Toughness and Hamiltonicity of a class of planar graphs

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Abstract

A graph \( G \) is called chordal if every cycle of \( G \) of length at least four has a chord. By a theorem of Böhme, Harant and Tkáč more than 1-tough chordal planar graphs are Hamiltonian. We prove that this is even true for more than 1-tough planar graphs under the weaker assumption that separating cycles of length at least four have chords.

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1. Introduction and results

All graphs considered are undirected, finite, connected and simple. We denote by \( V(G) \) the set of vertices and by \( E(G) \) the set of edges of a graph \( G \). A good reference for terminology and notation not defined here is [5].

Let the \( l \)-wheel \( W^l \) be the graph obtained from a cycle \( C \) on \( l \) vertices by adding a new vertex (called the central vertex of \( W^l \)) and joining it to all vertices of \( C \) by an edge.

Let \( G \) be a graph with induced subgraphs \( G_1, G_2 \) and \( S \), such that \( G = G_1 \cup G_2 \) and \( S = G_1 \cap G_2 \). If, additionally, \( G \) is planar and \( S \) is a facial 3-cycle of \( G_1 \) and \( G_2 \), we write \( G : f_\Delta(G_1, G_2) \).

Let \( C \) be a cycle of a graph \( G \), \( \varphi \) an orientation of \( C \) and \( u,v \in V(C) \). We denote by \( [u,v] \) the path from \( u \) to \( v \) of \( C \) following \( \varphi \). Let \( [u,v] := [u,v] - \{u,v\} \). If \( [u,v] \) consists only of the edge \( (uv) \), we define \( V([u,v]) = \emptyset \). For \( x \in V(G) - V(C) \) with \( |N_G(x) \cap V(C)| > 1 \) and \( y \in N_G(x) \cap V(C) \) we set \( y_x^1, y_x^2 \in V(C) \cap (N_G(x) - \{y\}) \), such that \( V(y_x^1, y_x^2) \cap N_G(x) = \emptyset \) respectively \( V(y_x^1, y_x^2) \cap N_G(x) = \emptyset \) (\( N_G(x) \) denotes the set of neighbours of \( x \) in \( G \)).

The number of components of a graph \( H \) is denoted by \( \omega(H) \). A subset \( S \) of \( V(G) \) separates \( G \), if \( \omega(G - S) > 1 \). We call a cycle \( C \) of a graph \( G \) a separating i-cycle of \( G \) (short: \( SIC \)) if \( |V(C)| = i \) and \( V(C) \) separates \( G \).

Let \( h(G) \) denote the length of a longest cycle of a graph \( G \). The shortness exponent of an infinite class of graphs \( I \) was introduced by Grünbaum and Walther [9] as a measure of the non-Hamiltonicity of the class, and is defined by

\[
\sigma(I) = \liminf_{G \in I} \left( \frac{\log h(G)}{\log |V(G)|} \right).
\]

It is not hard to prove that, if \( \sigma(I) < 1 \), then there is a sequence \( G_1, G_2, \ldots \) of graphs in \( I \) such that \( \frac{h(G_i)}{|V(G_i)|} \to 0 \) as \( i \to \infty \).

A graph \( G \) is said to be \( t \)-tough if for every separating set \( S \subseteq V(G) \) the number \( \omega(G - S) \) of components of \( G - S \) is at most \( \frac{|S|}{t} \). The toughness of a graph \( G \), denoted by \( \tau(G) \), is the maximum value of \( t \) for which \( G \) is \( t \)-tough. For a complete graph \( G \) we define \( \tau(G) = \infty \).

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The concept of toughness was introduced by Chvátal in 1973 [4]. Clearly, 1-toughness is a necessary condition for Hamiltonicity, but it is not sufficient. In [4], Chvátal made the following conjecture:

**Conjecture 1** (Chvátal [4]). There exists $t_0$ such that every $t_0$-tough graph is Hamiltonian.

In [4] Chvátal constructed $\frac{3}{2}$-tough graphs without Hamiltonian cycles. Based on this, Chvátal made the following stronger conjecture:

**Conjecture 2** (Chvátal [4]). Every $t$-tough graph with $t > \frac{3}{2}$ is Hamiltonian.

This conjecture was first disproved by Thomassen (see [2]). In 1985 Enomoto et al. [8] have shown that every 2-tough graph has a 2-factor, while for every $\varepsilon > 0$ there is a $(2 - \varepsilon)$-tough graph without a 2-factor, and hence without a Hamiltonian cycle, for arbitrary $\varepsilon > 0$. For a long time the following conjecture appeared to be both reasonable and best possible.

**Conjecture 3** (2-tough conjecture). Every $t$-tough graph with $t \geq 2$ is Hamiltonian.

In 1999 Bauer et al. [1] disproved the 2-tough conjecture by constructing a $(\frac{3}{4} - \varepsilon)$-tough non-Hamiltonian graph for arbitrary $\varepsilon > 0$. Conjecture 1 remains open.

More is known about the existence of Hamiltonian cycles in planar graphs. By a theorem of Tutte [14] Conjecture 2 is true for planar graphs. In 1993 Harant [10] has shown that for planar graphs Conjecture 2 is best possible, by constructing $\frac{3}{2}$-tough non-Hamiltonian planar graphs.

A planar graph is **maximal** if the insertion of any edge makes it non-planar. In 1979, Chvátal raised the following question: Is 1-toughness a sufficient condition for a maximal planar graph to be Hamiltonian? In 1980, Nishizeki [11] answered Chvátal’s question by constructing a 1-tough non-Hamiltonian maximal planar graph. Dillencourt [6] has shown that the shortness exponent of the class of 1-tough maximal planar graphs is at most $\log_2 \log_6 \log_7 T k$. Tkáč [13] improved the bound on the shortness exponent to $\log_5 \log_6$ and presented a non-Hamiltonian 1-tough maximal planar graph with a minimum number of vertices. $(\frac{3}{2} - \varepsilon)$-tough maximal planar non-Hamiltonian graphs were constructed by Owens [12] for arbitrary $\varepsilon > 0$.

A graph $G$ is **chordal** if it contains no chordless cycle of length at least four. It is easy to see that every chordal polyhedral graph is maximal planar, too. In 1999 Böhme et al. [3] have proved the following result:

**Theorem 1** (Böhme et al. [3]). Every chordal planar graph with toughness greater than one is Hamiltonian, while the shortness exponent of the class of all 1-tough chordal planar graphs is at most $\log_8 \log_9$.

A graph $G$ is **sep-chordal** if it contains no separating chordless cycle of length at least four. We generalize the result of Böhme, Harant and Tkáč to the following theorem:

**Theorem 2.** Every sep-chordal planar graph with toughness greater than one is Hamiltonian.

2. Proofs

Before proceeding with the proof of Theorem 2, two useful results on sep-chordal planar graphs are presented.

Let $\Gamma$ be the set of all polyhedral graphs without a $SiC$ for all $i \geq 3$. Furthermore, let $G_0$ be defined as in Fig. 1.

**Lemma 1.**

$$\Gamma = \left( \bigcup_{i=3}^{\infty} \{W_i\} \right) \cup \{G_0\}.$$  

**Proof of Lemma 1.** Let $G \in \Gamma$ and $x \in V(G)$ an arbitrary vertex of $G$. It is easy to see that there is an embedding of $G - \{x\}$ into the plane with a facial cycle $C^*$ such that $N_G(x) \subseteq V(C^*)$. Let $\phi$ be an orientation of $C^*$. Since $G$ has no $SiC$ for $i \geq 3$, it follows that $V(G - \{x\}) - V(C^*) = \emptyset$ and hence $V(G) = V(C^*) \cup \{x\}$. 


For $G_0(x) = V(C^*)$ we get $E(G) = E(C^*) \cup \{(xy) \mid y \in V(C^*)\}$ since $G$ has no $S3C$, and hence $G \simeq W^l$ with $l = |V(C^*)| \geq 3$.

If $G_0(x) \subset V(C^*)$ there are two vertices $u, v \in N_G(x) \cap V(C^*)$ with $v = u'$ and $G\{u', v\} \neq \emptyset$. Since $G - \{u, v\}$ is connected, it follows that there is a path from a vertex of $V\{u', v\}$ to a vertex of $V\{v, u\}$, which consists of only one edge $e$. Let $e = (yz)$ with $y \in V\{u, v\}$ and $z \in V\{v, u\}$. We consider the cases $z \in N_G(x)$ and $z \notin N_G(x)$.

(i) $z \in N_G(x)$. Since $G$ has no $SiC$ for $i \geq 3$ and $G - \{u', v\}$ is connected for any two vertices $u', v' \in V(G)$, we obtain successively $V\{u, v\} = \emptyset$, $V\{v, z\} = \emptyset$, $E(G) = E(C^*) \cup \{(ux), (xz)\} \cup \{(zy) \mid y \in V\{u, v\}\}$, and hence $G \simeq W^l$ with $l \geq 4$.

(ii) $z \notin N_G(x)$. Since $G$ is 3-connected, there exists another vertex $w \in (V(C^*) - V\{u, v\}) \cap N_G(x)$. W.l.o.g., let $z \in V\{u, w\}$. As $G$ has no $SiC$ for $i \geq 3$, $G - \{u, y\}$ is connected, and using the planarity of $G$, we obtain successively $V\{u, w\} = \emptyset$, $V\{v, y\} = \emptyset$, $V\{v, z\} = \emptyset$, $V\{z, w\} \cap N_G(x) = \emptyset$, $V\{u, y\} = \emptyset$, $V\{v, z\} = \emptyset$, and hence $G \simeq G_0$. □

Now, we want to generalize the perfect elimination order for chordal polyhedral graphs defined in [3]. Let $(G^0, G^{k-1}, \ldots, G^0)$ be a sequence of graphs with $G^i \in \Gamma$ for all $i = 0, \ldots, k$. Further, let $H^0 \simeq G^0$ and let $H^i$ be constructed from $G^i$ and $H^{i-1}$ by $H^i := f_\Delta(G^i, H^{i-1})$ for all $i = 1, \ldots, k$. If $G \simeq H^k$, then we call $(G^0, G^{k-1}, \ldots, G^0)$ a perfect $\Gamma$-elimination order of $G$ of length $k$. Clearly, every graph $G \in \Gamma$ has a trivial perfect $\Gamma$-elimination order of length 0. By a result of Dirac [7] a graph $G$ is a chordal polyhedral graph if and only if it has a perfect $\Gamma$-elimination order $(G^0, G^{k-1}, \ldots, G^0)$ with $G^i \simeq W^l$ for all $i = 0, \ldots, k$. We prove by induction on the length of a perfect $\Gamma$-elimination order that every graph $G$ with a perfect $\Gamma$-elimination order is a sep-chordal polyhedral graph. By definition a sep-chordal polyhedral graph has no $SiC$ for $i > 3$. Let $G$ be a polyhedral graph and $G^0, G^1, G^2$ two induced subgraphs of $G$ with $G \simeq f_\Delta(G^0, G^2)$. It is easy to see that, if $G$ is a sep-chordal polyhedral graph, then $G^0$ and $G^2$ are sep-chordal polyhedral graphs and, if $G^1$ and $G^2$ have a perfect $\Gamma$-elimination order, then $G$ has a perfect $\Gamma$-elimination order. By induction on the number of $S3C$s of a sep-chordal polyhedral graph we have:

**Lemma 2.** A polyhedral graph is sep-chordal if and only if it has a perfect $\Gamma$-elimination order.

**Proof of Theorem 2.** The proof is based on the ideas of the proof of Theorem 1 in [3]. Let $G$ be a sep-chordal planar graph with toughness greater than one. Hence, $G$ is 3-connected. Let $(G^0, G^{k-1}, \ldots, G^0)$ be a perfect $\Gamma$-elimination order of $G$ and let the graphs $H^0, \ldots, H^k$ be as in the definition of the perfect $\Gamma$-elimination order. We call a 3-cycle $D$ of $H^i$ an active 3-cycle of $H^i$, if $D$ is separating in $G$ and non-separating in $H^i$. Let $A_i$ be the set of all active 3-cycles of $H^i$. An extendable pair of $H^i$ is an ordered pair $(C, f)$, where $C$ is a Hamiltonian cycle of $H^i$ and $f$ is an injection that maps every active 3-cycle $D$ of $H^i$ onto an edge $f(D)$ such that $f(D) \in E(C \cap D)$.

We claim: There is an extendable pair $(C_0, f_0)$ of $H^i$ for all $i = 0, \ldots, k$.

The proof is by induction on $i$. By definition of the perfect $\Gamma$-elimination order we have $H^0 \in \Gamma$. It is easy to see that every graph $H^0 \in \Gamma$ is Hamiltonian and there exists, for every Hamiltonian cycle $C_0$ of $H^0$, an injection $f_0^* \subseteq E(C_0 \cap D)$ that maps every facial 3-cycle of $H^0$ onto an edge of $C_0$. Thus, if $f_0$ is the restriction of $f_0^*$ to $A_0$, the pair $(C_0, f_0)$ is an extendable pair of $H^0$.

We proceed with the induction step. Let $i \geq 1$ and $H^i = f_\Delta(G^i, H^{i-1})$ with $D_{i-1} = G^i \cap H^{i-1}$, $V(D_{i-1}) = \{a, b, c\}$, and $f_{i-1}(D_{i-1}) = \{ab\}$. Since $D_{i-1}$ is an $S3C$ of $H^i$, $D_{i-1}$ is not an active 3-cycle of $H^i$. For all active 3-cycles $D \in A_i \cap A_{i-1}$ we define $f_i(D) := f_{i-1}(D)$.
We consider the following cases:

(i) Case 1: $G' \simeq G_0$ (Fig. 2).

Let $C_i$ be the Hamiltonian cycle of $H'$ obtained from $C_{i-1}$ by replacing the edge $(ab)$ by the path $(defb)$. There is precisely one new 3-cycle $C_{def} = (defd)$ in $H'$. If $C_{def}$ is an active 3-cycle of $H'$ we define $f_i(C_{def}) := (df)$.

(ii) Case 2: $G' \simeq W^3$ (Fig. 3).

Let $C_i$ be the Hamiltonian cycle of $H'$ obtained from $C_{i-1}$ by replacing the edge $(ab)$ by the path $(axb)$. There are precisely three new 3-cycles in $H'$, $C_a = (axca), C_b = (bxcb), and C_c = (axba). Since G is more than 1-tough at most two of them are active 3-cycles of $H'$. Let $f_i(C_a) := (ax)$ if $C_a$ is an active 3-cycle of $H'$ and $f_i(C_b) := (xb)$ if $C_b$ is an active 3-cycle of $H'$. If $C_c$ is an active 3-cycle of $H'$ we define $f_i(C_c) := (xb)$ if $C_a$ is an active 3-cycle of $H'$ and $f_i(C_c) := (ax)$ if $C_a$ is not an active 3-cycle of $H'$.

(iii) Case 3: $G' \simeq W^l$ ($l \geq 4$)

Case 3.1: $c$ is the central vertex of $W^l$ (Fig. 4).

Let $C_i$ be the Hamiltonian cycle of $H'$ obtained from $C_{i-1}$ by replacing the edge $(ab)$ by the path $(ax_1 \ldots x_{l-2}b)$. There are precisely $l-1$ new 3-cycles in $H'$, $C_a = (ax_1ac), C_b = (bx_{l-2}bc), and C_c = (ax_1x_{j+1}c)$ for $j = 1, \ldots, l-3$. If $C_a$ is an active 3-cycle of $H'$ we define $f_i(C_a) := (ax_1)$ and if $C_b$ is an active 3-cycle of $H'$ we define $f_i(C_b) := (x_{l-2}b)$. For all active 3-cycles $C_{c_j}$ of $H'$ we set $f_i(C_{c_j}) := (x_jx_{j+1})$.

Case 3.2: $a$ is the central vertex of $W^l$ (Fig. 5).

Let $C_i$ be the Hamiltonian cycle of $H'$ obtained from $C_{i-1}$ by replacing the edge $(ab)$ by the path $(ax_1 \ldots x_{l-2}b)$. There are precisely $l-1$ new 3-cycles in $H'$, $C_b = (ax_{l-2}ba), C_c = (ax_1ca), and C_a = (ax_1x_{j+1}a)$ for $j = 1, \ldots, l-3$. If $C_b$ is an active 3-cycle of $H'$ we define $f_i(C_b) := (x_{l-2}b)$ and if $C_c$ is an active 3-cycle of $H'$ we define $f_i(C_c) := (ax_1a)$. For all active 3-cycles $C_{a_j}$ of $H'$ we set $f_i(C_{a_j}) := (x_jx_{j+1})$.

In all cases $(C_i, f_i)$ is an extendable pair of $H'$. This proves Theorem 2. □
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References