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Switching with more than two colours

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Dedicated to the memory of Jaap Seidel

Abstract

The operation of switching a finite graph was introduced by Seidel, in the study of strongly regular graphs. We may conveniently regard a graph as being a 2-colouring of a complete graph; then the extension to switching of an m-coloured complete graph is easy to define. However, the situation is very different. For $m > 2$, all m-coloured graphs lie in the same switching class. However, there are still interesting things to say, especially in the infinite case.

This paper presents the basic theory of switching with more than two colours. In the finite case, all graphs on a given set of vertices are equivalent under switching, and we determine the structure of the switching group and show that its extension by the symmetric group on the vertex set is primitive.

In the infinite case, there is more than one switching class; we determine all those for which the group of switching automorphisms is the symmetric group. We also exhibit some other cases (including the random m-coloured complete graph) where the group of switching-automorphisms is highly transitive.

Finally we consider briefly the case where not all switchings are allowed. For convenience, we suppose that there are three colours of which two may be switched. We show that, in the case of almost all finite random graphs, the analogue of the bijection between switching classes and two-graphs holds.

1. Two colours; finitely many vertices

The operation of switching a graph Γ with respect to a set X of vertices was introduced by Seidel [8]; it is often called *Seidel switching* or *Seidel equivalence*. The operation consists of exchanging adjacency and non-adjacency between X and its complement, while keeping adjacencies within or outside X unaltered. Seidel used an adjacency matrix with 0 on the diagonal, -1 for adjacency, and $+1$ for non-adjacency; then switching corresponds to conjugating this matrix by a diagonal matrix with entries ± 1. This representation arises in a geometric context as follows (see [9]).

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Suppose that we have a set of lines through the origin in Euclidean space, such that the acute angle α between any pair is the same. Choose a unit vector on each line. Then the Gram matrix of inner products of these vectors has the form $I + (\cos\alpha)A$, where A is the adjacency matrix of a graph (of the form just described). Replacing some of the unit vectors by their negatives corresponds to switching the graph.

For our purposes, it is more convenient to think of a complete graph with edges coloured red and blue; the switching operation σ_X with respect to a subset X of V involves interchanging colours of edges from X to its complement, leaving all other edges unaltered. For brevity, in what follows, the word "graph" will mean "edge-coloured complete graph" (with the appropriate number of colours).

We now give a very brief summary of the properties of switching. We consider graphs on a fixed set V of n vertices.

- The switching operations form a group of order 2^{n-1}, the *switching group*, whose orbits on graphs are called *switching classes*. Each switching class has size 2^{n-1}.
- Two graphs belong to the same switching class if and only if the parity of the number of red edges in any 3-subset of V is the same in both graphs.
- A set $\mathcal{T}$ of 3-subsets of V is realised as the set of triples containing an odd number of red edges of some graph on V if and only if every 4-set contains an even number of members of $\mathcal{T}$. (Such a set $\mathcal{T}$ is called a *two-graph*. The term was introduced by G. Higman (unpublished); see Seidel [9]. Thus two-graphs are essentially the same as switching classes of graphs.)

The unlabelled switching classes of graphs (or, equivalently, two-graphs) were enumerated by Mallows and Sloane [7]. The number of two-graphs is the same as the number of even graphs (that is, graphs with all valencies even) on the same set. A conceptual proof of this appears in [2].

We define the *extended switching group* to be the semidirect product $S \rtimes \mathrm{Sym}(n)$, where S is the switching group and $\mathrm{Sym}(n)$ the group of all permutations of the vertices. (For n odd, the extended switching group is isomorphic to the Weyl group of type D_n.) The *group* $\mathrm{SAut}(\Gamma)$ *of switching-automorphisms* of a graph Γ is the image of the stabiliser of Γ in the extended switching group, under the natural homomorphism to $\mathrm{Sym}(n)$. Equivalently, it is the group of permutations g of V for which there exists a switching operation $\sigma \in S$ with $\Gamma g = \Gamma\sigma$. It is easy to see that graphs in the same switching class have the same group of switching-automorphisms.

The group of switching-automorphisms of Γ coincides with the automorphism group of the two-graph associated with Γ, and may be 2-transitive, as many examples in Seidel [9] show. However, it cannot be 3-transitive (except for the switching class of the complete or null graph), since it preserves a ternary relation (the associated two-graph).

2. More than two colours; finitely many vertices

Suppose that Γ is a complete graph on V, with $|V| = n$, whose edges are coloured with m colours, where $m \geq 2$. If c, d are colours and X a set of vertices, we define the switching operation $\sigma_{c,d,X}$ to interchange colours c and d on edges between X and its complement, and leave all other colours on such edges and all colours on edges within or outside X unaltered. The *switching group* $S_{m,n}$ is the group generated by all such switchings; it is a permutation group on the set $\mathcal{G}_{m,n}$ of all such coloured complete graphs. Since $S_{m,n}$ is normalised by the symmetric group $\mathrm{Sym}(V)$, these groups generate their semidirect product $S^*_{m,n} = S_{m,n} \rtimes \mathrm{Sym}(n)$, the *extended switching group*. The group $\mathrm{SAut}(\Gamma)$ of switching-automorphisms is defined in the same way as for two colours.

The main difference between the cases of two or more colours is that there is only one switching class for $m \geq 3$:

Theorem 2.1. *The switching group $S_{m,n}$ is transitive on $\mathcal{G}_{m,n}$ if $m \geq 3$.*

Proof. Let c, d, e be three colours and x, y two vertices. Then the commutator of $\sigma_{c,d,x}$ and $\sigma_{d,e,y}$ induces the 3-cycle (c, d, e) on the colours on $\{x, y\}$, while fixing all other colours there and all colours on other edges. Thus, we can permute transitively the colours on any edge while fixing those on all other edges. Repeating for each edge, we can map any edge-coloured graph to any other. $\square$

Corollary 2.2. *If $m \geq 3$, then the group of switching-automorphisms of any graph in $\mathcal{G}_{m,n}$ is the symmetric group* $\mathrm{Sym}(n)$. $\square$

We can describe the structure of $S_{m,n}$ completely. Here $\mathrm{Alt}(m)$ denotes the alternating group of degree m.

Theorem 2.3.

$$S_{m,n} \cong \mathrm{Alt}(m)^{n(n-1)/2} \rtimes C_2^{n-1}.$$

Proof. In the preceding proof, the 3-cycles generate the alternating group on the colours on each edge. So $S_{m,n}$ contains the direct product of copies of the alternating group. This is also true for $m = 2$, since $\mathrm{Alt}(2)$ is the trivial group. This product N is clearly a normal subgroup of $S_{m,n}$.

There is a homomorphism from $S_{m,n}$ to $S_{2,n}$, where each generator $\sigma_{c,d,X}$ of $S_{m,n}$ maps to the generator σ_X of $S_{2,n}$. (The image of an arbitrary element $g \in S_{m,n}$ is σ_X, where the edges on which the parity of the permutation of the colours is odd are those from X to its complement.) The kernel of this homomorphism is N, while the image is $S_{2,n}$, which is elementary Abelian of order 2^{n-1} by Seidel's result. The extension clearly splits. $\square$

For the extended switching group, Theorem 2.1 can be strengthened as follows.

Theorem 2.4. *The extended switching group $S^*_{m,n}$ is primitive on $\mathcal{G}_{m,n}$ if $m \geq 3$.*

Proof. This group contains $\mathrm{Alt}(m) \wr \mathrm{Sym}(n)$, where $\mathrm{Sym}(n)$ has its action on 2-element subsets of $\{1, \ldots, n\}$, and the wreath product has its product action.

If $m \geq 4$, the bottom group is primitive and not regular, and the top group is transitive; so the primitivity follows from the analysis preceding the O'Nan–Scott theorem in Dixon and Mortimer [3, Lemma 2.7]. So suppose that $m = 3$.

Now our group has a regular normal subgroup which is elementary Abelian of order $3^{\binom{n}{2}}$. This group can be represented as the set of functions from the set of 2-subsets to the integers mod 3. A block of imprimitivity containing the zero element is a subgroup which is invariant under both switching (changing sign on all edges through a vertex) and permutation of vertices.

Suppose that H is a subgroup which is so invariant, and contains a non-zero function f. We can suppose that f is non-zero on some edge containing x. Switching at x and subtracting, we obtain a non-zero function f' which vanishes on all edges not containing x. If $f(\{x, y\}) \neq 0$, then switching at y and subtracting, we obtain a function f'' which is non-zero only on the edge $\{x, y\}$. Now the images of f'' under permutations generate the whole group. $\square$

3. Two colours; infinitely many vertices

Switching for infinite graphs is defined exactly as for finite graphs. The switching group is elementary Abelian, and is isomorphic to the group of subsets of V (with the operation of symmetric difference) modulo $\{\emptyset, V\}$. The main difference is that switchings with respect to singletons do not generate the group. It is similarly true that the group of switching-automorphisms of an infinite graph may be 2-transitive but cannot be 3-transitive except in the case of the complete or null graph.

4. More than two colours; infinitely many vertices

Unlike in the finite case, the switching group does not act transitively on the set of edge-coloured complete graphs on an infinite set.

We define a *switched c-clique* on V to be an edge coloured complete graph on V, such that there is a partition $V = V_1 \cup \cdots \cup V_k$ with the properties

(a) any edge within a part V_i has colour c;
(b) the colour of an edge with vertices in V_i and V_j depends only on i and j.

Proposition 4.1. *Let Γ be an m-coloured complete graph with $m \geq 3$. Then Γ is a switched c-clique if and only if it can be obtained from the graph with all edges of colour c by switching.*

Proof. Suppose that Γ is a switched c-clique. Form the finite graph Δ on $\{1, \ldots, k\}$, where the colour of the edge $\{i, j\}$ is the same as the colour of edges from V_i to V_j in Γ. By Theorem 2.1, Δ can be switched into a c-clique. The switchings lift in an obvious way to Γ, and also switch it into a c-clique.

Conversely, let Γ be obtained from a c-clique by the product $\sigma_1 \cdots \sigma_t$ of a sequence of switchings. Then there is a partition of V into parts given by intersections of the switching sets of $\sigma_1, \ldots, \sigma_t$ and their complements, which clearly satisfies (a) and (b). $\square$

Since a switched c-clique contains no infinite c'-clique for any colour $c' \neq c$, we see that a switched c-clique cannot also be a switched c'-clique for $c' \neq c$; hence the cliques of different colours lie in different switching classes.

Switching does not change the group of switching automorphisms; so $\mathrm{SAut}(\Gamma) = \mathrm{Sym}(V)$ holds if Γ is a switched c-clique. The converse is also true. This depends on a preliminary lemma. A *moiety* of an infinite set is an infinite subset whose complement is also infinite.

Lemma 4.2. *An infinite multicoloured graph is a switched c-clique if and only if the vertex set can be partitioned into three moieties such that the induced subgraph on the union of any two is a switched c-clique.*

Proof. The reverse implication is clear. So suppose that V is the disjoint union of W_1, W_2, W_3, and for each $i \neq j$, there is an equivalence relation $\equiv_{ij}$ on $W_i \cup W_j$ whose equivalence classes have properties (a) and (b) of the definition of a switched c-clique. Extend $\equiv_{ij}$ to an equivalence relation on V in which the remaining set W_k is a single class. Let $\equiv$ be the meet of these three equivalence relations.

We claim that $\equiv$ has properties (a) and (b). Certainly it has only finitely many classes. Take two points x, y in the same class. Then they belong to the same set W_i, say W_1 without loss of generality. Since $x \equiv_{12} y$, the edge $\{x, y\}$ has colour c. Now let z be any point in a different equivalence class. Suppose, without loss of generality, that $z \in W_1 \cup W_2$. Then the properties of $\equiv_{12}$ ensure that $\{x, z\}$ and $\{y, z\}$ have the same colour. □

Theorem 4.3. *Let Γ be an m-coloured complete graph with $m \geq 3$. Then* $\mathrm{SAut}(\Gamma) = \mathrm{Sym}(V)$ *if and only if Γ is a switched c-clique.*

Proof. Suppose that $\mathrm{SAut}(\Gamma) = \mathrm{Sym}(V)$. By the infinite form of Ramsey's theorem, there is a moiety W of V which is a c-clique for some colour c. Since $\mathrm{Sym}(V)$ is transitive on moieties, and $\mathrm{SAut}(\Gamma)$ induces a switching automorphism from every set to its image, it follows that every moiety is a switched c-clique. Now Lemma 4.2 gives the result. □

There are, however, other countable graphs whose switching automorphism groups are highly transitive. One type is given by the next theorem; we will see another in the next section. (In fact, we have no example Γ for which $\mathrm{SAut}(\Gamma)$ is not highly transitive.)

Theorem 4.4. *Let Γ be an m-coloured complete graph. Suppose that there is a finite partition $V = V_1 \cup \cdots \cup V_k$ such that the colour of an edge with vertices in V_i and V_j depends only on i and j. Then* $\mathrm{FSym}(V)$ *is contained in* $\mathrm{SAut}(\Gamma)$.

Proof. It is enough to show that an arbitrary transposition (x, y) belongs to $\mathrm{SAut}(\Gamma)$. Refine the partition so that $\{x\}$ and $\{y\}$ are parts. Now switch so that all edges with ends in different parts are red. It is clear that the transposition (x, y) is an automorphism of the switched graph, and so it is a switching automorphism of the original graph. □

Perhaps the converse is true too.

A permutation g of V is *almost an automorphism* of Γ if the set of edges e for which e and e^g have different colours is finite. The set of all almost-automorphisms of Γ is a group, the *almost-automorphism group*, denoted by $\mathrm{AAut}(\Gamma)$.

Proposition 4.5. *For any infinite m-coloured complete graph with $m \geq 3$, we have* $\mathrm{AAut}(\Gamma) \leq \mathrm{SAut}(\Gamma)$.

Proof. As in the finite case, we can change the colours of any finite number of edges arbitrarily, while fixing all other colours, by switching. □

5. Switching the random graph

The most important application of switching in the infinite case with two colours concerns the countable *random graph* R (otherwise known as the *Erdős–Rényi graph* or the *Rado graph*), see [4]. This is the unique countable graph with the property that a random countable graph (whose edges are chosen independently with probability $1/2$) is isomorphic to R with probability 1.

The graph R is homogeneous, and indeed is the unique countable universal homogeneous graph, by Fraïssé's theorem [5].

Now the group $\mathrm{SAut}(R)$ is 2-transitive, and is a transitive extension of $\mathrm{Aut}(R)$.

This group features in a remarkable theorem of Thomas [11]. To state this theorem we need some terminology. There is a natural topology on the symmetric group of countable degree, namely the topology of pointwise convergence. With respect to this topology, a subgroup of $\mathrm{Sym}(V)$ is closed if and only if it is the automorphism group of a first-order structure on V (and this structure may be taken to be purely relational). A *reduct* of a structure M on V is a closed subgroup of $\mathrm{Sym}(V)$ containing $\mathrm{Aut}(M)$. For example, $\mathrm{SAut}(R)$ is a reduct of R; it is closed because it is the automorphism group of the associated two-graph. We refer to Hodges [6] for further details.

An *anti-automorphism* of a graph Γ is an isomorphism from Γ to the complementary graph $\overline{\Gamma}$, while a *switching anti-automorphism* is a permutation g such that $\Gamma g = \overline{\Gamma}\sigma$ for some switching σ.

Now Thomas' theorem is as follows:

Theorem 5.1. *There are just five reducts of the random graph R. These are* $\mathrm{Aut}(R)$; *the group of automorphisms and anti-automorphisms of R; the group* $\mathrm{SAut}(R)$; *the group of switching-automorphisms and switching anti-automorphisms of R; and the symmetric group on the vertex set of R.* $\square$

In an analogous way, Fraïssé's theorem implies that there is a unique countable homogeneous m-coloured complete graph R_m for any (finite or countable) m. If m is finite, this is also the "random m-coloured complete graph" (in the sense that with probability 1 the random structure is isomorphic to it). These graphs and their automorphism groups have been studied by Truss [12].

Now, in contrast to the case of two colours, we have the following result:

Proposition 5.2. *For $m \geq 3$, the group* $\mathrm{SAut}(R_m)$ *is highly transitive; so this group is not a reduct of R_m.*

Proof. The group $\mathrm{AAut}(R_m)$ is highly transitive [13], and is contained in $\mathrm{SAut}(R_m)$, by Proposition 4.5. So $\mathrm{SAut}(R_m)$ is highly transitive. Now the closure of a highly transitive group is the symmetric group; but $\mathrm{SAut}(R_m)$ is not the symmetric group, by Theorem 4.3. $\square$

In fact, all the reducts of R_m have been determined by Bennett [1]. We sketch his result later in this paper.

6. Restricted switching

A variant on what we have considered is to allow only some possible switchings of colours. We consider in detail the situation where there are three colours called red, blue and green, and only blue–green switchings are permitted.

(This kind of switching has a geometrical interpretation. We are given a set of lines in Euclidean space making angles $\pi/2$ and α. Choose unit vectors along the lines; their Gram matrix has the form $I + (\cos\alpha)A$, where A is a matrix with entries 0 and ± 1. If colours red, blue, green correspond to entries 0, $+1$, -1 respectively, then changing the sign of a set of vectors corresponds to blue–green switching.)

Blue–green switching clearly leaves all red edges unchanged. It also preserves an analogue of a two-graph, namely, the parity of the number of green edges in any blue–green triangle. Is the converse true? Let us say that two 3-coloured complete graphs on V are *P-equivalent* if they have the same red edges and each blue–green triangle has the same parity of the number of green edges; and *S-equivalent* if one can be obtained from the other by blue–green switching.

P-equivalence does not imply S-equivalence in general. Suppose that Γ consists of a blue n-cycle (with $n \geq 4$), all other edges red. By switching, we can make any even number of edges in the cycle green; but any replacement of blue by green gives a P-equivalent graph. However, the following is true.

Theorem 6.1. (a) *Any* 3*-coloured complete graph which is P-equivalent to the countable random* 3*-coloured complete graph* R_3 *is S-equivalent to* R_3.

(b) *Let* Γ *be a random finite* 3*-coloured complete graph with* n *vertices. Then the probability of the event that every* 3*-coloured complete graph P-equivalent to* Γ *is S-equivalent to* Γ *tends to* 1 *as* $n \to \infty$.

Proof. (a) Suppose that Γ_1 is the random 3-coloured complete graph R_3, and Γ_2 is a graph which is P-equivalent to Γ_1. We begin with some notation. We let $c_i(xy)$ denote the colour of the edge $\{x, y\}$ in Γ_i, and $R_i(v)$, $B_i(v)$, $G_i(v)$ the sets of vertices joined to v by red, blue, or green edges respectively in Γ_i, for $i = 1, 2$. We let $BG_i(v) = B_i(v) \cup G_i(v)$. In the proof we shall modify the graph Γ_2 so that various colours or sets become the same; once we know that, for example, $c_1(xy) = c_2(xy)$, we drop the subscript. Note that we can immediately write $R(v)$ and $BG(v)$, by the definition of P-equivalence.

Let $\Delta(v)$ be the symmetric difference of $B_1(v)$ and $B_2(v)$. Switching Γ_2 with respect to $\Delta(v)$ gives a new graph Γ_2' such that all edges containing v have the same colour in Γ_1 and Γ_2'. Now replacing Γ_2 by Γ_2', we may assume that this holds for Γ_2.

Now the subgraphs on $\{v\} \cup BG(v)$ are identical in Γ_1 and Γ_2. Let $x, y \in BG(v)$. If $c_1(xy)$ is red, the result is clear. Otherwise, $c_1(vx) = c_2(vx)$ and $c_1(vy) = c_2(vy)$, and so $c_1(xy) = c_2(xy)$ by P-equivalence.

Next we claim that, for any two vertices $x, y \in R(v)$, the edges from x and y to $BG(v)$ are either of the same colour in the two graphs, or differ by an interchange of blue and green. Suppose that $c_1(xz) = c_2(xz)$ is blue or green for some $z \in BG(v)$. Let $z' \in BG(v)$ be another point such that $c_1(xz')$ is blue or green; we must show that $c_1(xz') = c_2(xz')$. If $c(zz')$ is blue or green, then this assertion follows from P-equivalence. But since $\Gamma_1 \cong R_3$, the blue–green graph on $BG(v) \cap BG(x)$ is connected. (The induced structure on this set is isomorphic to R_3, so any two vertices in $BG(v) \cap BG(x)$ are joined by a blue–green path of length at most 2.) So the claim follows.

Now $R(v) = R^+(v) \cup R^-(v)$, where, for $x \in R(v)$, the colours of edges from x to $BG(v)$ are the same in Γ_1 and Γ_2 if $x \in R^+(v)$, and differ by a blue–green exchange if $x \in R^-(v)$. Let Γ_2' be obtained by switching Γ_2 with respect to $R^-(v)$. This switching does not change the colours in $\{v\} \cup BG(v)$, and has the result that $R^-(v)$ is empty in the switched graph. Replacing Γ_2 by Γ_2', we may assume that edges between $R(v)$ and $BG(v)$ have the same colour in Γ_1 and Γ_2.

Finally, take $x, y \in R(v)$ with $c_1(xy)$ blue or green. Again, since $\Gamma_1 \cong R_3$, there exists $z \in BG(v)$ such that $c(xz)$ and $c(yz)$ are each blue or green. Then P-equivalence ensures that $c_1(xy) = c_2(xy)$. So $\Gamma_1 = \Gamma_2$. Since we switched the original Γ_2 twice in the course of the proof, the proposition is proved.

(b) The above argument only depends on the fact that, given any set S of at most four vertices, there is a vertex joined to every vertex in S by blue or green edges. The probability that this fails in an n-vertex graph is at most

$$\sum_{i=1}^{4} \binom{n}{i} \left(1 - \left(\frac{2}{3}\right)^i\right)^{n-i},$$

which tends to zero as $n \to \infty$, so the property holds in almost all random 3-coloured finite complete graphs. □

The theorem can be expressed in another way, following [2]. Suppose that we consider the red graph as given. Let $\mathcal{C}$ be the 2-dimensional complex whose simplices are the vertices, edges, and triangles in the blue–green graph. Then P- and S-equivalence classes of the colouring with no green edges are 1-cocycles and 1-coboundaries over $\mathbb{Z}/(2)$; so the cohomology group $H^1(\mathcal{C}, \mathbb{Z}/(2))$ measures the extent to which P-equivalence fails to imply S-equivalence.

Proposition 6.2. *For the infinite random graph, and for almost all finite random graphs,* $H^1(\mathcal{C}, \mathbb{Z}/(2)) = 0$ (*where* $\mathcal{C}$ *is as above*).

Proof. The only comment required is that we need to change the probabilities so that a red edge has probability $1/2$ instead of $1/3$. □

The most general type of restricted switching works as follows. Let B be a group of permutations on the set of colours (a subgroup of $\mathrm{Sym}(m)$). A switching operation has the form $\sigma_{g,X}$ for $g \in B$ and $X \subseteq V$; it applies the permutation g to the colours of edges between X and its complement, and leaves other colours unaltered. We refer to this operation as *B-restricted switching*. In the same way, if A is any subgroup of $\mathrm{Sym}(m)$, we define an *A-restricted duality* to be the operation of permuting the colours of all the edges according to some permutation in A.

Now we can state Bennett's classification [1] of reducts of R_m. He defines a reduct to be *reducible* if there are two colours which are indistinguishable (that is, we are colour-blind for some pair of colours). The classification of reducible reducts thus simply becomes the classification of reducts of R_{m-1}, which is done by induction. Bennett shows that the irreducible reducts are generated (as topological groups) by B-reduced switchings and

A-reduced dualities, where B is an Abelian subgroup of $\mathrm{Sym}(m)$ and A a subgroup of its normaliser. The special case of Theorem 6.1 for R_3 corresponds to the case where B interchanges colours blue and green (fixing red) and A is the trivial subgroup.

Problem. Do other reducts of R_m have analogues for finite random graphs similar to Theorem 6.1(b)? Do they have cohomological interpretations?

Remark. Most of the results of this paper are taken from the second author's Ph.D. thesis [10].

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