Invariant and attracting sets of impulsive delay difference equations with continuous variables

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Received 23 June 2007; received in revised form 11 September 2007; accepted 10 October 2007

Abstract

The aim of this paper is to study the invariant and attracting sets of impulsive delay difference equations with continuous variables. Some criteria for the invariant and attracting sets are obtained by using the decomposition approach and delay difference inequalities with impulsive initial conditions.

Keywords: Invariant and attracting sets; Decomposition approach; Delay; Difference inequality; Impulsive

1. Introduction

Difference equations with continuous variables are difference equations in which the unknown function is a function of a continuous variable. These equations appear as natural descriptions of observed evolution phenomena in many branches of the natural sciences (see, e.g., [1,2]). The book [2] presents an exposition of some unusual properties of difference equations, in particular, of difference equations with continuous variables.

Philos et al. [3] obtained some significant results on the asymptotic behaviour of scalar delay difference equations with continuous variables; Shaikhet [4,5] obtained some stable results of difference equations and stochastic difference equations with continuous variables by using Lyapunov Functionals, respectively; Romanenko [6] discussed the attractors of continuous difference equations. For some results on the oscillation of difference equations with continuous variables, one can refer to [7–13] and the references cited therein.

However, besides the delay effect, an impulsive effect likewise exists in a wide variety of evolutionary process, in which states are changed abruptly at certain moments of time. Moreover, equations with impulses may exhibit several real world phenomena, such as rhythmical beating, the merging of solutions, and non-continuity of solutions. Their study is assuming a greater importance [14,15]. In [16,17], some oscillatory results on impulsive difference equations with continuous (or discrete) variables are obtained. In [18–21], some stable conditions on impulsive difference equations with continuous (or discrete) variables are given. As is well known, stability is one of the major problems encountered in applications, and has attracted considerable attention due to its important role in applications. However,
under impulsive perturbation, an equilibrium point sometimes does not exist in many physical systems, especially, in nonlinear dynamical systems. Therefore, an interesting subject is to discuss the invariant sets and the attracting sets of impulsive systems. Some significant progress has been made in the techniques and methods of determining the invariant sets and attracting sets for delay difference equations with discrete variables, delay differential equations and impulsive functional differential equations [22–25]. Unfortunately, the corresponding problems for impulsive delay difference equations with continuous (or discrete) variables have not been considered.

Motivated by the above-mentioned papers and discussion, we here make a first attempt to arrive at results on the invariant sets and the attracting sets of impulsive systems. Some significant progress has been made in the techniques and methods of determining the invariant sets and the attracting sets of nonlinear dynamical systems. Therefore, an interesting subject is to discuss the invariant sets and the attracting sets of impulsive systems. Some significant progress has been made in the techniques and methods of determining the invariant sets and the attracting sets of impulsive systems. Therefore, an interesting subject is to discuss the invariant sets and the attracting sets of nonlinear dynamical systems.

2. Preliminaries

Consider the impulsive delay difference equation with continuous variable

\[
\begin{cases}
x_i(t) = a_i x_i(t - \sigma) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau)), & t \geq t_0, t \neq t_k, \\
x_i(t_k) = I_{ik}(x_i(t_k^-)), & k = 1, 2, \ldots,
\end{cases}
\]

(1)

where \(a_i, b_{ij} \quad (i, j = 1, 2, \ldots, n)\) are real constants, \(\sigma\) and \(\tau\) are positive real numbers such that \(\tau > \sigma\), \(t_k (k = 1, 2, \ldots)\) is an impulsive sequence such that \(t_1 < t_2 < \cdots < t_k < \cdots\) and \(\lim_{k \to \infty} t_k = \infty\). \(f_j, I_{ik} : \mathbb{R} \to \mathbb{R}\) are real-valued functions.

By a solution of (1), we mean a piecewise continuous real-valued function \(x_i(t)\) defined on the interval \([t_0 - \tau, \infty)\) which satisfies (1) for all \(t \geq t_0\).

In the sequel, by \(\Phi_i\) we will denote the set of all continuous real-valued functions \(\phi_i\) defined on an interval \([t_0 - \tau, t_0]\), which satisfies the “compatibility condition”

\[
\phi_i(t_0) = a_i \phi(t_0 - \sigma) + \sum_{j=1}^{n} b_{ij} f_j(\phi_j(t_0 - \tau)).
\]

(2)

By the method of steps, one can easily see that, for any given initial function \(\phi_i \in \Phi_i\), there exists a unique solution \(x_i(t), i = 1, \ldots, n,\) of (1) which satisfies the initial condition

\[
x_i(t) = \phi_i(t), \quad \text{for} \quad t \in [t_0 - \tau, t_0],
\]

(3)

this function will be called the solution of the initial problem (1)–(3).

For convenience, we rewrite (1) and (3) into the following vector form

\[
\begin{cases}
x(t) = Ax(t - \sigma) + Bf(x(t - \tau)), & t \geq t_0, t \neq t_k, \\
x(t_k) = I_k(x(t_k^-)), & k = 1, 2, \ldots, \\
x(t) = \phi(t), & t \in [t_0 - \tau, t_0],
\end{cases}
\]

(4)

where \(x(t) = (x_1(t), \ldots, x_n(t))^T, A = \text{diag}[a_1, \ldots, a_n], B = (b_{ij})_{n \times n}, f(x) = (f_1(x_1), \ldots, f_n(x_n))^T, I_k = (I_{1k}(x_1), \ldots, I_{nk}(x_n))^T, \phi = (\phi_1, \ldots, \phi_n)^T \in \Phi,\) in which \(\Phi = (\phi_1, \ldots, \phi_n)^T.\)

In what follows, we will introduce some notations and basic definitions.

Let \(\mathbb{R}^n\) be the space of \(n\)-dimensional real column vectors and \(\mathbb{R}^{m \times n}\) denote the set of \(m \times n\) real matrices. \(E\) denotes an identical matrix with appropriate dimensions. For \(A, B \in \mathbb{R}^{m \times n}\) or \(A, B \in \mathbb{R}^n, A \geq B (A > B)\) means that each pair of corresponding elements of \(A\) and \(B\) satisfies the inequality “\(\geq (>)\)”. Particularly, \(A\) is called a nonnegative matrix if \(A \geq 0\) and is denoted by \(A \in \mathbb{R}^{m \times n}_+\), and \(z\) is called a positive vector if \(z > 0\). \(\rho(A)\) denotes the spectral radius of \(A\).

\(C[X, Y]\) denotes the space of continuous mappings from the topological space \(X\) to the topological space \(Y\).

\(PC[I, \mathbb{R}^n] \triangleq \{\phi : I \to \mathbb{R}^n | \phi(t^+) = \phi(t) \text{ for } t \in I, \phi(t^-) \text{ exists for } t \in I \text{ and } \phi(t^-) = \phi(t) \text{ except for points } t_k \in I\}\), where \(I \subset \mathbb{R}\) is an interval, \(\phi(t^+)\) and \(\phi(t^-)\) denote the right limit and left limit of function \(\phi(t)\), respectively. Especially, let \(PC = PC([t_0 - \tau, t_0], \mathbb{R}^n)\).
**Definition 1.** The set $S \subset PC$ is called a positive invariant set of (4), if for any initial value $\phi \in S$, we have the solution $x(t) \in S$ for $t \geq t_0$.

**Definition 2.** The set $S \subset PC$ is called a global attracting set of (4), if for any initial value $\phi \in PC$, the solution $x(t)$ converges to $S$ as $t \to +\infty$. That is,

$$\text{dist}(x(t), S) \to 0, \quad \text{as} \quad t \to +\infty,$$

where $\text{dist}(\varphi, S) = \inf_{\psi \in S} \text{dist}(\varphi, \psi)$, $\text{dist}(\varphi, \psi) = \sup_{s \in [t_0 - \tau, t_0]} |\varphi(s) - \psi(s)|$ for $\varphi \in PC$. In particular, $S = \{0\}$ is called asymptotically stable.

Following [26], we split the matrices $A, B$ into two parts, respectively,

$$A = A^+ - A^-, \quad B = B^+ - B^-$$

with $a_i^+ = \max\{a_i, 0\}, a_i^- = \max\{-a_i, 0\}, b_{ij}^+ = \max\{b_{ij}, 0\}, b_{ij}^- = \max\{-b_{ij}, 0\}$.

Then the first equation of (4) can be rewritten as

$$x(t) = (A^+ - A^-)x(t - \sigma) + (B^+ - B^-)f(x(t - \tau)). \quad (5)$$

Now take the symmetric transformation $y = -x$. From (5), it follows that

$$x(t) = A^+x(t - \sigma) + A^-y(t - \sigma) + B^+f(x(t - \tau)) + B^-g(y(t - \tau)), \quad (6)$$

and

$$y(t) = A^+y(t - \sigma) + A^-x(t - \sigma) + B^+g(y(t - \tau)) + B^-f(x(t - \tau)), \quad (7)$$

where $g(u) = -f(-u)$.

Set

$$z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad h(z(t)) = \begin{bmatrix} f(x(t)) \\ g(y(t)) \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} A^+ & A^- \\ A^- & A^+ \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B^+ & B^- \\ B^- & B^+ \end{bmatrix}.$$

So, by (6) and (7), we have

$$z(t) = \mathcal{A}z(t - \sigma) + \mathcal{B}h(z(t - \tau)). \quad (8)$$

Set $J_k(v) = -I_k(-v)$; in view of the impulsive part of (4), we also have $x(t_k) = I_k(x(t_k^-)), y(t_k) = J_k(y(t_k^-))$, so

$$z(t_k) = \omega_k(z(t_k^-)), \quad k = 1, 2, \ldots, \quad (9)$$

where $\omega_k(z) = (I_k(x)^T, J_k(y)^T)^T$.

**Lemma 1** ([27]). Suppose that $M \in R^{n \times n}_+$ and $\rho(M) < 1$; then there exists a positive vector $z$ such that

$$(E - M)z > 0.$$

For $M \in R^{n \times n}_+$ and $\rho(M) < 1$, we denote

$$\Omega_\rho(M) = \{z \in R^n | (E - M)z > 0, z > 0\}.$$

By Lemma 1, we have the following result.

**Lemma 2.** $\Omega_\rho(M)$ is nonempty, and for any scalars $k_1 \geq 0, k_2 \geq 0, k_1k_2 \neq 0$ and vectors $z_1, z_2 \in \Omega_\rho(M)$, we have

$$k_1z_1 + k_2z_2 \in \Omega_\rho(M).$$

**Lemma 3.** Assume that $u(t) = (u_1(t), \ldots, u_n(t))^T \in C[[t_0, \infty), R^n]$ satisfy

$$\begin{cases} u(t) \leq Mu(t - \sigma) + Nu(t - \tau) + J, \quad t \geq t_0, \\ u(\theta) \in PC, \quad \theta \in [t_0 - \tau, t_0], \end{cases} \quad (10)$$

where $M = (m_{ij}), N = (n_{ij}) \in R^{n \times n}_+, J \in R^n_+$. 

If \( \rho(M + N) < 1 \), then there exists a positive vector \( v = (v_1, \ldots, v_n)^T \) such that
\[
u(t) \leq ve^{-\lambda(t-t_0)} + (E - M - N)^{-1}J, \quad t \geq t_0,
\] (11)
where \( \lambda > 0 \) is a constant and defined as
\[
(E - Me^{\lambda \tau} - Ne^{\lambda \tau})v \geq 0
\]
for the given \( v \).

**Proof.** Since \( M, N \in R_+^{n \times n} \) and \( \rho(M + N) < 1 \), by Lemma 1, there exists a positive vector \( p \in \Omega_\rho(M + N) \) such that \( (E - M - N)p > 0 \). By continuity, there is at least one positive solution \( \lambda \) for (12), i.e.,
\[
\sum_{j=1}^{n} (m_{ij}e^{\lambda \sigma} + n_{ij}e^{\lambda \tau}) p_j \leq p_i, \quad i = 1, \ldots, n.
\] (13)

For \( u(\theta) \in PC, \theta \in [t_0 - \tau, t_0] \), there exists a positive constant \( l > 1 \) such that
\[
u(\theta) \leq lpe^{-\lambda(\theta-t_0)} + W, \quad \theta \in [t_0 - \tau, t_0],
\] (14)
where \( W = (E - M - N)^{-1}J \).

By Lemma 2, \( lp \in \Omega_\rho(M + N) \); so, without loss of generality, we can find a \( v \in \Omega_\rho(M + N) \) such that
\[
\sum_{j=1}^{n} (m_{ij}e^{\lambda \sigma} + n_{ij}e^{\lambda \tau}) v_j \leq v_i.
\] (15)
and
\[
u(\theta) \leq ve^{-\lambda(\theta-t_0)} + W, \quad \theta \in [t_0 - \tau, t_0].
\] (16)

Set \( u(t) = v(t)e^{-\lambda(t-t_0)} + W, t \geq t_0 \); substituting this into (10), we have
\[
v(t)e^{-\lambda(t-t_0)} + W \leq M(v(t - \sigma)e^{-\lambda(t-\sigma-t_0)} + W) + N\left(v(t - \tau)e^{-\lambda(t-\tau-t_0)} + W\right) + J,
\] (17)
that is
\[
v(t) \leq Me^{\lambda \tau} v(t - \sigma) + Ne^{\lambda \tau} v(t - \tau).
\] (18)

By (16), we get that
\[
u(\theta) \leq v, \quad \theta \in [t_0 - \tau, t_0].
\] (19)

Next, we will prove for any \( t \geq t_0 \),
\[
u(t) \leq v.
\] (20)

To this end, we consider an arbitrary number \( \varepsilon > 0 \), we claim that
\[
u(t) < (1 + \varepsilon)v, \quad t \geq t_0.
\] (21)

Otherwise, by the continuity of \( u(t) \), there must exist a \( t^* > t_0 \) and an index \( r \) such that
\[
u(t) < (1 + \varepsilon)v, \quad \text{for } t \in [t_0, t^*), \quad v_r(t^*) = (1 + \varepsilon)v_r.
\] (22)

Then, by using (18), from (15), we obtain
\[
(1 + \varepsilon)v_r = v_r(t^*) \leq \sum_{j=1}^{n} (m_{rj}v_j(t^* - \sigma)e^{\lambda \sigma} + n_{rj}v_j(t^* - \tau)e^{\lambda \tau})
\[
< \sum_{j=1}^{n} (m_{rj}e^{\lambda \sigma} + n_{rj}e^{\lambda \tau}) (1 + \varepsilon)v_j
\]
\[
\leq (1 + \varepsilon)v_r,
\]
which is a contradiction. Hence, (21) holds for all numbers \( \varepsilon > 0 \); it follows immediately that (20) is always satisfied, which can easily be led to (11). This completes the proof. \( \square \)
3. Main results

For convenience, we introduce the following assumptions.

(H1) For any \(x, y \in \mathbb{R}^n\), there exists a nonnegative matrix \(P = (p_{ij})_{n \times n} \geq 0\) and a nonnegative vector \(\mu = (\mu_1, \ldots, \mu_n)^T \geq 0\) such that
\[
f(x) - f(y) \leq P(x - y) + \mu. \tag{23}\]

(H2) For any \(x, y \in \mathbb{R}^n\), there exist nonnegative matrices \(Q_k = (q_{ij}^k)_{n \times n} \geq 0\) and a nonnegative vector \(v = (v_1, \ldots, v_n)^T \geq 0\) such that
\[
I_k(x) - I_k(y) \leq Q_k(x - y) + v, \quad k = 1, 2, \ldots. \tag{24}\]

(H3) \(\rho(A + BP) < 1\) and \(\rho(Q_k) < 1, k = 1, 2, \ldots\), where \(P = \text{diag}(P, P)\), \(Q_k = \text{diag}(Q_k, Q_k)\).

(H4) \(\Omega = \bigcap_{k=1}^{\infty} \{\Omega_\rho(Q_k)\} \bigcap \Omega_\rho(A + BP)\) is nonempty.

**Theorem 1.** Assume that (H1)–(H4) hold. Then there exists a positive vector \(\eta = (\alpha^T, \beta^T)^T \in \Omega\), which will be given below by (33), such that \(S = \{\phi \in PC| - \beta \leq \phi \leq \alpha\}\) is a positive invariant set of (4), where \(\alpha \geq 0, \beta \geq 0, \alpha, \beta \in \mathbb{R}^n\).

**Proof.** From (23) and (24), we can claim that for any \(z \in \mathbb{R}^{2n}\),
\[
h(z) \leq \mathcal{P}z + \Lambda, \tag{25}\]
and
\[
w_k(z) \leq \mathcal{Q}_kz(t_k^-) + \Gamma, \quad k = 1, 2, \ldots, \tag{26}\]
where \(\Lambda = (\mu^T, \mu^T)^T\), \(\Gamma = (v^T, v^T)^T\).

So by using (8) and (9) and taking into account (25) and (26), we get
\[
z(t) \leq Az(t - \sigma) + B\mathcal{P}z(t - \tau) + B\Lambda, \tag{27}\]
and
\[
z(t_k) \leq \mathcal{Q}_kz(t_k^-) + \Gamma, \quad k = 1, 2, \ldots, \tag{28}\]
respectively.

By assumptions (H3), (H4) and Lemma 1, there exists a positive vector \(\eta_1 \in \Omega\) such that
\[
(E - A - B\mathcal{P})\eta_1 > 0, \tag{29}\]
and
\[
(E - \mathcal{Q}_k)\eta_1 > 0, \quad k = 1, 2, \ldots. \tag{30}\]

Since \(B\Lambda\) and \(\Gamma\) are positive constant vectors, by (29) and (30), there must exist two scalars \(k_1 > 0, k_2 > 0\) such that
\[
(E - A - B\mathcal{P})k_1\eta_1 \geq B\Lambda, \tag{31}\]
and
\[
(E - \mathcal{Q}_k)k_2\eta_1 \geq \Gamma, \quad k = 1, 2, \ldots. \tag{32}\]
respectively.

Set
\[
\eta = (\alpha^T, \beta^T)^T \triangleq \max\{k_1, k_2\}\eta_1, \tag{33}\]
by Lemma 2, clearly, \(\eta \in \Omega\) and
\[
(E - A - B\mathcal{P})\eta \geq B\Lambda, \tag{34}\]
and
\[ (E - Q_k)\eta \geq \Gamma, \quad k = 1, 2, \ldots . \] (35)

Next, we will prove, for any \(-\beta \leq \phi \leq \alpha\), i.e., \(z(t) \leq \eta, \quad t \in [t_0 - \tau, t_0]\),
\[ z(t) \leq \eta, \quad t \in [t_0, t_1). \] (36)

In order to prove (36), we first prove, for any \(\epsilon > 0\),
\[ z(t) < (1 + \epsilon)\eta, \quad t \in [t_0, t_1). \] (37)

If (37) is false, by the piecewise continuous nature of \(z(t)\), there must exist a \(t^* \in [t_0, t_1)\) and an index \(m\) such that
\[ z(t) < (1 + \epsilon)\eta, \quad \text{for } t \in [t_0, t^*), \quad z_m(t^*) = (1 + \epsilon)\eta_m. \] (38)

Denoting \(A = (c_{ij})_{2n \times 2n}, BP = (d_{ij})_{2n \times 2n}, B\Lambda = (\lambda_1, \ldots, \lambda_{2n})\), we get
\[
(1 + \epsilon)\eta_m = z_m(t^*) \leq \sum_{j=1}^{2n} \left( c_{mj}z_j(t^* - \sigma) + d_{mj}z_j(t^* - \tau) \right) + \lambda_m
\]
\[
< \sum_{j=1}^{2n} \left( c_{mj} + d_{mj} \right) (1 + \epsilon)\eta_j + \lambda_m
\]
\[
\leq (1 + \epsilon)(\eta_m - \lambda_m) + \lambda_m
\]
\[
= (1 + \epsilon)\eta_m - \epsilon\lambda_m
\]
\[< (1 + \epsilon)\eta_m. \]

The first inequality follows from (27), the second inequality from the inequality of (38), the third inequality from (34). This is a contradiction and hence (37) holds. From the fact that (37) is fulfilled for any \(\epsilon > 0\), it follows immediately that (36) is always satisfied.

On the other hand, using (28), (30) and (36), we obtain that
\[ z(t_1) \leq Q_1z(t_1^-) + \Gamma \leq Q_1\eta + \Gamma \leq \eta. \] (39)

Therefore, we can claim that
\[ z(t) \leq \eta, \quad t \in [t_1 - \tau, t_1]. \] (40)

In a similar way to the proof of (36), we can prove that (40) implies
\[ z(t) \leq \eta, \quad t \in [t_1, t_2). \] (41)

Hence, by the induction principle, we conclude that
\[ z(t) \leq \eta, \quad t \in [t_{k-1}, t_k), \quad k = 1, 2, \ldots, \] (42)
which implies \(z(t) \leq \eta\) holds for any \(t \geq t_0\), i.e., \(-\beta \leq x(t) \leq \alpha\) for any \(t \geq t_0\). This completes the proof of the theorem. \(\square\)

**Remark 1.** If \(\alpha = \beta\), then the invariant set \(S\) is symmetric. Of course, it is evident that the asymmetric invariant set may give a more accurate trajectory behaviour than a symmetric one.

**Remark 2.** In fact, under the assumptions of **Theorem 1**, the \(\eta\) must exist, for example, since \(\rho(A + B\Pi) < 1\) and \(\rho(Q_k) < 1\) imply \((E - A - B\Pi)^{-1} > 0\) and \((E - Q_k)^{-1} > 0\), respectively, so we may take \(\eta\) as the following
\[ \eta = \max\{(E - A - B\Pi)^{-1}B\Pi, (E - Q_k)^{-1}\Gamma\}. \] (43)

**Theorem 2.** If assumptions (H1)–(H4) hold, then the \(S = \{\phi \in PC| -\beta \leq \phi \leq \alpha\}\) is a global attracting set of (4), where \(\alpha \geq 0, \beta \geq 0, \alpha, \beta \in \mathbb{R}^n\), the vector \(\eta = (\alpha^T, \beta^T)^T\) is chosen as (43).
Proof. From the proof of Theorem 1, we see that assumptions (H1) and (H2) imply that (25) and (26) hold, respectively, and so do (27) and (28). From (27), assumption (H3) and Lemma 3, and taking into account the definition of \( \eta \), we obtain that

\[
    z(t) \leq z e^{-\lambda (t - t_0)} + (E - A - BP)^{-1} BA \leq z e^{-\lambda (t - t_0)} + \eta, \quad t \neq t_k, \ k = 1, 2, \ldots,
\]

where the positive vector \( z \in \Omega \) and \( \lambda > 0 \) satisfying

\[
    (E - A e^{\lambda \sigma} - BP e^{\lambda \tau}) z \geq 0.
\]

From (28) and taking into account the definitions of \( \eta, z \), we get that

\[
    z(t_1) \leq Q_1 z(t^-) + \Gamma' \leq Q_1 z e^{-\lambda (t_1 - t_0)} + Q_1 \eta + \Gamma' \leq z e^{-\lambda (t_1 - t_0)} + \eta.
\]

Therefore, we have that

\[
    z(t) \leq z e^{-\lambda (t - t_0)} + \eta, \quad t \in [t_1 - \tau, t_1].
\]

So using (44) and (47) and Lemma 3 again, we obtain that

\[
    z(t) \leq z e^{-\lambda (t - t_0)} + \eta, \quad t \in [t_1, t_2).
\]

Hence, by the induction principle, we conclude that

\[
    z(t) \leq z e^{-\lambda (t - t_0)} + \eta, \quad \text{for all} \ t \in [t_0, t_k), \ k = 1, 2, \ldots
\]

which implies that the conclusion holds. The proof is complete. \( \Box \)

References


