Note

An upper bound for the competition numbers of graphs

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A R T I C L E   I N F O

Article history:
Received 5 January 2009
Received in revised form 28 June 2009
Accepted 2 September 2009
Available online 24 September 2009

Keywords:
Competition graph
Competition number
Chordal graph
Hole

A B S T R A C T

A hole of a graph is an induced cycle of length at least 4. Kim (2005) [2] conjectured that the competition number \( k(G) \) is bounded by \( h(G) + 1 \) for any graph \( G \), where \( h(G) \) is the number of holes of \( G \). In Lee et al. [3], it is proved that the conjecture is true for a graph whose holes are mutually edge-disjoint. In Li et al. (2009) [4], it is proved that the conjecture is true for a graph, all of whose holes are independent. In this paper, we prove that Kim’s conjecture is true for a graph \( G \) satisfying the following condition: for each hole \( C \) of \( G \), there exists an edge which is contained only in \( C \) among all induced cycles of \( G \).

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1. Introduction and preliminaries

In this paper, all undirected/directed graphs are nontrivial, finite and simple. An undirected graph \( G = (V(G), E(G)) \) is simply called a graph, and a directed graph \( D = (V(D), A(D)) \) is called a digraph in short. Each element \((u, v)\) of \( A(D) \) is called an arc from \( u \) to \( v \). A digraph is acyclic if it contains no directed cycles. The competition graph of a digraph \( D = (V(D), A(D)) \) (see [6] for its background) is a graph \( C(D) \) on \( V(D) \) with the set of edges

\[ E(C(D)) = \{ uv \mid \text{there exists a vertex } x \in V(D) \text{ such that } (u, x), (v, x) \in A(D) \}. \]

Let \( G \) be a graph and \( I_k \) a set of \( k \) isolated vertices each of which is not a vertex of \( G \). It is not difficult to see that there exists an acyclic digraph \( D \) on \( V(G) \cup I_{E(G)} \) such that \( C(D) = G \cup I_{E(G)} \). The competition number of \( G \) is defined as

\[ \min \{ k \mid \text{there exists an acyclic digraph } D \text{ such that } C(D) = G \cup I_k \}, \]

and is denoted by \( k(G) \). Roberts [6] and Opsut [5] presented some upper and lower bounds for \( k(G) \) and determined the competition numbers for some classes of graphs. The following are results given in [6].

Proposition 1.1 (Roberts [6]).

1. For any chordal graph \( G \), we have \( k(G) \leq 1 \).
2. For any nontrivial connected triangle-free graph \( G \), we have the equality \( k(G) = |E(G)| - |V(G)| + 2 \).

A hole of \( G \) is an induced cycle of length at least 4. Let \( h(G) \) be the number of holes of \( G \). Recently, Cho and Kim [1] proved that \( k(G) \leq 2 \) for a graph \( G \) with exactly one hole. Then, in [2], Kim conjectured that the inequality \( k(G) \leq h(G) + 1 \) holds for any graph \( G \). In [3], Lee, Kim, Kim and Sano proved that the conjecture is true for a graph whose holes are mutually edge-disjoint. In [4], Li and Chang showed that the conjecture is true for a graph, all of whose holes are independent.
Let us consider the following condition on graphs.

\((*)\) For each hole \(C\) of a graph, there exists an edge which is contained only in \(C\) among all induced cycles of the graph.

We remark that all induced cycles in the condition \((*)\) include triangles. We prove that Kim’s conjecture is true if a graph satisfies the condition \((*)\) (see Theorem 2.1). We note that the condition \((*)\) and the one that all holes are mutually edge-disjoint do not imply each other. We also notice that the condition \((*)\) and the one that all holes are independent do not imply each other.

We examine the relation between \(h(G)\) and the following graph invariant

\[ l(G) := \min\{|E(G) \setminus E(H)| \mid H \text{ is a chordal subgraph of } G\}. \]

This parameter is inspired by the proof of Theorem 2.1 which tells us that the inequality \(k(G) \leq l(G) + 1\) holds for any graph \(G\) (see Remark 2.2).

2. The relation between upper bounds for the competition numbers of graphs

**Theorem 2.1.** If a graph \(G\) satisfies the condition \((*)\), then the inequality

\[ k(G) \leq h(G) + 1 \]

holds.

**Proof.** By the assumption, for each hole \(C\) of \(G\), there exists an edge which is contained only in \(C\) among all induced cycles of \(G\). We then pick one such edge \(e_C\) from each hole \(C\) of \(G\) and let

\[ E := \{e_C = u_iv_i \mid C_i \text{ is a hole of } G\}. \]

We note \(|E| = h(G)\). We prove the following:

(i) the subgraph \(G - E\) of \(G\) is chordal, and

(ii) there exists an acyclic digraph \(D\) such that \(C(D) = G \cup I_{h(G)+1}\).

To verify (i), suppose that \(G - E\) is not chordal. Then, we see that \(G - E\) contains a hole \(C'\) of \(G - E\). Note that \(C'\) is not a hole of \(G\). Thus, \(C'\) has a chord \(e_C \in E\) in \(G\) for some hole \(C\) of \(G\) and the chord \(e_C\) is contained in an induced cycle other than \(C\). This contradicts the choice of \(e_C\).

Next, we prove (ii). Since \(G - E\) is chordal, we have \(k(G - E) \leq 1\) by Proposition 1.1, (1). Then, there exists an acyclic digraph \(D' = (V(D'), A(D'))\) such that \(C(D') = (G - E) \cup \{x_0\}\). Let \(I_{h(G)+1} = \{x_0, \ldots, x_{h(G)}\}\) be a set of \((h(G) + 1)\) isolated vertices each of which is not a vertex of \(G\). We define a digraph \(D\) on \(V(G) \cup I_{h(G)+1}\) by

\[ A(D) = A(D') \cup \{(u_i, x_i), (v_i, x_i) \mid i = 1, \ldots, h(G)\}. \]

It is easy to see that \(D\) is acyclic and \(C(D) = G \cup I_{h(G)+1}\). Thus, our conclusion is proved. \(\square\)

**Remark 2.2.** In the proof of Theorem 2.1, the property that \(G - E\) is chordal is used only to obtain \(k(G - E) \leq 1\). In view of (ii) above, we define the following graph invariant

\[ l(G) := \min\{|E(G) \setminus E(H)| \mid H \text{ is a chordal subgraph of } G\} \]

for a graph \(G\). By reviewing the proof of (ii), we see that the inequality

\[ k(G) \leq l(G) + 1 \]

holds. We also notice that

\[ l(G) \leq |E(G) \setminus E(G - E)| = |E| = h(G) \]

under the condition \((*)\).

In what follows, we focus on the relation between \(l(G)\) and \(h(G)\) for a graph \(G\). The upper bound \(l(G) + 1\) given in (2.1) is sharp if \(G\) is chordal (\(l(G) = 0\)). First, we give an example of a graph with \(l(G) \geq 1\) such that the upper bound is sharp.

**Example 2.3.** Let \(G\) be the graph defined by

\[ V(G) = \{u, v, x_1, \ldots, x_n, y_1, \ldots, y_n\} \quad \text{and} \quad E(G) = \{uv, wx_i, vy_i, x_1y_1 \mid i = 1, \ldots, n\} \]

(see Fig. 1, \(n = 3\)). It is not difficult to see \(l(G) = n\). Actually, we notice that \(G - F\) is not chordal for any set \(F\) of \((n - 1)\) edges of \(G\). We also see that \(G - \{x_1y_1 \mid i = 1, \ldots, n\}\) is chordal. On the other hand, since \(G\) is triangle-free, we have

\[ k(G) = |E(G)| - |V(G)| + 2 = (3n + 1) - (2n + 2) + 2 = n + 1 = l(G) + 1 \]

by Proposition 1.1, (2). Notice that \(l(G) = n = h(G)\).
Fig. 1. An example of a graph $G$ such that $k(G) = l(G) + 1$ holds.

Fig. 2. An example of a graph $G$ satisfying $l(G) > h(G)$.

Fig. 3. An example of a graph $G$ such that $l(G) < h(G)$ holds but $G$ does not satisfy ($\ast$).

One cannot hope in general that $l(G) \leq h(G)$ holds (see the following example).

**Example 2.4.** Here we give a graph such that $l(G) > h(G)$ holds. Let $G$ be the graph with $h(G) = 1$ illustrated in Fig. 2. We can check that the graph $G - e$ is not chordal for any edge $e \in E(G)$, so we see $l(G) > 1 = h(G)$. Also we notice that $G - \{xy, yz\}$ is chordal. Hence, we see $l(G) = 2$.

Next we give a graph $G$ such that the inequality $l(G) \leq h(G)$ holds but $G$ does not satisfy the condition ($\ast$).

**Example 2.5.** Let $G$ be the graph with $h(G) = 6$ depicted in Fig. 3. We see that $G$ does not satisfy the condition ($\ast$) since every edge of the 4-cycle illustrated by thick lines is contained in two or more induced cycles. On the other hand, we notice that $G - \{v_3 v_7, v_5 v_6, v_6 v_7\}$ is chordal, so we have $l(G) \leq 3 < 6 = h(G)$.

In [5], for a graph $G$, the graph invariant $i(G)$ was defined as the minimum number of cliques which cover the edges of $G$. As a graph invariant similar to $i(G)$, it is natural to define another invariant

$$l'(G) := \min\{n \mid E(G) = \bigcup_{i=1}^{n} E(G_i), G_i \text{ is a connected chordal subgraph of } G\}.$$  

We see that the inequality $l'(G) \leq l(G) + 1$ holds for any connected graph $G$. However, we remark that the invariant $l'(G)$ is not an upper bound for $k(G)$ in general (see the following example).

**Example 2.6.** Let $G$ be the graph depicted in Fig. 4. Since $G$ is triangle-free, we have $k(G) = 12 - 8 + 2 = 6$ by Proposition 1.1, (2). We also see $l'(G) > 1$ since $G$ is not chordal. The graph $G$ can be represented as the union of two chordal graphs (see Fig. 4 below), so we have $l'(G) = 2$. 

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$G$

$\begin{array}{c}
\text{x} \\
\text{y} \\
\text{z}
\end{array}$

$h(G) = 1, l(G) = 2$

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$G$

$\begin{array}{c}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5 \\
v_6 \\
v_7 \\
v_8 \\
\text{...}
\end{array}$

---

$G$

$\begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
x_8 \\
\text{...}
\end{array}$

$l(G) = 3$
Fig. 4. An example of a graph $G$ satisfying $l'(G) < k(G)$.

Acknowledgments

The author would like to thank Y. Sano for giving some advice and helpful comments for this research.

References