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Toughness and Hamiltonicity of a class of planar graphs

T. Gerlach

Department of Mathematics, Technical University of Ilmenau, D-98684 Ilmenau, Germany

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Abstract

A graph G is called chordal if every cycle of G of length at least four has a chord. By a theorem of Böhme, Harant and Tkáč more than 1-tough chordal planar graphs are Hamiltonian. We prove that this is even true for more than 1-tough planar graphs under the weaker assumption that separating cycles of length at least four have chords.

Keywords: Hamiltonian cycles; Planar graph; Chordal graph; Toughness

1. Introduction and results

All graphs considered are undirected, finite, connected and simple. We denote by $V(G)$ the set of vertices and by $E(G)$ the set of edges of a graph G. A good reference for terminology and notation not defined here is [5].

Let the l-wheel W^l be the graph obtained from a cycle C on l vertices by adding a new vertex (called the *central vertex of* W^l) and joining it to all vertices of C by an edge.

Let G be a graph with induced subgraphs G_1, G_2 and S, such that $G = G_1 \cup G_2$ and $S = G_1 \cap G_2$. If, additionally, G is a planar graph and S is a facial 3-cycle of G_1 and G_2, we write $G := f_{\triangle}(G_1, G_2)$.

Let C be a cycle of a graph G, φ an orientation of C and $u, v \in V(C)$. We denote by $[u, v]$ the path from u to v of C following φ. Let $]u, v[:= [u, v] - \{u, v\}$. If $[u, v]$ consists only of the edge (uv), we define $V(]u, v[) = \emptyset$. For $x \in V(G) - V(C)$ with $|N_G(x) \cap V(C)| > 1$ and $y \in N_G(x) \cap V(C)$ we set $y_x^+, y_x^- \in V(C) \cap (N_G(x) - \{y\})$, such that $V(]y, y_x^+[) \cap N_G(x) = \emptyset$ respectively $V(]y_x^-, y[) \cap N_G(x) = \emptyset$ ($N_G(x)$ denotes the set of neighbours of x in G).

The number of components of a graph H is denoted by $\omega(H)$. A subset S of $V(G)$ *separates* G, if $\omega(G - S) > 1$. We call a cycle C of a graph G a *separating i-cycle of G* (short: SiC) if $|V(C)| = i$ and $V(C)$ separates G.

Let $h(G)$ denote the length of a longest cycle of a graph G. The *shortness exponent* of an infinite class of graphs Γ was introduced by Grünbaum and Walther [9] as a measure of the non-Hamiltonicity of the class, and is defined by

$$\sigma(\Gamma) = \liminf_{G \in \Gamma} \left(\frac{\log h(G)}{\log |V(G)|} \right).$$

It is not hard to prove that, if $\sigma(\Gamma) < 1$, then there is a sequence $G_1, G_2, \ldots$ of graphs in Γ such that $\frac{h(G_i)}{|V(G_i)|} \to 0$ as $i \to \infty$.

A graph G is said to be *t-tough* if for every separating set $S \subseteq V(G)$ the number $\omega(G - S)$ of components of $G - S$ is at most $\frac{|S|}{t}$. The *toughness* of a graph G, denoted by $\tau(G)$, is the maximum value of t for which G is t-tough. For a complete graph G we define $\tau(G) = \infty$.

E-mail address: tobias.gerlach@tu-ilmenau.de (T. Gerlach).

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The concept of toughness was introduced by Chvátal in 1973 [4]. Clearly, 1-toughness is a necessary condition for Hamiltonicity, but it is not sufficient. In [4], Chvátal made the following conjecture:

Conjecture 1 (Chvátal [4]). There exists t_0 such that every t_0-tough graph is Hamiltonian.

In [4] Chvátal constructed $\frac{3}{2}$-tough graphs without Hamiltonian cycles. Based on this, Chvátal made the following stronger conjecture:

Conjecture 2 (Chvátal [4]). Every t-tough graph with $t > \frac{3}{2}$ is Hamiltonian.

This conjecture was first disproved by Thomassen (see [2]). In 1985 Enomoto et al. [8] have shown that every 2-tough graph has a 2-factor, while for every $\varepsilon > 0$ there is a $(2-\varepsilon)$-tough graph without a 2-factor, and hence without a Hamiltonian cycle, for arbitrary $\varepsilon > 0$. For a long time the following conjecture appeared to be both reasonable and best possible.

Conjecture 3 (2-tough conjecture). Every t-tough graph with $t \geqslant 2$ is Hamiltonian.

In 1999 Bauer et al. [1] disproved the 2-tough conjecture by constructing a $(\frac{9}{4}-\varepsilon)$-tough non-Hamiltonian graph for arbitrary $\varepsilon > 0$. Conjecture 1 remains open.

More is known about the existence of Hamiltonian cycles in planar graphs. By a theorem of Tutte [14] Conjecture 2 is true for planar graphs. In 1993 Harant [10] has shown that for planar graphs Conjecture 2 is best possible, by constructing $\frac{3}{2}$-tough non-Hamiltonian planar graphs.

A planar graph is *maximal* if the insertion of any edge makes it non-planar. In 1979, Chvátal raised the following question: Is 1-toughness a sufficient condition for a maximal planar graph to be Hamiltonian? In 1980, Nishizeki [11] answered Chvátal's question by constructing a 1-tough non-Hamiltonian maximal planar graph. Dillencourt [6] has shown that the shortness exponent of the class of 1-tough maximal planar graphs is at most $\frac{\log 6}{\log 7}$. Tkáč [13] improved the bound on the shortness exponent to $\frac{\log 5}{\log 6}$ and presented a non-Hamiltonian 1-tough maximal planar graph with a minimum number of vertices. $(\frac{3}{2}-\varepsilon)$-tough maximal planar non-Hamiltonian graphs were constructed by Owens [12] for arbitrary $\varepsilon > 0$.

A graph G is *chordal* if it contains no chordless cycle of length at least four. It is easy to see that every chordal polyhedral graph is maximal planar, too. In 1999 Böhme et al. [3] have proved the following result:

Theorem 1 (Böhme et al. [3]). *Every chordal planar graph with toughness greater than one is Hamiltonian, while the shortness exponent of the class of all* 1*-tough chordal planar graphs is at most* $\frac{\log 8}{\log 9}$.

A graph G is *sep-chordal* if it contains no separating chordless cycle of length at least four. We generalize the result of Böhme, Harant and Tkáč to the following theorem:

Theorem 2. *Every sep-chordal planar graph with toughness greater than one is Hamiltonian.*

2. Proofs

Before proceeding with the proof of Theorem 2, two useful results on sep-chordal planar graphs are presented.

Let Γ be the set of all polyhedral graphs without a SiC for all $i \geqslant 3$. Furthermore, let G_0 be defined as in Fig. 1.

Lemma 1.

$$\Gamma = \left(\bigcup_{l=3}^{\infty}\{W^l\}\right) \cup \{G_0\}.$$

Proof of Lemma 1. Let $G \in \Gamma$ and $x \in V(G)$ an arbitrary vertex of G. It is easy to see that there is an embedding of $G - \{x\}$ into the plane with a facial cycle $C^\star$ such that $N_G(x) \subseteq V(C^\star)$. Let φ be an orientation of $C^\star$. Since G has no SiC for $i \geqslant 3$, it follows that $V(G-\{x\}) - V(C^\star) = \emptyset$ and hence $V(G) = V(C^\star) \cup \{x\}$.

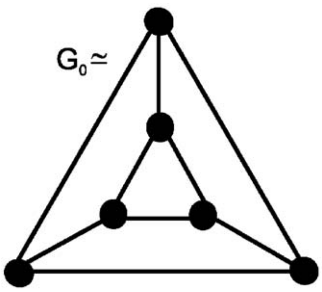

Fig. 1. The graph G_0.

For $N_G(x) = V(C^\star)$ we get $E(G) = E(C^\star) \cup \{(xy) \mid y \in V(C^\star)\}$ since G has no $S3C$, and hence $G \simeq W^l$ with $l = |V(C^\star)| \geqslant 3$.

If $N_G(x) \subset V(C^\star)$ there are two vertices $u, v \in N_G(x) \cap V(C^\star)$ with $v = u_x^+$ and $V(]u, v[) \neq \emptyset$. Since $G - \{u, v\}$ is connected, it follows that there is a path from a vertex of $V(]u, v[)$ to a vertex of $V(]v, u[)$, which consists of only one edge e. Let $e = (yz)$ with $y \in V(]u, v[)$ and $z \in V(]v, u[)$. We consider the cases $z \in N_G(x)$ and $z \notin N_G(x)$.

(i) $z \in N_G(x)$. Since G has no SiC for $i \geqslant 3$ and $G - \{u', v'\}$ is connected for any two vertices $u', v' \in V(G)$, we obtain successively $V(]z, u[) = \emptyset$, $V(]v, z[) = \emptyset$, $E(G) = E(C^\star) \cup \{(xu), (xv), (xz)\} \cup \{(zy) \mid y \in V(]u, v[)\}$, and hence $G \simeq W^l$ with $l \geqslant 4$.

(ii) $z \notin N_G(x)$. Since G is 3-connected, there exists another vertex $w \in (V(C^\star) - V([u, v])) \cap N_G(x)$. W.l.o.g., let $z \in V(]v, w[)$. As G has no SiC for $i \geqslant 3$, $G - \{u, y\}$ is connected, and using the planarity of G, we obtain successively $V(]w, u[) = \emptyset$, $V(]y, v[) = \emptyset$, $V(]v, z[) = \emptyset$, $V(]z, w[) \cap N_G(x) = \emptyset$, $V(]u, y[) = \emptyset$, $V(]z, w[) = \emptyset$, and hence $G \simeq G_0$. □

Now, we want to generalize the perfect elimination order for chordal polyhedral graphs defined in [3]. Let $\langle G^k, G^{k-1}, \ldots, G^0 \rangle$ be a sequence of graphs with $G^i \in \Gamma$ for all $i = 0, \ldots, k$. Further, let $H^0 \simeq G^0$ and let H^i be constructed from G^i and H^{i-1} by $H^i := f_\triangle(G^i, H^{i-1})$ for all $i = 1, \ldots, k$. If $G \simeq H^k$, then we call $\langle G^k, G^{k-1}, \ldots, G^0 \rangle$ a *perfect Γ-elimination order of G of length k*. Clearly, every graph $G \in \Gamma$ has a trivial perfect Γ-elimination order of length 0. By a result of Dirac [7] a graph G is a chordal polyhedral graph if and only if it has a perfect Γ-elimination order $\langle G^k, G^{k-1}, \ldots, G^0 \rangle$ with $G^i \simeq W^3$ for all $i = 0, \ldots, k$. We prove by induction on the length of a perfect Γ-elimination order that every graph G with a perfect Γ-elimination order is a sep-chordal polyhedral graph. By definition a sep-chordal polyhedral graph has no SiC for $i > 3$. Let G be a polyhedral graph and G^1, G^2 two induced subgraphs of G with $G = f_\triangle(G^1, G^2)$. It is easy to see that, if G is a sep-chordal polyhedral graph, then G^1 and G^2 are sep-chordal polyhedral graphs and, if G^1 and G^2 have a perfect Γ-elimination order, then G has a perfect Γ-elimination order. By induction on the number of $S3Cs$ of a sep-chordal polyhedral graph we have:

Lemma 2. *A polyhedral graph is sep-chordal if and only if it has a perfect Γ-elimination order.*

Proof of Theorem 2. The following proof is based on the ideas of the proof of Theorem 1 in [3]. Let G be a sep-chordal planar graph with toughness greater than one. Hence, G is 3-connected. Let $\langle G^k, G^{k-1}, \ldots, G^0 \rangle$ be a perfect Γ-elimination order of G and let the graphs $H^0, \ldots, H^k$ be as in the definition of the perfect Γ-elimination order. We call a 3-cycle D of H^i an *active 3-cycle* of H^i, if D is separating in G and non-separating in H^i. Let Δ_i be the set of all active 3-cycles of H^i. An *extendable pair* of H^i is an ordered pair (C, f), where C is a Hamiltonian cycle of H^i and f is an injection that maps every active 3-cycle D of H^i onto an edge $f(D)$ such that $f(D) \in E(C \cap D)$.

We claim: There is an extendable pair (C_i, f_i) of H^i for all $i = 0, \ldots, k$.

The proof is by induction on i. By definition of the perfect Γ-elimination order we have $H^0 \in \Gamma$. It is easy to see that every graph $H^0 \in \Gamma$ is Hamiltonian and there exists, for every Hamiltonian cycle C_0 of H^0, an injection f_0^* that maps every facial 3-cycle of H^0 onto an edge of C_0. Thus, if f_0 is the restriction of f_0^* to Δ_0, the pair (C_0, f_0) is an extendable pair of H^0.

We proceed with the induction step. Let $i \geqslant 1$ and $H^i = f_\triangle(G^i, H^{i-1})$ with $D_{i-1} = G^i \cap H^{i-1}$, $V(D_{i-1}) = \{a, b, c\}$, and $f_{i-1}(D_{i-1}) = (ab)$. Since D_{i-1} is an $S3C$ of H^i, D_{i-1} is not an active 3-cycle of H^i. For all active 3-cycles $D \in \Delta_i \cap \Delta_{i-1}$ we define $f_i(D) := f_{i-1}(D)$.

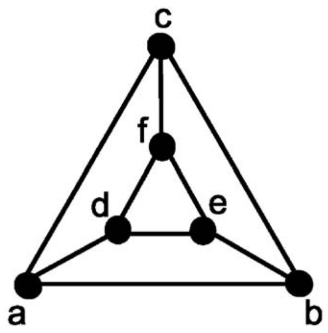

Fig. 2. Case 1: $G^i \simeq G_0$.

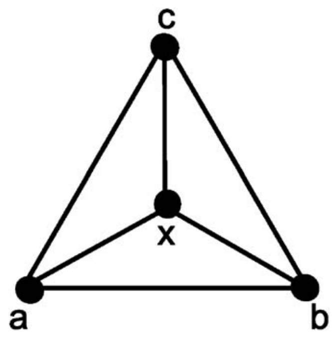

Fig. 3. Case 2: $G^i \simeq W^3$.

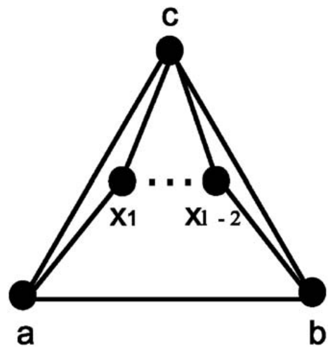

Fig. 4. Case 3.1: c is the central vertex of W^l.

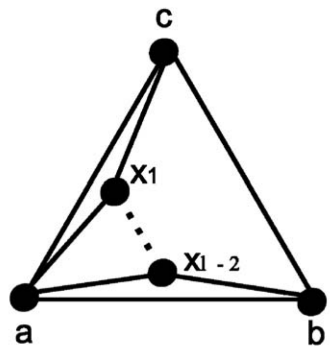

Fig. 5. Case 3.2: a is the central vertex of W^l.

We consider the following cases:

(i) *Case* 1: $G^i \simeq G_0$ (Fig. 2).
Let C_i be the Hamiltonian cycle of H^i obtained from C_{i-1} by replacing the edge (ab) by the path $(adfeb)$. There is precisely one new 3-cycle $C_{def} = (defd)$ in H^i. If C_{def} is an active 3-cycle of H^i we define $f_i(C_{def}) := (df)$.

(ii) *Case* 2: $G^i \simeq W^3$ (Fig. 3).
Let C_i be the Hamiltonian cycle of H^i obtained from C_{i-1} by replacing the edge (ab) by the path (axb). There are precisely three new 3-cycles in H^i, $C_a = (axca)$, $C_b = (bxcb)$, and $C_c = (axba)$. Since G is more than 1-tough at most two of them are active 3-cycles of H^i. Let $f_i(C_a) := (ax)$ if C_a is an active 3-cycle of H^i and $f_i(C_b) := (xb)$ if C_b is an active 3-cycle of H^i. If C_c is an active 3-cycle of H^i we define $f_i(C_c) := (xb)$ if C_a is an active 3-cycle of H^i and $f_i(C_c) := (ax)$ if C_a is not an active 3-cycle of H^i.

(iii) *Case* 3: $G^i \simeq W^l$ $(l \geqslant 4)$
Case 3.1: c is the central vertex of W^l (Fig. 4).
Let C_i be the Hamiltonian cycle of H^i obtained from C_{i-1} by replacing the edge (ab) by the path $(ax_1 \dots x_{l-2}b)$. There are precisely $l-1$ new 3-cycles in H^i, $C_a = (cx_1ac)$, $C_b = (cx_{l-2}bc)$, and $C_c^j = (cx_jx_{j+1}c)$ for $j = 1, \dots, l-3$. If C_a is an active 3-cycle of H^i we define $f_i(C_a) := (ax_1)$ and if C_b is an active 3-cycle of H^i we define $f_i(C_b) := (x_{l-2}b)$. For all active 3-cycles C_c^j of H^i we set $f_i(C_c^j) := (x_jx_{j+1})$.
Case 3.2: a is the central vertex of W^l (Fig. 5).
Let C_i be the Hamiltonian cycle of H^i obtained from C_{i-1} by replacing the edge (ab) by the path $(ax_1 \dots x_{l-2}b)$. There are precisely $l-1$ new 3-cycles in H^i, $C_b = (ax_{l-2}ba)$, $C_c = (ax_1ca)$, and $C_a^j = (ax_jx_{j+1}a)$ for $j = 1, \dots, l-3$. If C_b is an active 3-cycle of H^i we define $f_i(C_b) := (x_{l-2}b)$ and if C_c is an active 3-cycle of H^i we define $f_i(C_c) := (ax_1)$. For all active 3-cycles C_a^j of H^i we set $f_i(C_a^j) := (x_jx_{j+1})$.

In all cases (C_i, f_i) is an extendable pair of H^i. This proves Theorem 2. □

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