrightarrow{f} \tau_A Y \xrightarrow{\alpha} X \xrightarrow{(\alpha)} Z \xrightarrow{\tau_A^{-1} \circ g} P$$

gives the required path and the claim is proven.

(a) $\Rightarrow$ (b) Suppose that $A$ is weakly shod. By Remarks 2.4, we know that each path in $\text{ind} A$ from an injective module to a projective module can be refined to a path of irreducible maps. We then infer that there are only finitely many indecomposable modules lying in paths in $\text{ind} A$ from an injective module to a projective one. Therefore, $L_A \cup R_A$ is cofinite in $\text{ind} A$ and (i) is proven. Also, any module not lying in $L_A \cup R_A$ has to belong to a pip-bounded component by Remark 2.4.

To prove (ii) just observe that, since $A$ is weakly shod, then any nonsemiregular component of $\Gamma_A$ is pip-bounded and, therefore, without oriented cycles (by Section 1.6).

(b) $\Rightarrow$ (a) Suppose now that $L_A \cup R_A$ is cofinite in $\text{ind} A$ and that none of the non-semiregular components of $\Gamma_A$ has oriented cycles. Denote by $X = \text{ind} A \setminus (L_A \cup R_A)$ and suppose $|X| = r$. If $A$ is not weakly shod, there would exist an indecomposable injective $I$ and an indecomposable projective $P$ such that there are paths from $I$ to $P$ of arbitrary length. Then, for each $t \geq 1$, there exists a path $(\xi_t)$:

$$I = X_0 \xrightarrow{f_1} X_1 \to \cdots \xrightarrow{f_t} X_t \xrightarrow{\theta_t} Y_1 \xrightarrow{g_t} Y_{t-1} \to \cdots \to Y_1 \xrightarrow{g_1} Y_0 = P$$

where $f_1, \ldots, f_t, g_1, \ldots, g_t$ are irreducible morphisms and $(\theta_t)$ is a path in $\text{ind} A$ of length greater than $r + 1$. Denote $n = \text{rk}(k_0(A))$ and choose such a path $(\xi_t)$ for a $t > n^2 + 1$.

Claim. $X_t \not\in L_A$.

We shall show that there exists a module of projective dimension at least 2 which is a predecessor of $X_t$. Denote by $(\ast)$ the subpath of $(\xi_t)$ from $I$ to $X_t$. If $(\ast)$ is not sectional, there exists an $l$, $1 \leq l \leq t - 1$ which is a hook in $(\ast)$. Without loss of generality we can assume that there are no hooks in $\{1, \ldots, l - 1\}$. So, since $I = Z_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{l-1}} Z_{l-1}$ is sectional, we infer that $f_{l-1} \cdot \cdots \cdot f_1 \neq 0$ and then, by Section 1.3, $\text{pd}_A Z_{l+1} \geq 2$ and the claim is proven in this case. Suppose now that $(\ast)$ is sectional. In particular, the modules in $(\ast)$ are pairwise nonisomorphic (by [3]). Since $t > n^2$, we infer that there exists an $u < t - n$ such that the subpath

$$X_u \to X_{u+1} \to \cdots \to X_{u+n} \to X_{u+n+1} \quad (\ast\ast)$$

of $(\ast)$ has no injective modules. Let now $u \leq i \leq u + n + 1$. Since $\text{Hom}_A(I, X_i) \neq 0$ for each $i = 0, \ldots, t$, we infer that $\text{pd}_A(\tau_A^{-1} X_i) \geq 2$ (Section 1.3). It follows now from [16,
Lemma 2] that there are \( u \leq p, q \leq u + n + 1 \) such that \( \text{Hom}_A(\tau_A^{-1}X_p, X_q) \) has a nonzero map \( \alpha \). The path

\[
\tau_A^{-1}X_p \xrightarrow{\alpha} X_q \xrightarrow{f_{q+1}} X_{q+1} \rightarrow \cdots \rightarrow X_{r-1} \xrightarrow{f_i} X_i
\]

shows the claim.

By a dual argument, one can show that \( Y_t \notin \mathcal{R}_A \). In particular, the modules in the path (\( \theta_t \)) belong to \( \mathcal{X} \). Suppose now (\( \theta_t \)) does not pass through a morphism in \( \text{rad}^\omega(\text{mod} A) \). Then the path (\( \xi_t \)) can be refined to a path in \( \Gamma_A \) and then \( I \) and \( P \) belong to a same (nonsemiregular) component \( \Gamma \). By hypothesis, \( \Gamma \) has no oriented cycles and then, in the path (\( \theta_t \)), there are at least \( r + 1 \) nonisomorphic indecomposable modules belonging to \( \mathcal{X} \), a contradiction. In case (\( \theta_t \)) pass through a morphism \( h: Z' \rightarrow Z'' \) in \( \text{rad}^\omega(\text{mod} A) \), it yields that, for each \( s \geq 0 \), (\( \theta_t \)) can be refined to a path

\[
X_t \leadsto Z' = Z_0 \xrightarrow{h_1} Z_1 \rightarrow \cdots \rightarrow Z_s = Z'' \leadsto Y_t
\]

in ind \( A \), where \( h_s \cdots h_1 \neq 0 \). Denote by \( b = \max\{l(W) : W \in \mathcal{X}\} \). If \( s > 2^b + 1 \), then by the Harada–Sai lemma, there exists an \( j \) such that \( Z_j \notin \mathcal{X} \), again a contradiction and the result is proven. 

2.6. The next example shows that \( \mathcal{L}_A \cup \mathcal{R}_A \) being cofinite in ind \( A \) does not imply that \( A \) is weakly shod.

Example. Let \( K \) be a field and let \( A \) be the radical square zero \( K \)-algebra given by the quiver \( \Delta \):

```
     3
    / \  \
 2--1--3
    \  / \\
     4--5
```

The Auslander–Reiten quiver \( \Gamma_A \) of \( A \) consists of:

(i) the postprojective component and the family of orthogonal homogeneous tubes corresponding to the Kronecker algebra given by the full subquiver of \( \Delta \) containing only the vertices 1 and 2;

(ii) the preinjective component and the family of orthogonal homogeneous tubes corresponding to the Kronecker algebra given by the full subquiver of \( \Delta \) containing only the vertices 4 and 5; and
(iii) a component $\Gamma$ with the following shape:

where the two copies of $S_3$ are identified.

Observe that there are no morphisms from a component described in (ii) or $\Gamma$ to a component described in (i). So, there are no morphisms from injective modules to any of the components described in (i). In particular, those components are contained in $\mathcal{L}_A$. Dually, the components described in (ii) are contained in $\mathcal{R}_A$. Concerning the component $\Gamma$, it is not difficult to see that $\mathcal{L}_A$ contains all the modules in $\Gamma$ which are predecessors of $S_2$ and $\mathcal{R}_A$ contains all the modules in $\Gamma$ which are successors of $S_4$. We then infer that $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in ind $A$. On the other hand, it is not difficult to see that there are paths from $I_2$ to $P_4$ of arbitrary length since they both belong to a cycle passing through $S_3$.

3. On the subcategories $\mathcal{L}_A$ and $\mathcal{R}_A$

3.1. For a weakly shod algebra $A$, denote by $\mathcal{P}_A^f$ the set of all indecomposable projective $A$-modules $P$ such that there exists a path in ind $A$ from an injective module to $P$.

We are now interested in relating the subcategories $\mathcal{L}_A$ and $\mathcal{R}_A$ and the components of the Auslander–Reiten quiver $\Gamma_A$. It is not difficult to see that if $\mathcal{P}_A^f = \emptyset$ (or even if any path from an injective module to a projective module is sectional), then $A$ is quasitilted [9] and, in this case, such relation has been established in [7]. We shall now consider the general case and, as we shall see, many of the results in this section generalize those of [7].

3.2. We start with the following lemmata.

Lemma. Let $A$ be a weakly shod algebra and let $\Gamma$ be a component of $\Gamma_A$.

(a) If $\Gamma \cap \mathcal{R}_A \neq \emptyset$, then each right stable $\tau_A$-orbit of $\Gamma$ has an indecomposable module in $\mathcal{R}_A$.

(b) If $\Gamma \cap \mathcal{L}_A \neq \emptyset$, then each left stable $\tau_A$-orbit of $\Gamma$ has an indecomposable module in $\mathcal{L}_A$.

Proof. We shall prove only (a) since the proof of (b) is dual.

(a) Let $X \in \Gamma \cap \mathcal{R}_A$ and let $Y$ be a right stable module in $\Gamma$. It is not difficult to see that there exists an integer $m$ such that $\tau_A^m Y$ is a successor of $X$ or a successor of an indecomposable injective module (Section 1.11). In the former case, the result follows from the fact that $\mathcal{R}_A$ is closed under successors.
Suppose then that there is a path \( I^e \tau^n A Y \) in \( \text{ind} A \), where \( I \) is an injective. Now, since \( A \) is weakly shod, there exists an \( n_0 \) such that any path in \( \text{ind} A \) from an injective module to a projective module has length at most \( n_0 \). Clearly, the path

\[
I^e \tau^n A Y \sim \tau^{n-1} A Y \sim \tau^{n-2} A Y \sim \cdots \sim \tau^{m-n_0} A Y = Z
\]

has length greater than \( n_0 \). So, there is no successor of \( Z \) which is projective. In particular, all successors of \( Z \) have injective dimension at most one and then the \( \tau A \)-orbit of \( Y \) has a module in \( R_A \), as required. \( \blacksquare \)

3.3. Lemma. Let \( A \) be an algebra and let \( \Gamma \) be a component of \( \Gamma_A \).

(a) If \( \Gamma \) has projective modules, then each left stable \( \tau A \)-orbit of \( \Gamma \) has a module which does not lie in \( R_A \).

(b) If \( \Gamma \) has injective modules, then each right stable \( \tau A \)-orbit of \( \Gamma \) has a module which does not lie in \( L_A \).

Proof. We shall prove only (a) since the proof of (b) is dual.

(a) Let \( X' \in \text{ind} A \) lying in a left stable \( \tau A \)-orbit of \( \Gamma \). Since \( \Gamma \) is connected and contains a projective module, there exists a walk \( X' \sim \cdots \sim P \) of minimal length between a module \( X' \) in the \( \tau A \)-orbit of \( X' \) and a projective module \( P \) in \( \Gamma \). By minimality, each module in \( X' \sim \cdots \sim P \) is left stable. Hence, \( \tau A X' \) is a predecessor of the module \( \tau A X_t \) which has injective dimension greater than one by Section 1.3. In particular, \( \tau A X \notin R_A \), which proves the result. \( \blacksquare \)

3.4. Corollary. Let \( A \) be a weakly shod algebra and let \( \Gamma \) be a pip-bounded component of \( \Gamma_A \). Then \( \Gamma \cap L_A \neq \emptyset \) and \( \Gamma \cap R_A \neq \emptyset \).

Proof. If \( \Gamma \) is postprojective, then there exists a simple projective module lying in \( \Gamma \), which clearly belongs to \( L_A \). Suppose \( \Gamma \) is not postprojective. Then there exists a left stable module \( X \in \Gamma \). Since \( \Gamma \) has no oriented cycles, the set \( \{ \tau^i A X : i \geq 0 \} \) is infinite. By Lemma 3.3, there is an \( r \) such that \( \tau^r A X \notin R_A \). So, for each \( s \geq r \), \( \tau^s A X \notin R_A \). Since \( L_A \cup R_A \) is cofinite in \( \text{ind} A \), we infer that there exists an \( l \) such that \( \tau^l A X \in L_A \), as required. The proof that \( \Gamma \cap R_A \neq \emptyset \) is similar. \( \blacksquare \)

3.5. The next proposition together with Proposition 3.7 are the main results of this section.

Proposition. Let \( A \) be a weakly shod algebra and \( \Gamma \) be a component of \( \Gamma_A \) with finitely many \( \tau A \)-orbits and no oriented cycles.
(a) If $\Gamma$ is left stable and $\Gamma \cap R_A \neq \emptyset$, then $\Gamma \subset R_A$.
(b) If $\Gamma$ is right stable and $\Gamma \cap L_A \neq \emptyset$, then $\Gamma \subset L_A$.

Proof. We shall prove only (a) since the proof of (b) is dual.

(a) Suppose $\Gamma$ has a module $Y$ which does not lie in $R_A$. Then there is a path $Y \sim Y$ in ind $A$ from $Y$ to an indecomposable module $Z$ of injective dimension greater than 1. If $Z$ is not in $\Gamma$, we infer that ($\ast$) has a morphism in rad $A$. Using Section 1.5, given $t \geq 1$, there is a path

$$Y = Y_0 \to Y_1 \to \ldots \to Y_t \sim Z$$

in ind $A$, where $f_i$ is irreducible for $i = 1, \ldots, t$. Since $\Gamma$ has only finitely many $\tau_A$-orbits and no oriented cycles, there exists a right stable indecomposable module $M$ such that given $l \geq 0$, there exists a path from $\tau^{-l}_A M$ to $Z$. Therefore, no module in the $\tau_A$-orbit of $M$ lies in $R_A$, a contradiction to Section 3.2.

Suppose now $Z \in \Gamma$. So, there is a nonzero morphism from $\tau^{-1}_A Z$ to an indecomposable projective $P$. Clearly, such a morphism lies in rad $A$ because $\Gamma$ is left stable. Then, by Section 1.5, for each $t \geq 1$, there is a path

$$\tau^{-1}_A Z = Z_0 \to Z_1 \to \ldots \to Z_t \sim P$$

in ind $A$ where $f_1, \ldots, f_t$ are irreducible.

Again, since there are only finitely many $\tau_A$-orbits in $\Gamma$, we infer that there exists a module $N$ such that given $l \geq 0$, there is a module of injective dimension greater than one which is a successor of $\tau^{-l}_A N$ (Section 1.3). In particular, the $\tau_A$-orbit of $N$ has no modules in $R_A$, again a contradiction to Section 3.2. \hfill $\Box$

3.6. Proposition. Let $A$ be a weakly shod algebra and $\Gamma$ be a regular component of $\Gamma_A$.

(a) If $\Gamma \cap R_A \neq \emptyset$, then $\Gamma \subset R_A$.
(b) If $\Gamma \cap L_A \neq \emptyset$, then $\Gamma \subset L_A$.

Proof. We shall prove only (a) since the proof of (b) is dual.

(a) Let $X \in \Gamma \cap R_A$. Suppose first that $\Gamma$ has oriented cycles. Then, by [11], $\Gamma$ is an stable tube and so any module in it is a successor of $X$. The result now follows from the fact that $R_A$ is closed under successors. Suppose now that $\Gamma$ has infinitely many $\tau_A$-orbits, and let $Y \in \Gamma$. By Section 3.2, there exists an $m$ such that $\tau^{-m}_A Y \in R_A$ because $\Gamma$ is right stable. Now, by Corollary 1.10, there is a path from $\tau^{-m}_A Y$ to $Y$ and so $\Gamma \in R_A$.

The remaining case follows now from the above proposition. \hfill $\Box$

3.7. Proposition. Let $A$ be a weakly shod algebra and let $\Gamma$ be a component of $\Gamma_A$ which is not generalized standard or contains oriented cycles.
(a) If $\Gamma$ contains a projective module, then $\Gamma \subseteq \mathcal{L}_A \setminus \mathcal{R}_A$.

(b) If $\Gamma$ contains an injective module, then $\Gamma \subseteq \mathcal{R}_A \setminus \mathcal{L}_A$.

**Proof.** We shall prove only (a) since the proof of (b) is dual.

(a) Suppose $\Gamma$ is a component as in the statement and containing a projective module. Observe that $\Gamma$ is not a pip-bounded component (by Section 1.6), and so $\Gamma$ is semiregular. Moreover, by Section 2.5, $\Gamma \subseteq \mathcal{L}_A \cup \mathcal{R}_A$. So it suffices to show that $\Gamma$ has no modules in $\mathcal{R}_A$.

**Case 1.** $\Gamma$ has oriented cycles.

Then, $\Gamma$ is a semiregular tube with projective modules (by [12]). Observe that $\Gamma$ contains an infinite sectional path \[
\cdots \to Y_t \to Y_{t-1} \to \cdots \to Y_1 \to Y_0, \quad (*)
\] where $\tau_A^{-1}Y_0$ is a summand of the radical of a projective module. Clearly, then, $\text{id}_A Y_0 \geq 2$ by Section 1.3 and so, $Y_j \notin \mathcal{R}_A$, for each $j$. Also, any module in $\Gamma$ is a predecessor of one of the modules in $(\ast)$ and so $\Gamma$ has no modules in $\mathcal{R}_A$.

**Case 2.** $\Gamma$ is not generalized standard and has no oriented cycles.

Suppose first that $\Gamma$ has infinitely many $\tau_A$-orbits. Then, there exists a stable module $M \in \Gamma$ such that the length of a shortest walk from any module in the $\tau_A$-orbit of $M$ to a nonstable module is greater than $2n$, where $n = \text{rk}_0 K_0(A)$. By Section 3.3, there exists an integer $s$ such that $\tau_s^A M \notin \mathcal{R}_A$. If now $\Gamma \cap \mathcal{R}_A \neq \emptyset$, then there exists an integer $r$ such that $\tau_r^A M \in \mathcal{R}_A$ (Section 3.2). Using Secton 1.9, we infer that there is a path in $\text{ind}_A$ from $\tau_r^A M$ to $\tau_s^A M$, a contradiction to the fact that $\mathcal{R}_A$ is closed under successors.

It remains to consider the case where $\Gamma$ has only finitely many $\tau_A$-orbit.

**Claim.** There exist $X,Y \in \Gamma$, with $Y \notin \mathcal{R}_A$, such that $\text{rad}_A^{\infty}(X,Y) \neq 0$.

Since $\Gamma$ is not generalized standard, there exists a nonzero morphism $f \in \text{rad}_A^{\infty}(X',Y')$ with $X',Y' \in \Gamma$. By Section 1.5, given $t \geq 1$, there is a path

\[
X' \xrightarrow{h_t} Y_t \xrightarrow{f_t} Y_{t-1} \to \cdots \to f_1 \to Y_0 = Y'
\]

in $\text{ind}_A$, where $f_i$ is irreducible for $i = 1, \ldots, t$ and $h_t \in \text{rad}_A^{\infty}(X', Y_t)$. Since $\Gamma$ has no oriented cycles and only finitely many $\tau_A$-orbits, there exists a left stable module $Y'' \in \Gamma$ such that $\text{rad}_A^{\infty}(X, \tau_A^r Y'') \neq 0$ for infinitely many positive integers $r$. The claim now follows again from Section 3.3.

Let now $g$ be a nonzero morphism in $\text{rad}_A^{\infty}(X, Y)$ with $X, Y \in \Gamma$ and $Y \notin \mathcal{R}_A$. Again by Section 1.5, given $t \geq 1$, there is a path

\[
X = X_0 \xrightarrow{g_1} X_1 \xrightarrow{g_2} \cdots \xrightarrow{g_t} X_t \to Y
\]

in $\text{ind}_A$, where $g_i$ is irreducible for $i = 1, \ldots, t$. Using again the facts that $\Gamma$ has no oriented cycles and only finitely many $\tau_A$-orbits, we infer that there exists a right stable
module $X'' \in \Gamma$ such that $\text{rad}^\infty_A(\tau_A^{-r}X'', Y) \neq 0$ for infinitely many positive integers $r$. Since $Y \notin R_A$, we infer that the $\tau_A$-orbit of $X''$ has no modules in $R_A$. It then follows from Section 3.2 that $\Gamma \cap R_A = \emptyset$, which proves the result.

3.8. We end this section with the following result which will also be useful later on.

**Corollary.** Let $A$ be a weakly shod algebra.

(a) Any indecomposable projective module not belonging to $L_A$ lies in a pip-bounded component.

(b) Any indecomposable injective module not belonging to $R_A$ lies in a pip-bounded component.

**Proof.** (a) Let $P$ be an indecomposable projective module not belonging to $L_A$ and let $\Gamma$ be the component of $\Gamma_A$ where $P$ lies. Assume that $\Gamma$ is not pip-bounded. Since $A$ is weakly shod, we infer that $\Gamma$ is a right stable semiregular component. By Section 3.7, we get that $\Gamma$ is generalized standard and has no oriented cycles. Then, by Section 3.5, we get that $\Gamma \cap L_A = \emptyset$. Observe now then that $\Gamma$ is not a postprojective component (because otherwise the simple projectives on it would belong to $L_A$). Hence, there exists an indecomposable projective module $P'$ whose radical has a nonprojective indecomposable direct summand $X$. Since $\text{id}_A \tau_A X \geq 2$, we get that $\Gamma$ has a module not lying in $L_A \cup R_A$, a contradiction to Section 2.5. The item (b) follows from dual arguments. □

4. Iterated one-point extensions of tilted algebras

4.1. In this section, we will show that a weakly shod algebra $A$ with $\mathcal{P}_A \neq \emptyset$ can be seen as an iteration of one-point extensions starting at a product of tilted algebras. This description will allow us to, amongst other things, describe the Auslander–Reiten quiver of such algebras. We fix a field $K$. From now on, all algebras are finite-dimensional $K$-algebras.

4.2. Given an algebra $B$ and a $B$-module $M$, the algebra

$$ A = B[M] = \begin{pmatrix} K & 0 \\ M & B \end{pmatrix} $$

is called the one-point extension of $B$ by $M$. It is well known that the $A$-modules can be described as triples $(K', X, f)$, where $X$ is a $B$-module and $f : K' \otimes_K M \to X$ is a $B$-homomorphism. The indecomposable projective $A$-modules can then be described as: (i) $(0, P, 0)$ where $P$ is an indecomposable projective $B$-module; and (ii) the extended projective $A$-module $(K, M, 0)$, whose radical is the module $(0, M, 0)$. Also, if $I$ is an injective $B$-module, then $(0, I, 0)$ is an injective $A$-module if and only if $\text{Hom}_B(M, I) = 0$. 
When there is no danger of confusion, we shall also indicate the $A$-module $(0, X, 0)$ simply by $X$, because $\text{mod } B$ can be naturally embedded into $\text{mod } A$. For more details on this construction we refer the reader, for instance, to [1].

4.3. Let $A$ be a weakly shod algebra with $\mathcal{P}_A^f \neq \emptyset$. Observe that none of the projective modules in $\mathcal{P}_A^f$ lie in a cycle since otherwise there would exist paths in $\text{ind } A$ from an injective to a projective with arbitrary length. We can then define in $\mathcal{P}_A^f$ an order as follows:

$$P \leq P' \iff \exists \text{ a path } P \sim P' \text{ for } P, P' \in \mathcal{P}_A^f.$$  

4.4. Lemma. Let $A = B[M]$ be a weakly shod algebra. Then there exists an $r \geq 0$ such that $\tau_B^r M \in \text{add } \mathcal{L}_B$ or $\tau_B^r M$ is projective.

Proof. Since $A$ is weakly shod, there exists an $n_0$ such that any path in $\text{ind } A$ from an injective $A$-module to a projective $A$-module has length at most $n_0$. If now there is an $r \geq 0$ such that $\tau_B^r M$ is projective, we are done. Suppose then that there exists an indecomposable summand $M'$ of $M$ such that, for each $r \geq 0$, $\tau_B^r M'$ is not projective and does not lie in $\mathcal{L}_B$. Let $s > n_0$ and consider a predecessor $X \in \text{ind } B$ of $\tau_A^s M'$ with $\text{pd}_B X \geq 2$. In particular, there exists a path $\xi : X \sim M'$ in $\text{ind } B$ of length greater than $n_0$ which can be lifted to a path $\hat{\xi} : (0, X, 0) \sim (0, M', 0)$ in $\text{ind } A$ of length greater than $n_0$. Observe that $\text{pd}_A (0, X, 0) \geq 2$ and then, there exists an injective $I$ and a nonzero morphism $\alpha : I \to \tau_A (0, X, 0)$. The path

$$I \xrightarrow{\alpha} \tau_A (0, X, 0) \sim (0, X, 0) \xrightarrow{\hat{\xi}} (0, M', 0) \to (K, M, \text{Id})$$

gives a path in $\text{ind } A$ from an injective module to a projective module with length greater than $n_0$, a contradiction.

4.5. Lemma. Let $A = B[M]$ be a weakly shod algebra with $\mathcal{P}_A^f \neq \emptyset$, where the extended projective is a maximal element in $\mathcal{P}_A^f$. Then $B$ is a product of weakly shod algebras.

Proof. Let $\Gamma$ be the component of $\Gamma_A$, where the extended projective lies. Clearly, $\Gamma$ is a pip-bounded component. Suppose $B$ is not weakly shod. Then, there are paths in $\text{ind } B$ from an injective $I$ to a projective $Q$ of arbitrary length. Consider, for each $t \geq 0$, a path $\theta_t : I \sim Q$ in $\text{ind } B$ of length greater than $t$. Suppose first that $\text{Hom}_B (M, I) = 0$. Then $(0, I, 0)$ is an injective module in $\text{ind } A$. Since the paths $\theta_t$ can be lifted to paths $\tilde{\theta}_t$ in $\text{ind } A$, we get paths there from the injective module $(0, I, 0)$ to the projective $(0, Q, 0)$ of arbitrary length, a contradiction. Suppose now that there exist an indecomposable summand $M'$ of $M$ and a nonzero morphism $\alpha : M' \to I$. Hence, there exists, for each $t \geq 0$, a path $\tilde{\xi}_t : M' \xrightarrow{\alpha} I \xrightarrow{\tilde{\theta}_t} Q$ in $\text{ind } B$ of length greater than $t$. Lifting them to $\text{ind } A$, one gets paths $(0, M', 0) \sim (0, P, 0)$ in $\text{ind } A$ of arbitrary length. Clearly, $(0, M', 0) \in \Gamma$ and since by Section 1.6 there are no paths of arbitrary length between two modules in $\Gamma$, we infer
that \((0, Q, 0) \notin \Gamma\). So, there exists a path from \((0, M', 0)\) to \((0, Q, 0)\) with a morphism in \(\text{rad}^\infty(\text{mod} A)\). Using Section 1.5, for each \(s \geq 1\), there is a path

\[
(0, M', 0) = Y_0 \xrightarrow{f_1} Y_1 \rightarrow \cdots \rightarrow Y_\ell \xrightarrow{f_\ell} Y_\ell \xrightarrow{h_\ell} (0, Q, 0)
\]

where \(f_1, \ldots, f_\ell\) are irreducible. Clearly, using Section 1.6 and Section 1.11, there exists an \(j\) such that \(Y_j\) is either a successor of \(P\) or a successor of an injective. In the former case we get a contradiction with the maximality condition on \(P\) while in the latter case, we get a path from an injective to a projective passing through a morphism in \(\text{rad}^\infty(\text{mod} A)\), contradicting Section 1.6.

\[\blacksquare\]

4.6. Next result is essential in our considerations.

**Proposition.** Let \(A = B[M]\) be a weakly shod algebra with \(\mathcal{P}^f_A \neq \emptyset\), and where the extended projective is a maximal element in \(\mathcal{P}^f_A\). Let \(\Gamma'\) be a component of \(\Gamma_B\) containing a summand of \(M\). Then

(a) \(\Gamma' \cap \mathcal{L}_B \neq \emptyset\).
(b) \(\Gamma' \cap \mathcal{R}_B \neq \emptyset\).
(c) \(\Gamma'\) has no oriented cycles.
(d) \(\Gamma'\) is generalized standard.

**Proof.** Since \(A\) is weakly shod, there exists an \(n_0\) such that any path in \(\text{ind} A\) from an injective module to a projective module has length at most \(n_0\). Denote by \(\Gamma\) the component of \(\Gamma_A\) containing the extended projective \(P\). If now \(\Gamma'\) is pip-bounded, then the result follows from Section 1.6 and Corollary 3.4. So, we can assume that \(\Gamma'\) is a semiregular component. Observe first that if \(\Gamma' \cap \mathcal{L}_B \neq \emptyset\), then, by Lemma 4.4, \(\Gamma'\) has a projective module not lying in \(\mathcal{L}_B\), a contradiction to (a), which proves (c) and (d). By Section 3.8, \(\Gamma'\) is a pip-bounded component, a contradiction and (a) is proven.

**Case 1.** \(\Gamma'\) has injective modules.

Observe that if \(\Gamma'\) is not generalized standard or if \(\Gamma'\) has oriented cycles, then, by Proposition 3.7, \(\Gamma' \subset \mathcal{R}_B \setminus \mathcal{L}_B\), a contradiction to (a), which proves (c) and (d). By Section 3.8, the injective modules in \(\Gamma'\) lie in \(\mathcal{R}_B\), which proves (b).

**Case 2.** \(\Gamma'\) has no injective modules.

Let \(M'\) be an indecomposable summand of \(M\) lying in \(\Gamma'\). Observe that if \(M'\) lies in an oriented cycle, there would also exist a path in \(\text{ind} A\) from \((0, M', 0)\) to itself, a contradiction to the fact that such a module lies in a pip-bounded component of \(\Gamma_A\). So, \(M'\) does not lie in an oriented cycle, and in particular, it is not \(\tau_B\)-periodic.

**Claim.** There exists an \(s\) such that \(\tau_B^s M' \in \mathcal{R}_B\).

Suppose this is not true. Then, for each \(s \geq 0\), \(\tau_B^s M'\) has a successor of injective dimension at least 2, and therefore, a successor which is a projective \(B\)-module. Since
$M'$ is not $\tau_B$-periodic and right stable, we infer that, for each $r \geq 0$, there exists a path $(\theta_r)$ in ind $B$ from $M'$ to a projective $P'$ of length greater than $r$. Lifting the paths $(\theta_r)$ to paths in ind $A$ one gets that, for each $r \geq 0$, there exists a path $\tilde{\theta}_r$ from $(0, M', 0)$ to the projective $A$-module $(0, P', 0)$ of length greater than $r$. Using Section 1.5 if necessary, it yields that for each $t \geq 0$, there exists a path in ind $A$

\[
(0, M', 0) = Y_0 \xrightarrow{f_1} Y_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} Y_t \xrightarrow{(\xi_t)} (0, P', 0)
\]

where $f_1, \ldots, f_t$ are irreducible and $(\xi_t)$ is a path of length greater than $n_0$. Since $\Gamma$ has no oriented cycles and only finitely many $\tau$-orbits, there exists a $t$ such that $Y_t$ is either a successor of the extended projective $P$ or a successor of an injective. In the former case, we get a path from $P$ to $(0, P', 0)$ a contradiction to the maximality condition on $P$ in $\mathcal{P}_A^f$, while in the latter case, one gets a path from an injective to the projective $(0, M', 0)$ with length greater than $n_0$, and the claim is proven.

In particular, $\Gamma' \cap L_B \neq \emptyset$, which proves (b). Suppose $\Gamma'$ has projective modules. Then, since $\Gamma' \subseteq L_A \setminus L_B$, it follows from Proposition 3.7 that (c) and (d) hold. It remains the case where $\Gamma'$ is regular. By Section 4.6, $\Gamma'$ is also directed and then clearly $M'$ would lie in a cycle, again a contradiction and the proof is complete. $\square$

4.7. We recall the following result from [7, (5.2)].

**Theorem.** Let $A$ be a quasitilted algebra which is not tilted, and $\Gamma$ be a component $\Gamma_A$.

(a) If $\Gamma$ contains a projective module, then $\Gamma \subseteq L_A \setminus R_A$.
(b) If $\Gamma$ contains an injective module, then $\Gamma \subseteq R_A \setminus L_A$.

4.8. **Lemma.** Let $A = B[M]$ be a weakly shod algebra with $\mathcal{P}_A^f \neq \emptyset$, where the extended projective $A$-module is a maximal element in $\mathcal{P}_A^f$ and let $B'$ be an indecomposable summand of $B$. If $\mathcal{P}_B^f = \emptyset$, then $B'$ is a tilted algebra.

**Proof.** By Lemma 4.5, we know that $B$ is a product of connected weakly shod algebras. Since, by hypothesis, $\mathcal{P}_B^f = \emptyset$, it yields that $B'$ is in fact a quasitilted algebra. We shall show that $B'$ is a tilted algebra.

Let $M'$ be an indecomposable summand of $M$ which is a $B'$-module and let $\Gamma'$ be the component of $\Gamma_B$ where $M'$ lies. Hence, by Section 4.6, $\Gamma' \cap L_B = \emptyset$. If now $\Gamma'$ has an injective module, we infer by Section 4.7 that $B'$ is tilted. Suppose then that $\Gamma'$ has no injective modules. By Section 4.6, we have that $\Gamma' \cap R_B \neq \emptyset$. Therefore, if $\Gamma'$ has a projective module, then, by Section 4.7, $B'$ is tilted and we are done. Suppose finally that $\Gamma'$ is regular. By Section 4.6, $\Gamma'$ is also directed and then, by [4], $\Gamma'$ is a connecting component and $B'$ is tilted. The result is proven. $\square$

4.9. We shall now prove the main result of this paper.
Theorem. Let $A$ be a connected weakly shod algebra with $\mathcal{P}_A^f \neq \emptyset$. Then there are algebras $B = A_t, \ldots, A_0 = A$ and $A_1$-modules $M_i$ for each $i = 1, \ldots, t$ such that

(a) $B$ is a product of tilted algebras.
(b) $A_i = A_{i+1}[M_{i+1}]$ for each $i = 0, \ldots, t - 1$.
(c) For each $i = 0, \ldots, t - 1$, the algebra $A_i$ is weakly shod and the extended projective $A_i$-module is a maximal element in $\mathcal{P}_A^f$.

Proof. The proof is now very easy. Let $A = A_0$ be a weakly shod algebra with $\mathcal{P}_A^f \neq \emptyset$ and let $P_0$ be a maximal element in $\mathcal{P}_A^f$. Clearly, there exists an algebra $A_1$ and an $A_1$-module $M_1$ such that $A_0 = A_1[M_1]$ and the extended projective $A_0$-module is $P_0$. For each indecomposable summand $A'_1$ of $A_1$ which is weakly shod with $\mathcal{P}_{A'_1}^f \neq \emptyset$, we proceed again as above. Iterating this procedure, one ends up with algebras $B = A_t, \ldots, A_0 = A$ and $A_i$-modules $M_i$ for each $i = 1, \ldots, t$ such that conditions (b) and (c) hold and such that $\mathcal{P}_B^f = \emptyset$. By Lemma 4.8, we have that $B$ is a product of tilted algebras and the result is proven. $\blacksquare$

5. Uniqueness and faithfulness

5.1. Let $A$ be a connected weakly shod algebra with $\mathcal{P}_A^f \neq \emptyset$. Using the results of Section 4, we shall show that $\Gamma_A$ has a unique pip-bounded component $\Gamma$ which is moreover faithful. Recall that in [17], Skowroński had studied the algebras containing a generalized standard component without oriented cycles. From his main result, one can get the following.

Theorem. Let $A$ be an Artin algebra and let $\Gamma$ be a connected component of $\Gamma_A$. If $\Gamma$ is faithful, generalized standard and without oriented cycles, then there exist tilted algebras $A^{(l)}$ and $A^{(r)}$ such that any component $\Gamma'$ of $\Gamma_A$ different of $\Gamma$, satisfies one and only one of the following conditions:

(a) $\Gamma'$ is a component of $\Gamma_A^{(l)}$ and $\text{Hom}_A(X, Y) \neq 0$ for some $X \in \Gamma'$ and $Y \in \Gamma$;
(b) $\Gamma'$ is a component of $\Gamma_A^{(r)}$ and $\text{Hom}_A(X, Y) \neq 0$ for some $X \in \Gamma$ and $Y \in \Gamma'$.

5.2. Applying this to our context, we have the following.

Proposition. Let $A$ be a weakly shod algebra and assume that $\Gamma_A$ has a faithful pip-bounded component $\Gamma$. Then

(a) if $\Gamma''$ is a component of $\Gamma_A$ different of $\Gamma$, then $\Gamma'' \subset \mathcal{L}_A \setminus \mathcal{R}_A$ or $\Gamma'' \subset \mathcal{R}_A \setminus \mathcal{L}_A$;
(b) $\Gamma$ is the unique pip-bounded component in $\Gamma_A$;
(c) the intersection $\mathcal{L}_A \cap \mathcal{R}_A$ is finite and it is contained in $\Gamma$.
Proof. (a) By Theorem of Section 5.1, if $\Gamma'$ is a component different of $\Gamma$, then either $\text{Hom}_A(\Gamma, \Gamma') \neq 0$ or $\text{Hom}_A(\Gamma', \Gamma) \neq 0$ but not both. Assume the former. We will show that $\Gamma' \subseteq \mathcal{R}_A \setminus \mathcal{L}_A$. Observe that $\Gamma'$ has no projective modules since otherwise $\text{Hom}_A(\Gamma', \Gamma) \neq 0$ because $\Gamma$ is faithful, a contradiction to our hypothesis.

Claim. There exist modules $X \in \Gamma \cap (\mathcal{R}_A \setminus \mathcal{L}_A)$ and $X' \in \Gamma'$ with $\text{rad}_A^\infty(X, X') \neq 0$.

Indeed, since $\text{Hom}_A(\Gamma, \Gamma') \neq 0$, there exists a nonzero morphism $h \in \text{rad}_A^\infty(Y, X')$ with $Y \in \Gamma$ and $X' \in \Gamma'$. By Section 1.5, for each $i \geq 0$, there exists a path $(\theta_i)$:

$$Y = Y_0 \xrightarrow{f_1} Y_1 \rightarrow \cdots \xrightarrow{f_i} Y_i \xrightarrow{h'} X', $$

where $f_1, \ldots, f_i$ are irreducible and $0 \neq h' \in \text{rad}_A^\infty(Y_i, X')$. So, there exists a right stable module $M$ such that $\text{rad}_A^\infty(\tau^{-i}_A M, X') \neq 0$ for infinitely many positive integers $r$. Since $\Gamma'$ has injective modules, by Lemma 3.3, there exists an $l_i \geq 0$ such that $\tau^{-l_i}_A M \notin \mathcal{L}_A$. So the set $\mathcal{B} = \{\tau^{-i}_A M : i \geq l_i \}$ is infinite and has no modules in $\mathcal{L}_A$. Hence $\mathcal{B} \cap \mathcal{R}_A \neq \emptyset$ because $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in $\text{ind} A$. This proves the claim.

Clearly, also, $X' \in \Gamma \cap (\mathcal{R}_A \setminus \mathcal{L}_A)$. If now $\Gamma'$ is regular, then by Proposition 3.6, $\Gamma' \subseteq \mathcal{R}_A \setminus \mathcal{L}_A$ as required. It remains to consider the case where $\Gamma'$ is a semiregular component with injective modules. In case $\Gamma'$ is not generalized standard or contains oriented cycles, the result follows from Proposition 3.7. Suppose then that $\Gamma'$ is generalized standard and has no oriented cycles. Hence, by Section 3.5, $\Gamma' \subseteq \mathcal{R}_A$. In order to show that $\Gamma' \cap \mathcal{L}_A = \emptyset$, consider a nonzero morphism $g \in \text{rad}_A^\infty(X, X') \neq 0$ with $X \in \Gamma \cap (\mathcal{R}_A \setminus \mathcal{L}_A)$ and $X' \in \Gamma'$ as in the claim. Using Section 1.5 and the above hypothesis on $\Gamma'$, there exists a left stable module $Z \in \Gamma'$ such that for each $i \geq 0$, $\tau^{-i}_A Z$ is a successor of $X$. In particular, the $\tau_A$-orbit of $Z$ has no modules in $\mathcal{L}_A$. By Section 3.2, $\Gamma' \cap \mathcal{L}_A = \emptyset$ and then $\Gamma' \subseteq \mathcal{R}_A \setminus \mathcal{L}_A$. If now $\text{Hom}_A(\Gamma', \Gamma) \neq 0$, a similar argument shows that $\Gamma' \subseteq \mathcal{L}_A \setminus \mathcal{R}_A$, which proves (a).

Observe now that (b) follows from (a), and (c) is a clear consequence of (a), (b), and Lemma 3.3. \qed

5.3. Lemma. Let $A = B[M]$ be a weakly shod algebra with $\mathcal{P}_A^f \neq \emptyset$, where the extended projective module $P$ is maximal in $\mathcal{P}_A^f$. Let $B'$ be an indecomposable summand of $B$ and let $M'$ be an indecomposable summand of $M$ which is a $B'$-module.

(a) If $\Gamma_B$ has a faithful pip-bounded component, then $M'$ lies in it.
(b) If $B'$ is tilted, then $M'$ lies in a connecting component.
(c) If $P'$ is an indecomposable projective $B'$-module lying in the same component of $\Gamma_B$ as $M'$, then $(0, P', 0)$ and $P$ lie in the same component of $\Gamma_A$.

Proof. (a) By Section 4.6, we get that the component of $\Gamma_B$ where $M'$ lies has to intersect $\mathcal{L}_{B'}$ and $\mathcal{R}_{B'}$. The result now follows from Section 5.2.

(b) Let $\Gamma'$ be the component of $\Gamma_B$ where $M'$ lies and suppose it is not connecting. Since $\Gamma'$ is directed (Section 4.6), then it is either postprojective or preinjective. Suppose
\( \Gamma' \) is postprojective. Since \( \Gamma' \) is not connecting, it does not contain all the projective modules. Hence, there are indecomposable projective \( B' \)-modules \( P' \in \Gamma' \) and \( P'' \notin \Gamma' \) and a morphism \( f : P' \rightarrow P'' \) (which clearly belongs to \( \text{rad}^\infty(\text{mod} B') \)). By Section 1.5, for each \( t \geq 0 \), there exists a path

\[ P' = X_0 \xrightarrow{f_1} X_1 \rightarrow \cdots \rightarrow X_{t-1} \xrightarrow{f_t} X_t \sim P'', \]

where \( f_1, \ldots, f_t \) are irreducible maps. Hence there exists a module \( Y \in \Gamma' \) whose \( \tau_{B'} \)-orbit has infinitely many modules of projective dimension greater than 1. Hence, this \( \tau_{B'} \)-orbit has no module in \( \mathcal{R}_{B'} \) and so by Section 3.2, \( \Gamma' \cap \mathcal{R}_{B'} = \emptyset \), contradicting Section 4.6.

If \( \Gamma' \) is a preinjective component but not connecting, a similar argument using injective modules leads to a contradiction. This proves the result.

(c) Let \( \Gamma' \) be the component of \( \Gamma_{B'} \) where \( M' \) lies, let \( P' \in \Gamma' \) be an indecomposable projective module, and let \( \Gamma \) be the component of \( \Gamma_A \) where the extended projective module \( P \) lies. If there is no path from an indecomposable summand of \( M \) to \( P' \), then \((0, M', 0)\) and \( P' \) lie both in \( \Gamma \). In case there is such a path and \((0, P', 0)\) lies in a component \( \Gamma'' \) different of \( \Gamma \), then \( f \in \text{rad}_A^\infty(X, Y) \). Using Section 1.5 if necessary, one can assume that \( X \in \Gamma, Y \in \Gamma'' \) and a nonzero morphism \( f \in \text{rad}_A^\infty \). Using again Section 1.5 if necessary, we can also assume that \( Y \) is a successor of a projective \( P'' \in \Gamma'' \), leading to either a path from \( P \) to \( P'' \), a contradiction to the maximality condition on \( P \), or to a path from an injective to \( P'' \) passing through a morphism in \( \text{rad}_A^\infty \), also a contradiction (Remarks 2.4). \( \square \)

5.4. We shall now prove our main result of this section.

Theorem. Let \( A \) be a connected weakly shod algebra with \( \mathcal{P}_A^f \neq \emptyset \). Then \( \Gamma_A \) has a unique pip-bounded component which is moreover faithful.

Proof. We shall use here the notations of the statement of Theorem of Section 4.9. Observe that the algebra \( A_{t-1} = B[M_t] \) is a product of a connected weakly shod algebra \( A'_{t-1} \) with \( \mathcal{P}_{A'_{t-1}}^f \) and tilted algebras (which are clearly summands of \( B \)). The Auslander–Reiten quiver \( \Gamma_{A_{t-1}} \) has a pip-bounded component which is clearly faithful by Lemma 5.3. At each further step of the one-point extension towards \( A \), one can use again Lemma 5.3 to get a product of an algebra with a unique and faithful pip-bounded component with tilted algebras. The result now follows from the fact that \( A \) is connected. \( \square \)

6. Consequences and example

6.1. We shall discuss here some direct consequences of our main results from Sections 4 and 5.
Proposition. Let $A$ be a weakly shod algebra. Then

(a) the ordinary quiver of $A$ is directed;
(b) $\text{gl.dim } A < \infty$.

Proof. (a) If $P_A^f = \emptyset$, then $A$ is quasitilted and the result follows from [9]. Suppose $P_A^f \neq \emptyset$. As we have seen, $A$ is then built up from a (product of) tilted algebra(s) by iterating one-point extensions. The result will now follow from the following easily verified remarks: (i) the ordinary quiver of a tilted algebra is directed; and (ii) the process of one-point extending an algebra does not produce cycles in its ordinary quiver.

(b) Is a direct consequence of (a).  

6.2. Let $A$ be a connected weakly shod algebra with $P_A^f \neq \emptyset$, and let $\Gamma$ be the unique pip-bounded component of $\Gamma_A$. We have seen in Section 5.1 that if $\Gamma'$ is a component of $\Gamma_A$ different from $\Gamma$, then it is a component of a tilted algebra. Using now the well-known description of the Auslander–Reiten quiver of tilted algebras, we have the following. For a weakly shod algebra $A$ with $P_A^f \neq \emptyset$, the components of $\Gamma_A$ are of the following shape (using the notations of Section 5.1):

(i) postprojective component(s) (those of $\Gamma_A(l)$);
(ii) preinjective component(s) (those of $\Gamma_A(r)$);
(iii) a unique and faithful pip-bounded component which is the unique nonsemiregular component;
(iv) stable tubes;
(v) components of type $Z_A\infty$;
(vi) components constructed from tubes or from $Z_A\infty$ by ray or coray insertions.

Observe moreover that the components of $\Gamma_A(l)$ (or $\Gamma_A(r)$) which are embedded in $\Gamma_A$ are semiregular without injective (respectively projective) modules and are contained in $L_A\setminus R_A$ (respectively, in $R_A\setminus L_A$) (see Section 5.2).

6.3. We finish the paper with an example to illustrate the construction discussed in the previous sections.

Examples. Let $A = A_0$ be the $K$-algebra given by the quiver $\Delta$:
with relations $\alpha_i \beta = \varepsilon \psi = 0$ for $i = 1, 2$, $\gamma \delta \varepsilon = \theta \xi = 0$. The Auslander–Reiten quiver $\Gamma_{A_0}$ has a pip-bounded component with the following shape:

![Diagram](image)

Clearly, $A_0$ is a representation-infinite weakly shod algebra. Observe that $\mathcal{P}_A^f = \{P_3, P_5, P_6, P_7, P_8, P_9\}$, where $P_i$ indicates the indecomposable projective associated to the vertex $i$. Also, $P_7$ is the unique maximal element of $\mathcal{P}_A^f$. Therefore, $A$ can be seen as one-point extension $A_0 = (A_1' \times A_1'')[S_6 \otimes S_6 \otimes S_8]$ where $A_1'$ is the algebra given by the quiver:

![Quiver](image)

with relations $\alpha_i \beta = 0$ for $i = 1, 2$, $\gamma \delta \varepsilon = 0$, while $A_1''$ is the algebra given by the quiver:

![Quiver](image)

with $\theta \xi = 0$.

Observe that $A_1''$ is a tilted algebra of type $A_3$ while $A_1'$ is a weakly shod algebra whose Auslander–Reiten quiver has a pip-bounded component with the following shape:

![Diagram](image)
Now, the projective $P_6$ is a maximal element of $\mathcal{P}_f^{A_1'} = \{P_3, P_5, P_6\}$. Therefore, $A_1' = A_2'[\tau_{A_1'}^{-2}S_2]$, where $A_2'$ is the algebra given by the following quiver:

```
1 \alpha_1 \rightarrow 2 \beta \rightarrow 3 \gamma \rightarrow 5 \delta
```

with relations $\alpha_i\beta = 0$ for $i = 1, 2$. Clearly, $A_2'$ is a tilting algebra. Using the notations of Section 4.9, we have $B = A_2 = A_2' \times A_1''$, $A_1 = A_1' \times A_1''$, $M_1 = \tau_{A_1'}^{-2}S_2$, and $M_0 = S_6 \otimes S_6 \otimes S_8$.

References